APPLICATIONS OF GAUSS BONNET THEOREM

LECTURE 28

- **FLAT TORUS** (we put plane geometry on torus in Example 2.2 of Chpt. 7). We calculated explicitly that the point-set with induced geometry also had $\iint_T K\,dM = 0$ because $K<0$ and $K>0$ symmetrically. Well, the Gauss Bonnet Th^m shows $\iint_T K\,dM = 0$ merely due to $\chi(T) = 0$.

  $\chi(\Sigma(h)) = 2-2h = 0 \Rightarrow \underline{h=1}$. (one handle)

- Deleting point from sphere Σ. In Example 2.4 (1) of Chpt 7 we saw $\exists$ a flat-metric for the ~~deleted~~ punctured sphere (this means $K=0$ as calculated from metric). However,

  $$\boxed{\nexists \text{ a metric on whole sphere } \Sigma \text{ for which } K \leq 0}$$

  If there was then $\iint K\,dM \leq 0$ however, the Euler characteristic is <u>independent</u> of the choice of metric, it's topological, $2\pi\chi(\Sigma) > 0$.

> If a compact, orientable, geometric surface M has $K>0$ then M is diffeomorphic to a sphere.

Proof: $K>0 \Rightarrow \chi(M)>0$. But, $\chi(M) = 2-2h \Rightarrow h=0 \Rightarrow$ M sphere. //

APPLICATIONS OF GAUSS BONNET CONTINUED

- Here I pick summary pts. of § 7.7 pgs. 376 – 384 of Oneil.

Integration naturally extends to oriented polygons: $\iint_{\mathcal{P}} d\phi = \int_{\partial \mathcal{P}} \phi$.

Corollary (7.4) (p. 378): The following properties of a compact oriented surface are equivalent

(1.) $\exists$ a nonvanishing tangent vector field on M

(2.) $\chi(M) = 0$

(3.) M diffeomorphic to a torus.

Proof: (1) $\Rightarrow$ (2) Let V be a nonvanishing vector field on M. Then, for any geometric structure on M, the associated frame field:

$$E_1 = \frac{1}{\|V\|} V \quad , \quad E_2 = J(E_1)$$

is a global frame on M. If ω_{12} is its connection form then $d\omega_{12} = -K\, dM$. Apply GB-Thm & generalized Stokes' to obtain:

$$2\pi \chi(M) = \iint_M K\, dM = -\iint_M d\omega_{12} = -\underbrace{\int_{\partial M}}_{\text{empty for compact oriented surface.}} \omega_{12} = 0$$

(2) $\Rightarrow$ (3) We argued on 48 that $\chi(M) = 0 \Rightarrow \underline{h = 1}$. $\therefore$ M is torus.

(3) $\Rightarrow$ (1) For torus with usual parametrization either $\mathbf{X}_u$ or $\mathbf{X}_v$ provide global, nonzero tangent fields. //

Our work with rectangles naturally transfers to similar arguments with a polygonal decomp. $\mathcal{P}$ for M. In Lee you'll see an explicit computation of GB for triangulization. In any event. The following is true:

$\boxed{\text{}}$

Th$^{\text{m}}$ (7.5) (Gauss Bonnet): If $\mathcal{P}$ is oriented polygonal region in a geometric surface then

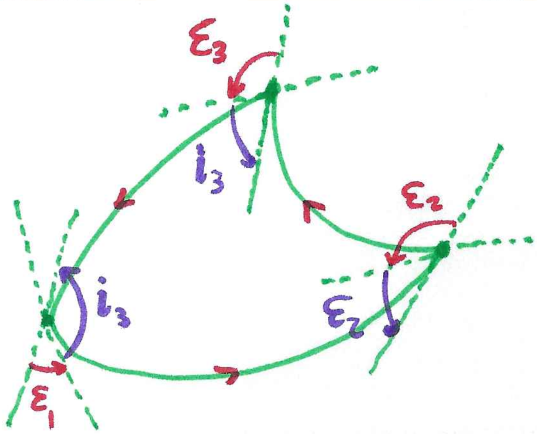
$$\iint_{\mathcal{P}} K\, dM + \int_{\partial \mathcal{P}} \kappa_g\, ds + \sum{}' \varepsilon_j = 2\pi \chi(\mathcal{P})$$

where $\sum{}' \varepsilon_j$ is sum of exterior angles at all the boundary curves comprising $\partial \mathcal{P}$

Classically, the following is of great interest,

Corollary (7.6): If Δ is a triangle in an oriented geometric surface M, then

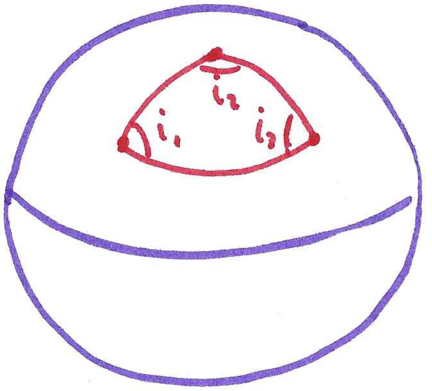
$$\iint_{\Delta} K\, dM + \int_{\partial \Delta} \kappa_g\, ds = 2\pi - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$$
$$= i_1 + i_2 + i_3 - \pi$$



If the ~~sides~~ sides are geodesic then $\kappa_g = 0$. If $K =$ constant then

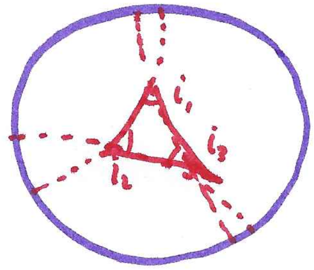
$$\boxed{K\, \text{Area}_{\Delta} = i_1 + i_2 + i_3 - \pi}$$

For example, $K = 0$ yields $\underline{i_1 + i_2 + i_3 = \pi}$.



Σ

$K > 0$



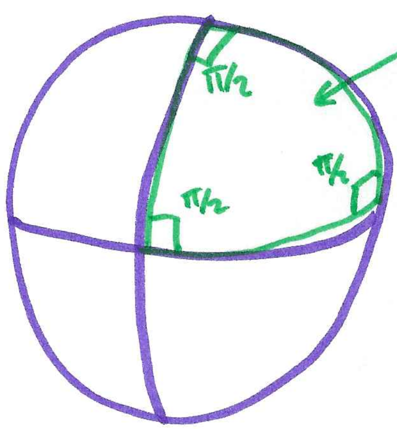
oh well, I tried.

H

$K < 0$

Δ can have an excess angle beyond π for sum of interior angles.

Δ can have a sum of interior angles less than π.



$i_1 + i_2 + i_3 = \pi + \frac{\pi}{2}$.

due to curvature of Σ. If $R = 1$ Note $A = \frac{1}{8}(4\pi R^2) = \frac{\pi}{2}$ and $K = 1$ for $R = 1$ So we get $KA = \frac{\pi}{2}$.

- For Σ_R have $K = \frac{1}{R^2}$ and $A = \frac{\pi R^2}{2}$ So again $KA = \frac{1}{R^2}\left(\frac{\pi R^2}{2}\right) = \frac{\pi}{2}$ independent of R, this example is good $\forall R > 0$.