LECTURE 29 : §3A-3B OF Kühnel

We examine the classical $\mathrm{I}, \mathrm{II}, \mathrm{III}$ form notations, this is $\approx$ Chpt. 5 of O'neill, I change Kühnel's notation to fit with our current notational scheme. Mathematically our context is that of an embedded surface in $\mathbb{R}^3$ given the <u>induced metric</u> (this is the I-form)

$$\mathrm{I}(V,W) = V \cdot W$$

$$\mathrm{II}(V,W) = \mathrm{I}(S(V),W)$$

$$\mathrm{III}(V,W) = \mathrm{I}(S(V),S(W))$$

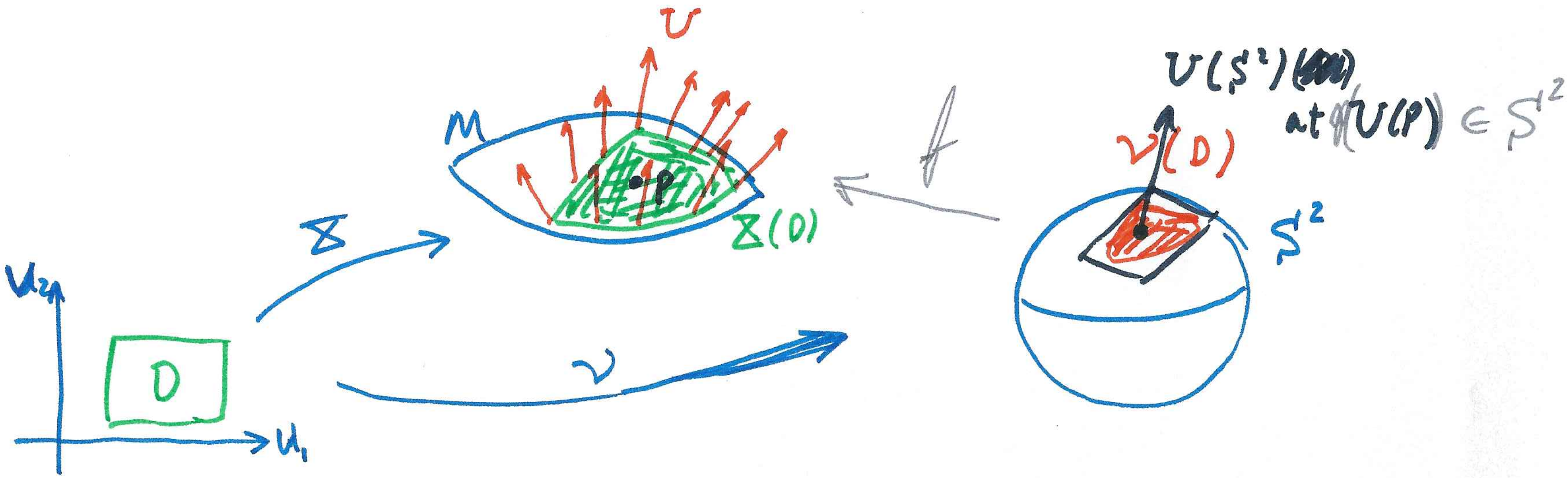
Here $\mathrm{I}, \mathrm{II}, \mathrm{III} : T_pM \times T_pM \longrightarrow \mathbb{R}$ are <u>symmetric</u> <u>bilinear</u> <u>forms</u> and w.r.t. $X : (u_1, u_2) \longmapsto X(u_1,u_2) \in M$ we have:

$$\mathrm{I}(X_{u_i}, X_{u_j}) = g_{ij} \qquad (g_{ij}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

$$\mathrm{II}(X_{u_i}, X_{u_j}) = h_{ij} \qquad (h_{ij}) = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

$$\mathrm{III}(X_{u_i}, X_{u_j}) = e_{ij}$$

SHAPE OPERATOR GIVEN BY GAUSS MAP BASED FORMULA:



$$\nu = \frac{1}{\|X_{u_1} \times X_{u_2}\|} \mathcal{P}(X_{u_1} \times X_{u_2})$$

where $\mathcal{P}(aU_1 + bU_2 + cU_3) = (a, b, c) \in \mathbb{R}^3$.
$\mathcal{P}$ converts vectors to corresponding points.

$$\boxed{\text{Def}^{n}/\quad S = -f_* \circ \nu_* \circ (X_*)^{-1} = -f \circ d\nu \circ (dX)^{-1}}$$

(SHAPE OPERATOR or Weingarten Map)

Relation between matrix of Shape Operator and II

$$(\mathrm{II}) = (h_{ij}) \Rightarrow h_{ij} = U \cdot \frac{\partial^2 X}{\partial u_i \partial u_j} = -\frac{\partial U}{\partial u_i} \cdot \frac{\partial X}{\partial u_j}$$

However,

$$S\left(\frac{\partial X}{\partial u_i}\right) = \sum_j h_i^{\ j} \frac{\partial X}{\partial u_j} \quad \text{that is} \quad S(X_{u_i}) = \sum_j h_i^{\ j} X_{u_j}$$

Thus, observe,

$$h_{ik} = S(X_{u_i}) \cdot X_{u_k} = \left(\sum_j h_i^{\ j} X_{u_j}\right) \cdot X_{u_k}$$

$$\Rightarrow h_{ik} = \sum_j h_i^{\ j} g_{jk}$$

$$\therefore \boxed{h_i^{\ j} = \sum_k h_{ik} g^{kj}} \text{ where } (g^{ij}) = (g_{ij})^{-1}$$

matrix of shape operator has one-index raised by the inverse of the metric. These calculations abound in classical physics...

Fun with the metric:

$$I = \underbrace{E^2 du\otimes du + 2F du\otimes dv + G^2 dv\otimes dv}_{\text{aka } E^2du^2 + 2F du dv + G^2 dv^2 = ds^2} = \sum_{i,j} g_{ij}\, du^i \otimes du^j$$

Integration involves the determinant of the metric as follows:

$$\iint_{f(Q)} \alpha\, dA = \iint_{Q} (\alpha\circ f)(u,v)\sqrt{\operatorname{Det}(g_{ij})}\, du\, dv$$

$$dA = \sqrt{g}\, du_1 \wedge du_2$$

Symmetry of Shape Operator:

$$\left(S(X_{u_j}) = -(D\nu)\big(\overbrace{DX^{-1}(X_{u_j})}^{\frac{\partial}{\partial u_j}}\big) = \frac{-\partial \nu}{\partial u_j} \right)$$

$$I(S(X_{u_i}), X_{u_j}) = \frac{-\partial \nu}{\partial u_i}\cdot X_{u_j} = \frac{-\partial}{\partial u_i}\big(\cancelto{0}{\nu\cdot X_{u_j}}\big) + \nu\cdot\frac{\partial^2 X}{\partial u_i \partial u_j}$$

$$I(X_{u_i}, S(X_{u_j})) = I(S(X_{u_j}), X_{u_i}) = \nu\cdot\frac{\partial^2 X}{\partial u_j \partial u_i}$$

Hence, as $\frac{\partial^2 X}{\partial u_i \partial u_j} = \frac{\partial^2 X}{\partial u_j \partial u_i} \Rightarrow I(S(v), w) = I(v, S(w))$ //