LECTURE 3: DIFFERENTIAL FORMS

$d_pf : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$, $p \longmapsto d_pf \rightarrow df$ a one-form

Wedge Product

(i.) $\wedge$ is an associative product; $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

(ii.) $\wedge$ is anticommutative on differentials; $dx^i \wedge dx^j = - dx^j \wedge dx^i$

(iii) $\wedge$ is distributive; $\alpha \wedge (c\beta + \gamma) = c\alpha \wedge \beta + \alpha \wedge \gamma$

Let $\alpha = \sum_i \alpha_i dx^i$ & $\beta = \sum_j \beta_j dx^j$

$$\alpha \wedge \beta = \left(\sum_i \alpha_i dx^i\right) \wedge \left(\sum_j \beta_j dx^j\right)$$

$$= \sum_{i,j} \alpha_i \beta_j dx^i \wedge dx^j \qquad (dx^i \wedge dx^j = -dx^j \wedge dx^i)$$

$$\alpha_i \beta_j = \beta_j \alpha_i$$

$$= -\left(\sum_j \beta_j dx^j\right) \wedge \left(\sum_i \alpha_i dx^i\right)$$

$$= -\beta \wedge \alpha.$$

0-form : function $f : \mathrm{dom}(f) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

1-form : $\alpha = \alpha_1 dx^1 + \cdots + \alpha_n dx^n = \sum_i \alpha_i \, dx^i$

2-form : $\beta = \sum_{i,j} \frac{1}{2} \beta_{ij} \, dx^i \wedge dx^j$ $\qquad (\beta_{ij} = -\beta_{ji})$

3-form : $\gamma = \sum_{i,j,k} \frac{1}{3!} \gamma_{ijk} \, dx^i \wedge dx^j \wedge dx^k$

k-form : $\omega = \sum_{i_1, \dots, i_k} \frac{1}{k!} \omega_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{I \in \mathcal{I}_k(n)} \omega_I \, dx^I$

$$\alpha \wedge \beta = \left(\sum_{I \in \mathcal{I}_p} \alpha_I \, dx^I \right) \wedge \sum_{J \in \mathcal{I}_q} \left(\beta_J \, dx^J \right)$$

$$= \sum_{\substack{I \in \mathcal{I}_p \\ J \in \mathcal{I}_q}} \alpha_I \beta_J \underbrace{dx^I \wedge dx^J}_{p+q \text{ differentials}} \leftarrow (p+q)\text{-form.}$$

$$= \sum_{\substack{I \in \mathcal{I}_p \\ J \in \mathcal{I}_q}} \beta_J \alpha_I (-1)^{pq} \, dx^J \wedge dx^I \quad (\text{see } 3)$$

$$= (-1)^{pq} \beta \wedge \alpha$$

$$\boxed{\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha}$$

$dx^i \wedge dx^j = -dx^j \wedge dx^i$

$\boxed{\underline{\text{Lemma}}: \ dx^i \wedge dx^i = 0}$

in $\mathbb{R}^n$

$\oint \underbrace{dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n}_{\substack{n\text{-form} \\ (\text{top-form})}}$

$$dx^I \wedge dx^J = (dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$
$$= dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge dx^J \wedge dx^{i_p} \, (-1)^q$$
$$= dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \wedge dx^J \wedge dx^{i_{p-1}} \wedge dx^{i_p} \, (-1)^q (-1)^q$$
$$\vdots$$
$$= dx^J \wedge dx^I \underbrace{(-1)^q (-1)^q \cdots (-1)^q}_{p\text{-fold}} = (-1)^{pq} \, dx^J \wedge dx^I$$

Exterior Derivatives

Ex ① $f = x^2 + yz$

$df = 2x\,dx + z\,dy + y\,dz$

Ex ② $\alpha = x^2\,dy + y\,dz$

$d\alpha = d(x^2) \wedge dy + dy \wedge dz = 2x\,dx \wedge dy + dy \wedge dz$

Ex ③ $\gamma = (e^{x^1} dx^2 + 2^{x^2} dx^3) \wedge dx^7 = e^{x^1} dx^2 \wedge dx^7 + 2^{x^2} dx^3 \wedge dx^7$

$d\gamma = e^{x^1} dx^1 \wedge dx^2 \wedge dx^7 + \ln(2)\, 2^{x^2} dx^2 \wedge dx^3 \wedge dx^7.$

$$\gamma = \sum_{i_1 \cdots i_k = 1}^{n} \frac{1}{k!} \gamma_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad k\text{-form}$$

$$d\gamma = \sum_{i_1 \cdots i_k = 1}^{n} \frac{1}{k!} \underbrace{d\gamma_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}}_{(k+1)} \leftarrow (k+1)\text{-form}$$

$$d\gamma_I = \sum_{j=1}^{n} \frac{\partial \gamma_I}{\partial x^j} dx^j$$

Prop/ ① $d(\alpha+\beta) = d\alpha + d\beta$

② $d(c\alpha) = c\, d\alpha$

③ $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d\beta$ ← graded product by the degree of form.

$\alpha\wedge\beta = (-1)^{|\alpha||\beta|}\beta\wedge\alpha$ 　　 α is p-form then $|\alpha| = p$.

Proof of ③ similar to

0-forms f, g, 　 $d(fg) = g\,df + f\,dg = (df)g + f(dg)$

1-forms α, β, 　 $d(\alpha\wedge\beta) = d\alpha\wedge\beta - \alpha\wedge d\beta$

$\boxed{d^2 = 0}$

$$d(d\gamma) = d\Big(\sum_{j_1, I} \frac{\partial \gamma_I}{\partial x^{j_1}} dx^{j_1} \wedge dx^I\Big)$$

$$= \sum_{j_1, j_2, I} \underbrace{\frac{\partial^2 \gamma_I}{\partial x^{j_2} \partial x^{j_1}}}_{\substack{\text{symmetric} \\ j_1 \,\&\, j_2}} \underbrace{dx^{j_2} \wedge dx^{j_1}}_{\substack{\text{antisym.} \\ \text{in } j_1 \,\&\, j_2}} \wedge dx^I = 0.$$

Ex 1 $\omega_{\vec{F}} = -d\phi$ for conservative force $\vec{F}$

If $\omega_{\vec{F}} = y^2 dx - dz$ then $d\omega_{\vec{F}} = 2y\, dy \wedge dx \neq 0 \Rightarrow \vec{F}$ not conservative.

Ex 2 $F = dA \leftarrow$ "(electric & magnetic fields in 2-form) = d (voltage vector-potential)"

$A' = A + d\lambda \hookrightarrow \underline{\underline{dA'}} = dA + d^2\lambda = \underline{\underline{dA}}$

give same F

we have gauge freedom to modify $A \longmapsto A'$ w/o changing the $\vec{E}, \vec{B}$ which result.

DIFFERENTIAL FORMS IN $\mathbb{R}^3$

$1,\ dx,\ dy,\ dz$

- 0-form $\quad f$
- 1-form $\quad \alpha = a\,dx + b\,dy + c\,dz = \omega_{\langle a,b,c\rangle}$
- 2-form $\quad \beta = a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy = \Phi_{\langle a,b,c\rangle}$
- 3-form $\quad \gamma = g\,dx\wedge dy\wedge dz$

Properties

$$\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$$

$$\omega_{\vec{A}} \wedge \omega_{\vec{B}} \wedge \omega_{\vec{C}} = \vec{A}\cdot(\vec{B}\times\vec{C})\,dx\wedge dy\wedge dz$$

$$\omega_{\vec{A}} \wedge \Phi_{\vec{B}} = (\vec{A}\cdot\vec{B})\,dx\wedge dy\wedge dz$$

Exterior Derivatives

f a function : $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \omega_{\langle \partial_x f, \partial_y f, \partial_z f\rangle} = \omega_{\nabla f}$

$\alpha = \omega_{\vec{F}}$: $d\alpha = \Phi_{\nabla\times\vec{F}}$

$\beta = \Phi_{\vec{G}}$: $d\beta = (\nabla\cdot\vec{G})\,dx\wedge dy\wedge dz$

$\boxed{d^2 = 0}$

$$d(df) = d\omega_{\nabla f} = \Phi_{\nabla\times\nabla f} = 0 \Rightarrow \underline{\nabla\times\nabla f = 0}.$$

$$d(d\alpha) = d(d\omega_{\vec{F}}) = d(\Phi_{\nabla\times\vec{F}}) = \nabla\cdot(\nabla\times\vec{F}) = 0 \Rightarrow \underline{\nabla\cdot(\nabla\times\vec{F}) = 0}.$$

<u>Example</u>: $x = r\cos\theta \qquad dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$y = r\sin\theta \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$$\underline{dx\wedge dy} = (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta)$$

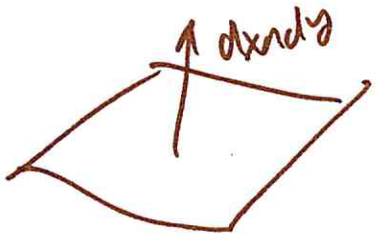
$$= \sin\theta\cos\theta\, \underbrace{\cancel{dr\wedge dr}}_{0} + r\cos^2\theta\, dr\wedge d\theta - r\sin^2\theta\, \underbrace{d\theta\wedge dr}_{-dr\wedge d\theta} - r\sin\theta\cos\theta\, \underbrace{\cancel{d\theta\wedge d\theta}}_{0}$$

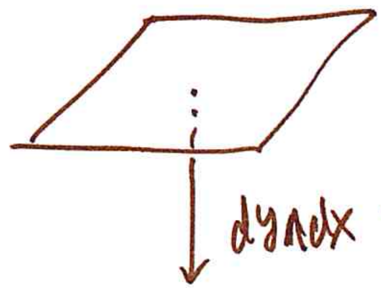
$$= r(\sin^2\theta + \cos^2\theta)\, dr\wedge d\theta$$

$$= \underline{r\, dr\wedge d\theta}$$

$$\iint_D f\, dx\, dy$$

$\underbrace{dx\wedge dy}$ vs. $dy\wedge dx$





$dy\wedge dx = -dx\wedge dy$

THE IDENTITY FOR $\nabla\cdot(\vec{A}\times\vec{B})$ via differential forms

$$\omega_{\vec{A}}\wedge\omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}$$

$$d\,\Phi_{\vec{A}\times\vec{B}} = \nabla\cdot(\vec{A}\times\vec{B})\, dx\wedge dy\wedge dz$$

$$d(\omega_{\vec{A}}\wedge\omega_{\vec{B}}) = d\omega_{\vec{A}}\wedge\omega_{\vec{B}} - \omega_{\vec{A}}\wedge d\omega_{\vec{B}}$$

$$= \Phi_{\nabla\times\vec{A}}\wedge\omega_{\vec{B}} - \omega_{\vec{A}}\wedge\Phi_{\nabla\times\vec{B}}$$

$$= \left((\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})\right) dx\wedge dy\wedge dz$$

$$\boxed{\nabla\cdot(\vec{A}\times\vec{B}) = (\nabla\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\nabla\times\vec{B})}$$