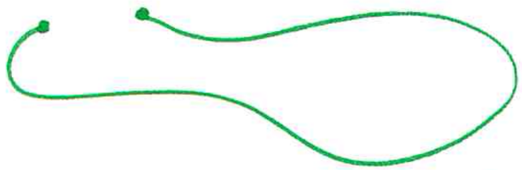
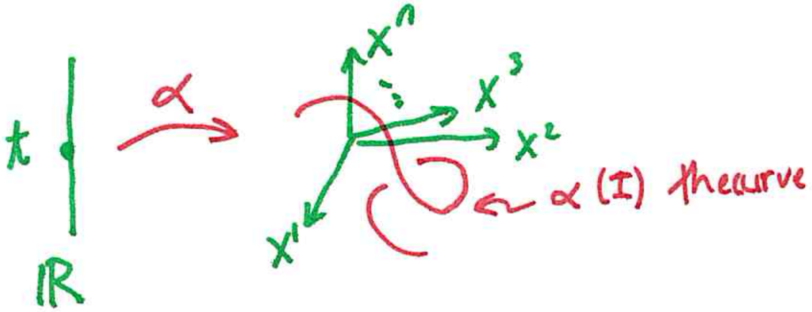
LECTURE 4: CURVES & THE PUSH-FORWARD (§1.4 AND 1.7 of O'NEILL)

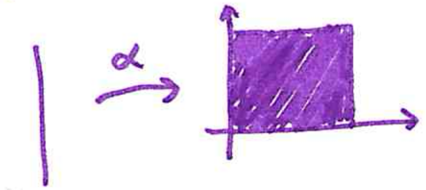


<u>smooth</u>, parametrized curve

$\alpha : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$



Ex $\alpha : \mathbb{R} \to [0,1]^2 \subseteq \mathbb{R}^2$



Hans Sagan

Velocity
of α
is an <u>operator</u>
on functions defined
near $\alpha(I)$

Ex $\alpha(t) = P$

$\alpha'(t) = 0 \quad \forall t$

Defⁿ/ $\alpha : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ smooth
and such that $\boxed{\alpha'(t) \neq 0}$ $\forall t$
is called a <u>regular parametrized curve</u>

Defⁿ/ for a smooth paramⁿ curve α and $t \in \mathrm{dom}(\alpha)$ we define

$$\alpha'(t) = \frac{d\alpha^1}{dt}\frac{\partial}{\partial x^1}\Big|_{\alpha(t)} + \cdots + \frac{d\alpha^n}{dt}\frac{\partial}{\partial x^n}\Big|_{\alpha(t)}$$

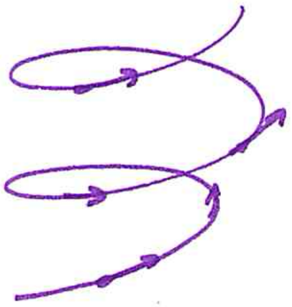
So, $\alpha'(t) \in T_{\alpha(t)}\mathbb{R}^n$

$$\alpha = (\alpha^1, \alpha^2, .., \alpha^n) \in \mathbb{R}^n$$

Ex $\alpha(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, t)$

① $\alpha'(t) = \frac{1}{\sqrt{2}}\left[-\sin t\frac{\partial}{\partial x} + \cos t\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]\Big|_{\alpha(t)}$: $\underbrace{\|\alpha'(t)\|}_{\text{speed}} = 1$

$\alpha'(t)[z] = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$



$\alpha'(t)[x^2+y^2] = \frac{-2x\sin t + 2y\cos t}{\sqrt{2}}\Big|_{\substack{x = \cos t/\sqrt{2} \\ y = \sin t/\sqrt{2}}}$

$= \frac{-2\cos t\cancel{\sin t} + 2\sin t\cancel{\cos t}}{2}$

$= 0.$

$\boxed{x^2+y^2 = 1/2}$

$\alpha'(t)\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{\sqrt{2}}\left(-\sin t\left[\frac{1}{1+y^2/x^2}\left[\frac{-y}{x^2}\right]\right] + \cos t\left[\frac{1}{1+y^2/x^2}\frac{1}{x}\right]\right)\Big|_{\alpha(t)}$

$\theta \nearrow$

$= \frac{y\sin t + x\cos t}{\sqrt{2}(x^2+y^2)}\Big|_{\alpha(t)}$

$= \frac{1/2}{\frac{1}{2}}$

$= 1.$

Ex ② $\alpha(t) = (t, t^2, t^3)$

$$\alpha'(t) = \left(1\frac{\partial}{\partial x} + \underline{\underline{2t}}\frac{\partial}{\partial y} + \underline{3t^2}\frac{\partial}{\partial z}\right)\Bigg|_{(t,t^2,t^3)}$$

$$\alpha'(t)\left[x^6 + y^3 - 2z^2\right] = \left[(1)6x^5 + \underline{\underline{2t}}(3y^2) + \underline{3t^2}(-4z)\right]\Bigg|_{\substack{x=t\\ y=t^2\\ z=t^3}}$$

$$= 6t^5 + 6tt^4 - 12t^2t^3$$

$$= 0.$$

$F(x,y,z) = x^6 + y^3 - 2z^3 = 0$ ← rep. some surface S in $\mathbb{R}^3$
and $\alpha(\mathbb{R}) \subseteq S = F^{-1}\{0\}$

Def^n/ We say $h : J \to I$ is a reparametrization of smooth param. curve $\alpha : I \longrightarrow \mathbb{R}^n$ and $\beta : J \longrightarrow \mathbb{R}^n$ where $\beta = \alpha \circ h$ is the reparametrization of α by h.

Ex ② $\|\alpha'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$ $\quad \underbrace{s(t) = \int_0^t \sqrt{1 + 4u^2 + 9u^4}\, du}_{\text{Solve for } t = h(s)}$

~~Solve [illegible]~~

then $\beta(s) = \alpha(h(s))$ is the <u>arclength para. of α</u>

Prop. 1.4.7 α a param. smooth curve and $\beta = \alpha \circ h$ a reparam. of α by h then

$$\underbrace{\beta'(s)}_{\text{operator}} = \underbrace{\frac{dh}{ds}}_{\text{scalar fnct.}} \underbrace{\alpha'(h(s))}_{\text{operation}}$$

Proof $\beta(s) = \alpha(h(s)) \longrightarrow \beta^j(s) = \alpha^j(h(s))$

$$\frac{d\beta^j}{ds} = \frac{d\alpha^j}{ds}(h(s))\frac{dh}{ds} = (\alpha^j)'(h(s))\frac{dh}{ds} \quad \text{for } j = 1, 2, \ldots, n$$

$$\beta'(s) = \sum_{j=1}^{n} \frac{d\beta^j}{ds} \underbrace{\boxed{\left.\frac{\partial}{\partial x^j}\right|_{\beta(s)}}}_{\hat{X}_j} = \sum_{j=1}^{n} (\alpha^j)'(h(s)) \boxed{\frac{dh}{ds}} \left.\underline{\frac{\partial}{\partial x^j}}\right|_{\beta(s)}$$

$$= \frac{dh}{ds} \underbrace{\sum_{j=1}^{n} (\alpha^j)'(h(s)) \left.\underline{\frac{\partial}{\partial x^j}}\right|_{\alpha(h(s))}}_{\alpha'(h(s))}$$

$$= \frac{dh}{ds} \underline{\alpha'(h(s))} \quad //$$

§1.5: THE PUSH-FORWARD IN $\mathbb{R}^n$

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$F = (F^1, F^2, \dots, F^m)$

$F^j : \mathbb{R}^n \longrightarrow \mathbb{R}$

Def^n/ $J_F = \left[\frac{\partial F}{\partial x^1} \middle| \frac{\partial F}{\partial x^2} \middle| \cdots \middle| \frac{\partial F}{\partial x^n} \right] = \begin{bmatrix} (\nabla F^1)^T \\ \hline (\nabla F^2)^T \\ \hline \vdots \\ \hline (\nabla F^m)^T \end{bmatrix} \leftarrow$ $m \times n$ matrix of functions

$$F(a+h) \cong F(a) + J_F(a)h$$

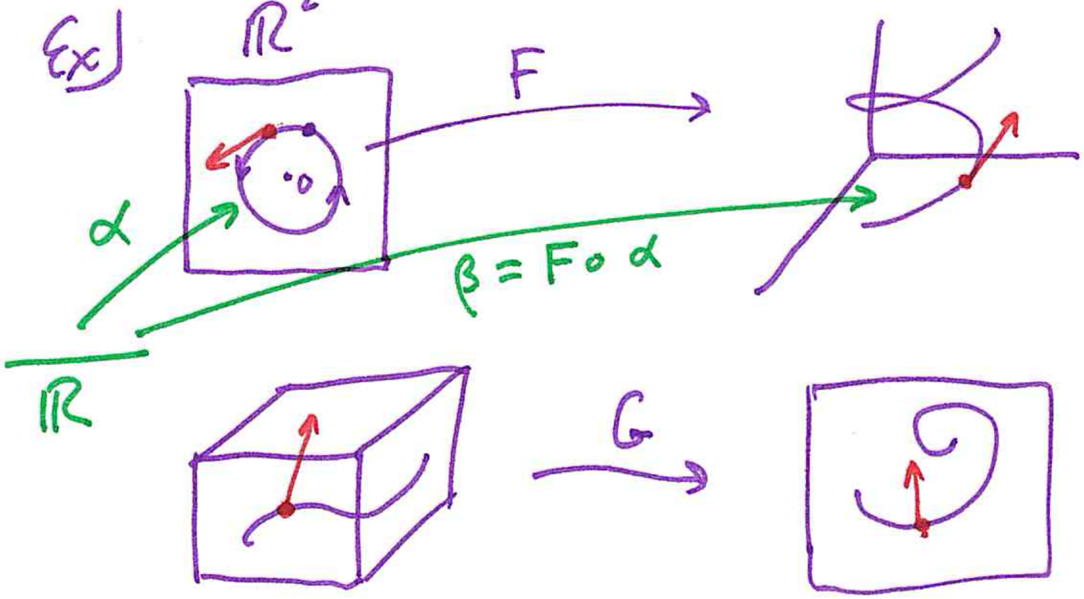
$$\Delta F = F(a+h) - F(a) = J_F(a)h = \begin{bmatrix} (\nabla F^1)(a)^T \\ \hline \vdots \\ \hline (\nabla F^m)(a)^T \end{bmatrix} \begin{bmatrix} h^1 \\ \vdots \\ h^n \end{bmatrix} = \underbrace{\begin{bmatrix} \nabla F^1(a) \cdot h \\ \hline \vdots \\ \hline \nabla F^m(a) \cdot h \end{bmatrix}}_{\text{vector of directional derivatives}}$$

Ex] $F(x,y) = (x^2+y^2, y, \cos(xy)) = \begin{bmatrix} x^2+y^2 \\ y \\ \cos(xy) \end{bmatrix}$

$$J_F = \left[\begin{array}{cc} 2x & 2y \\ 0 & 1 \\ -y\sin xy & -x\sin xy \end{array}\right] \begin{array}{l} \nabla(x^2+y^2) \\ \nabla(y) \\ \end{array}$$

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \hookrightarrow J_F \in \mathbb{R}^{3\times 2}$

Geometry of $F:\mathbb{R}^n \longrightarrow \mathbb{R}^m$



$$F(x,y) = (x, y, \tan^{-1}(y/x))$$

Def^n/ Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at p and suppose $X \in T_p\mathbb{R}^n$ where $x^j : j=1,2,.,n$ for $\mathbb{R}^n$ and $y^i : i=1,2,...,m$ then the <u>differential</u> <u>of</u> <u>F</u> at p is denoted d_pF and we define $d_pF : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^m$

$$d_pF(X) = \sum_{j=1}^{m} d_pF^j(X)\frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$= \sum_{j=1}^{m} X[F^j]\frac{\partial}{\partial y^j}\Big|_{F(p)} = \sum_{j=1}^{m}\sum_{i=1}^{n} X[x^i]\frac{\partial F^j}{\partial x^i}(p)\frac{\partial}{\partial y^j}\Big|_{F(p)}$$

$$\overline{\quad /\!/ \quad}$$

$$X = \sum_{i=1}^{n} X[x^i]\frac{\partial}{\partial x^i}\Big|_p$$

$$F_{*p}(X) = d_pF(X)$$

<u>Ex</u> $f : \overset{t}{\mathbb{R}} \longrightarrow \mathbb{R}^n$

$$d_pf\left(\frac{\partial}{\partial t}\Big|_{t_0}\right) = \sum_{i=1}^{n}\frac{\partial f^i}{\partial t}(t_0)\frac{\partial}{\partial x^i}\Big|_{f(t_0)}$$

$f = \alpha$

<u>Ex</u> $g : \mathbb{R}^n \longrightarrow \mathbb{R}^t$

$$dg(X) = \underline{X(g)}\frac{\partial}{\partial t}$$

$1 = \frac{\partial}{\partial t}$ scalar function

<u>Ex</u> $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$F(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$d_{(r,\theta)}F\left(\frac{\partial}{\partial r}\Big|_{(r,\theta)}\right) = \cos\theta\frac{\partial}{\partial x}\Big|_{F(r,\theta)} + \sin\theta\frac{\partial}{\partial y}\Big|_{F(r,\theta)}$$

Implicit and Inverse Functions Th^{m}'s

$\vec{F}(x,y) = \vec{C}$

then can solve for

$x \in \mathbb{R}^r, y \in \mathbb{R}^n, \vec{C} \in \mathbb{R}^n$

$y = G(x)$, $G : \mathbb{R}^r \longrightarrow \mathbb{R}^n$

$\vec{F}(x, G(x)) = \vec{C}$

Ex] $F(x,y,z) = \underline{x^2 + y + z^3 = 3}$

$\frac{\partial F}{\partial z} = 3z^2 \neq 0$

$\frac{\partial F}{\partial y} = 1$ $\quad J_F = [2x, 1, 3z^2]$

$\frac{\partial F}{\partial x} = 2x$

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$J_F(a)^{-1}$ exists

$\hookrightarrow \exists U, V$ with $a \in U$

$\underbrace{F|_U : U \longrightarrow V}_{\text{invertible with smooth inverse}}$

$F(r,\theta) = (r\cos\theta, r\sin\theta)$

$$J_F = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det J_F = r \neq 0$ away from $r = 0$.