LECTURE 5

$$\mathbb{R}^3 \to v\cdot w = v^1w^1 + v^2w^2 + v^3w^3 = \langle v, w\rangle$$

$$\mathbb{R}^3 \to \|v\| = \sqrt{v\cdot v}$$

$$\mathbb{R}^3 \to d(p,q) = \|q-p\|$$

Theft of these to $T_p\mathbb{R}^3 = \left\{ (p,v) \mid v\in\mathbb{R}^3 \right\} = \left\{ v^1 \frac{\partial}{\partial x^1}\Big|_p + v^2\frac{\partial}{\partial x^2}\Big|_p + v^3\frac{\partial}{\partial x^3}\Big|_p \;\Big|\; v^1, v^2, v^3 \in \mathbb{R} \right\}$

$$(p,v)\cdot(p,w) = v\cdot w$$

$$(p,v)\times(p,w) = (p, v\times w)$$

$$\|(p,v)\| = \|v\|$$

$$|v\cdot w| \leq \|v\|\,\|w\|$$

$\hookrightarrow$ Def$^{\underline{n}}$ θ between $v, w \neq 0$

$$v\cdot w = \|v\|\,\|w\|\cos\theta$$

$$\|v\times w\|^2 = \|v\|^2\|w\|^2 - (v\cdot w)^2 = \|v\|^2\|w\|^2(\underbrace{1-\cos^2\theta}_{\sin^2\theta}) = (\|v\|\,\|w\|\sin\theta)^2$$

$$\hookrightarrow \underline{\|v\times w\| = \|v\|\,\|w\|\,|\sin\theta|}.$$

Def$^{\underline{n}}$/ $S \subseteq T_p\mathbb{R}^3$ is orthogonal if $v, w \in S \Rightarrow v\cdot w = 0$
If $\|v\| = 1$ for each $v \in S$ an orthog. set then we say S is an <u>orthonormal set</u>.

Prop $S = \{V_1, V_2, V_3\} \subseteq T_p\mathbb{R}^3$ and $V_i \cdot V_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

then S is LI.

$V_i \cdot V_i = \|V_i\|^2 = 1$

$\therefore \|V_i\| = 1$.

Proof: Suppose $c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$ take dot-product with V_j

$$c_1 V_1 \cdot V_j + c_2 V_2 \cdot V_j + c_3 V_3 \cdot V_j = 0$$

$$\Rightarrow c_j \cdot 1 = 0 \quad \text{for } j = 1, 2, 3 \Rightarrow c_1 = c_2 = c_3 = 0.$$

Hence S is a basis for $T_p\mathbb{R}^3$. //

$\psi(p, v) \equiv v$

$\psi : T_p\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Prop If $X \in T_p\mathbb{R}^3$ and $S = \{V_1, V_2, V_3\}$ is orthonormal set of vectors then $X = \sum_{j=1}^{3} (X \cdot V_j) V_j$

We know, $\exists\, c_1, c_2, c_3 \in \mathbb{R}^3$ s.t. $X = c_1 V_1 + c_2 V_2 + c_3 V_3$

then $X \cdot V_j = c_j V_j \cdot V_j = c_j \quad \therefore \underline{X = (X \cdot V_1) V_1 + (X \cdot V_2) V_2 + (X \cdot V_3) V_3}$ //

Def^{n}/ A **FRAME** for $T_p\mathbb{R}^3$ is a set $\{E_1, E_2, E_3\}$ of orthonormal vectors. We def^{n} components w.r.t. this frame $X = \sum_{j=1}^{3} f^j E_j$ as f^1, f^2, f^3

$$\boxed{\text{Prop: } f^j = X \cdot E_j}$$

$$E_j \cdot X = E_j \cdot \sum_{k=1}^{3} f^k E_k = \sum_{k=1}^{3} f^k \overbrace{E_j \cdot E_k}^{\delta_{jk}} = f^j //$$

Ex] $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ forms frame on $\mathbb{R}^3$

at each p, $\underbrace{(p, \langle 1,0,0\rangle)}_{U_1(p)}, \underbrace{(p, \langle 0,1,0\rangle)}_{U_2(p)}, \underbrace{(p, \langle 0,0,1\rangle)}_{U_3(p)}$

$$X = X^1 U_1 + X^2 U_2 + X^3 U_3 \qquad X^j = X \cdot U_j$$

$$X[x] = X^1 \underbrace{U_1[x]}_{\frac{\partial x}{\partial x} = 1} + X^2 \underbrace{U_2[x]}_{\frac{\partial x}{\partial y} = 0} + X^3 \underbrace{U_3[x]}_{\frac{\partial x}{\partial z} = 0} = X^1$$

$$\underline{X^j = X[x^j]}.$$

$\boxed{\text{Prop}}$ If $V = \sum_{j=1}^{3} f^j E_j$ & $W = \sum_{i=1}^{3} g^j E_j$ then

$$V \cdot W = \sum_{i=1}^{3} f^i g^i = \sum_{i=1}^{3} v^i w^i$$

Proof: $V \cdot W = \left(\sum f^j E_j\right) \cdot \left(\sum g^i E_i\right)$

$$= \sum f^j g^i \underbrace{E_j \cdot E_i}_{\delta_{ji}}$$

$$= \underline{\sum f^i g^i} \;//$$

CROSS PRODUCT

Given $\{E_1, E_2, E_3\}$ orthonormal at $p \in \mathbb{R}^3$ $(E_j \in T_p\mathbb{R}^3$

$\boxed{\text{Prop} \quad E_1 \times E_2 = \pm E_3}$

$E_1 \times E_2 \in T_p\mathbb{R}^3 = \text{span}\{E_1, E_2, E_3\}$

$$E_1 \times E_2 = \big(\underbrace{(E_1 \times E_2) \cdot E_1}_{\pm \cancel{(E_1 \times E_1)} \cdot E_2 \,=\, 0}\big)E_1 + \big(\underbrace{(E_1 \times E_2) \cdot E_2}_{\cancel{(E_2 \times E_2)} \cdot E_1 \,=\, 0}\big)E_2 + \big(\overset{\pm E_3}{\overbrace{(E_1 \times E_2) \cdot E_3}}\big)E_3 = \underline{\underline{\pm E_3}}$$

$E_1 \cdot E_2 = 0 \quad \theta = 90°$

$\|E_1 \times E_2\| = \|E_1\| \|E_2\| \sin 90°$

$= 1.$

Prop If E_1, E_2, E_3 is frame and $\underline{E_1 \times E_2 = E_3}$ then $E_2 \times E_3 = E_1$ and $E_3 \times E_1 = E_2$ (positively oriented E_1, E_2, E_3)

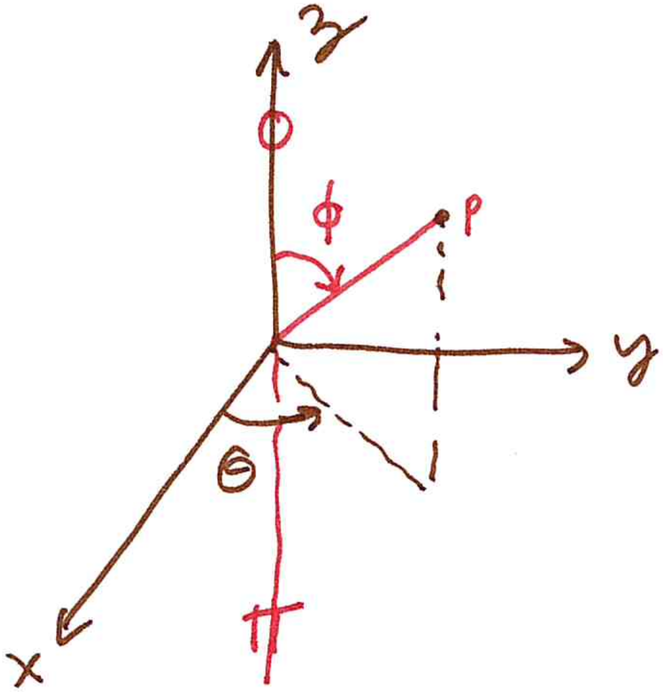
Proof
$$E_2 \times E_3 = ((E_2 \times E_3)\cdot E_1)E_1 + \underbrace{\cancel{((E_2 \times E_3)\cdot E_2)}}_{0} E_2 + \underbrace{\cancel{((E_2 \times E_3)\cdot E_3)}}_{0} E_3$$
$$= ((E_1 \times E_2)\cdot E_3)E_1$$
$$= (E_3 \cdot E_3)E_1$$
$$= E_1$$
similar, $\underline{E_3 \times E_1 = E_2}$. //

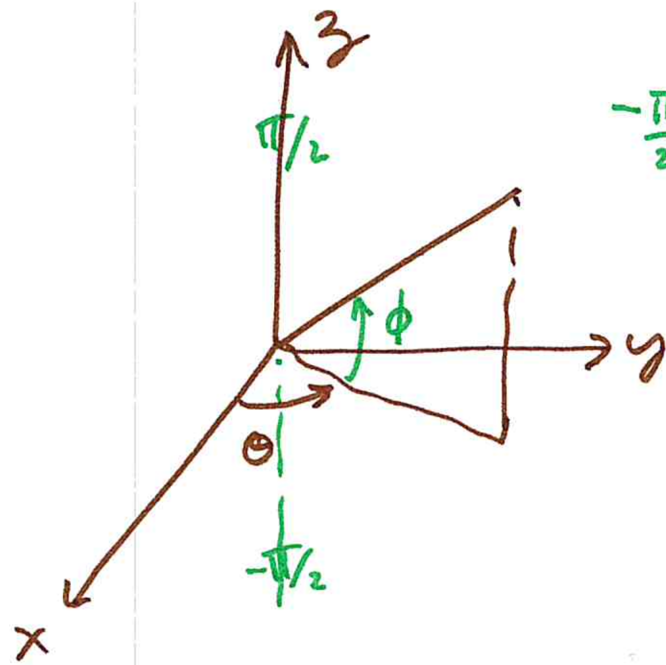
$$\underline{\sum_{k=1}^{3} \epsilon_{ijk} E_k = E_i \times E_j} \iff \text{pos. } \underline{\text{oriented}} \ \underline{\text{frame}}$$

Prop $V = \sum f^i E_i$, $W = \sum g^j E_j$ for $E_1 \times E_2 = \pm E_3$ then
$$V \times W = \pm \sum_{i,j,k} \epsilon_{ijk} \underbrace{f^i g^j}_{V^i W^j} E_k$$

Proof
$$V \times W = \sum_i \sum_j f^i g^j E_i \times E_j = \sum_{i,j} f^i g^j \left(\pm \sum_k \epsilon_{ijk} E_k\right)$$
$$= \pm \sum_{i,j,k} \underline{\epsilon_{ijk} f^i g^j E_k}. //$$

⑥





$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$

Def^{n}/ Attitude of frame $E_1, E_2, E_3 \in T_p\mathbb{R}^3$

$$E_i = a_{i1} U_1 + a_{i2} U_2 + a_{i3} U_3$$

$$A = (a_{ij}) = (E_i \cdot U_j)$$

Ex/
$$\left.\begin{aligned} E_1 &= aU_1 + bU_2 + cU_3 \\ E_2 &= dU_1 + eU_2 + fU_3 \\ E_3 &= gU_1 + hU_2 + iU_3 \end{aligned}\right\} \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Th^{m}/ attitude matrix is an orthogonal matrix: $A^TA = I \quad (A \in O(n))$

$$(A^TA)_{ij} = \sum_{k=1}^{3} (A^T)_{ik} A_{kj} = \sum_{k=1}^{3} A_{ki} A_{kj} \quad : \quad A_{ki} = E_k \cdot U_i, \quad A_{kj} = E_k \cdot U_j$$

$$= \sum_{k=1}^{3} (U_i)^k (U_j)^k$$

components of U_i w.r.t. E_1, E_2, E_3

$$U_i = \sum_{j=1}^{3} \underbrace{(U_i \cdot E_j)}_{E_j \cdot U_i = A_{ji}} E_j$$

$$= U_i \cdot U_j$$

$$= \delta_{ij} \;//$$

Ex 2.2.13

$$\left.\begin{aligned} E_1 &= \frac{1}{\sqrt{3}}(v_1 + v_2 + v_3) \\ E_2 &= \frac{1}{\sqrt{2}}(v_1 - v_3) \\ E_3 &= \frac{1}{\sqrt{6}}(v_1 - 2v_2 + v_3) \end{aligned}\right\} \longrightarrow A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$A^T A \neq I$$

Ex

$$\left.\begin{aligned} E_1 &= v_1 \\ E_2 &= v_2 \\ E_3 &= v_3 \end{aligned}\right\} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex

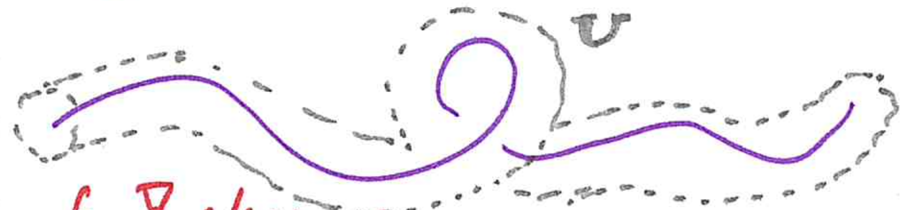
$$\left.\begin{aligned} E_1 &= \cos\theta\, v_1 + \sin\theta\, v_2 \\ E_2 &= -\sin\theta\, v_1 + \cos\theta\, v_2 \\ E_3 &= v_3 \end{aligned}\right\} \quad A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{x^2+y^2}} \begin{bmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & \sqrt{x^2+y^2} \end{bmatrix}$$

CALCULUS OF VECTORFIELDS

CARTESIAN COMP FNCTS OF Y^{*}

$\alpha : I \longrightarrow \mathbb{R}^3 \qquad \alpha = (\alpha^1, \alpha^2, \alpha^3)$

$Y = \sum_i Y^i \partial_i \in \mathfrak{X}(\alpha) \hookrightarrow Y^i : U \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$



$$Y \circ \alpha = \sum_{i=1}^{3} (\underbrace{Y^i \circ \alpha}_{\text{parametrized comp. of } Y \text{ along } \alpha}) \frac{\partial}{\partial x^i}\Big|_{\alpha}$$

Ex) $\alpha(t) = (t, t^2, t^3)$

$Y = x^2 \partial_x + (y + \sin(z))\partial_z$

$Y^1 = x^2 \longrightarrow (Y^1 \circ \alpha)(t) = \underline{t^2}.$

$Y^2 = 0 \longrightarrow (Y^2 \circ \alpha)(t) = \underline{0}.$

$\underbrace{Y^3 = y + \sin z}$

$(Y^3 \circ \alpha)(t) = \underline{t^2 + \sin t^3}.$

$$(Y \circ \alpha)(t) = t^2 \frac{\partial}{\partial x}\Big|_{\alpha(t)} + (t^2 + \sin t^3)\frac{\partial}{\partial z}\Big|_{\alpha(t)}$$

$$Y'(t) = 2t\, U_1 + (2t + 3t^2 \cos t^3)\, U_3 = 2t\, \partial_x\Big|_{(t,t^2,t^3)} + 3t^2 \cos(t^3)\, \partial_z\Big|_{(t,t^2,t^3)}$$

$$\underline{Y''(t) = 2 U_1 + (2 + 6t\cos t^3 - 9t^4 \sin t^3)\, U_3}.$$

Def^n/ If $\alpha : I \longrightarrow \mathbb{R}^3$ is smooth param. curve and $Y \in \mathfrak{X}(\alpha)$ then $Y', Y'', \dots, Y^{(k)} \in \mathfrak{X}(\alpha)$ defined by

$$Y^{(j)}(t) = \sum_{i=1}^{3} \frac{d^j (Y^i \circ \alpha)(t)}{dt} \left.\frac{\partial}{\partial x^i}\right|_{\alpha(t)} \in T_{\alpha(t)}\mathbb{R}^3.$$

Prop 2.3.5

i. $\frac{d}{dt}(c_1 Y + c_2 Z) = c_1 Y' + c_2 Z'$ for $c_1, c_2 \in \mathbb{R}$

ii. $\frac{d}{dt}(Y \cdot Z)(\alpha(t)) = \frac{dY}{dt} \cdot Z(\alpha(t)) + Y(\alpha(t)) \cdot \frac{dZ}{dt}$ in $T_{\alpha(t)}\mathbb{R}^3$

iii. $\frac{d}{dt}(Y \times Z)\cancel{(\alpha(t))} = Y' \times Z(\alpha(t)) + Y(\alpha(t)) \times Z'$

Proof: (iii) $(Y \circ \alpha)(t) = \underline{\sum_i a^i v_i}$ and $(Z \circ \alpha)(t) = \underline{\sum b^i v_i}$

$$(Y \times Z)(\alpha(t)) = \sum_k a^i b^j \underline{\underline{\epsilon_{ijk}}}\, v_k$$

$$(Y \times Z)' = \sum_k \underbrace{\frac{d}{dt}(a^i b^j)}_{\frac{da^i}{dt} b^j + a^i \frac{db^j}{dt}} \epsilon_{ijk}\, v_k = Y' \times Z + Y \times Z'. //$$

velocity — acceleration

Th^m/ Let $\alpha : I \to \mathbb{R}^3$ be smooth para. curve $\qquad \alpha'(t) = \sum_{i=1}^{3} \frac{d\alpha^i}{dt} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)}$

$\alpha''(t) = (\alpha')'(t) = \sum_{i=1}^{3} \frac{d^2\alpha^i}{dt^2} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)}$

(i) $\alpha' = 0 \iff \alpha(t) = P \quad \forall t \in I$

(ii.) $\alpha'' = 0 \iff \alpha(t) = tV + P \quad \forall t \in I$ for some $V \in \mathbb{R}^3$ (α a line)

(iii) Let $Y \in \mathfrak{X}(\alpha)$ then $Y' = 0 \iff Y \circ \alpha = c^1 U_1 + c^2 U_2 + c^3 U_3$ for constants c^1, c^2, c^3

Proof $\Leftarrow$ for i, ii, iii by diff. & def^n's involved.

$\Rightarrow$ for (i.) $\ni$ $\alpha' = 0$. Let $\alpha = (\alpha^1, \alpha^2, \alpha^3) \to \frac{d\alpha^j}{dt} = 0$ for $j = 1, 2, 3$. We Assume I is connected $\hookrightarrow \alpha^j(t) = c^j \quad \forall t \in I$

However, $P = (c^1, c^2, c^3)$ then $\alpha(t) = P \quad \forall t \in I$.

$\Rightarrow$ for (ii) $\ni$ $\alpha'' = 0 \Rightarrow \frac{d^2\alpha^j}{dt^2} = 0 \quad \forall t \in I \Rightarrow \alpha^j(t) = c^j t + e^j$

Identify $(c^j) = V$ and $(e^j) = P \hookrightarrow \underline{\alpha(t) = tV + P}$.

$\Rightarrow$ for (iii) $\underline{Y' = 0}$, Let $(Y \circ \alpha)^j = a^j \Rightarrow \frac{da^j}{dt} = 0$

Hence $a^j(t) = c^j \quad \forall t \in I \Rightarrow \underline{Y = c^1 U_1 + c^2 U_2 + c^3 U_3}$.

$\underbrace{\frac{d}{dt}\left(\|Y(\alpha(t))\|\right) = 0}_{\text{Given,}}$ $\qquad Y' = 0 \iff Y(\alpha(t_1)) \underset{\text{distantly // to}}{=} Y(\alpha(t_2))$