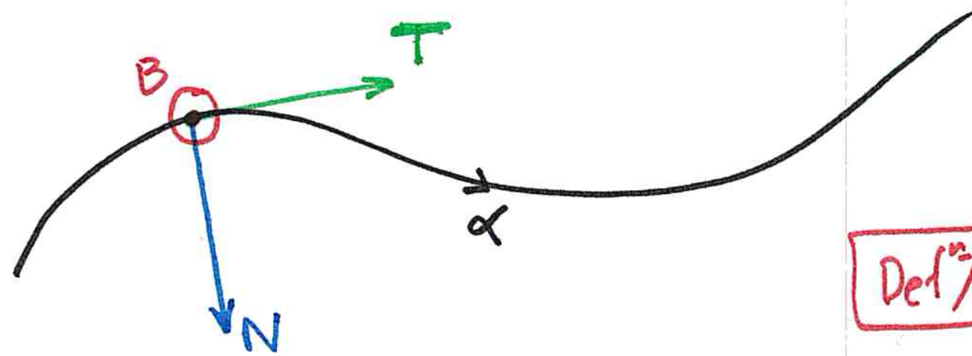
LECTURE 6: FRENET SERRET FOR CURVES IN $\mathbb{R}^3$



$\alpha : I \longrightarrow \mathbb{R}^3$

regular $\alpha'(s) \neq 0 \quad \forall s \in I$

arclength $\|\alpha'(s)\| = 1 = \frac{ds}{ds}$.

Def^n/ $B = T \times N$ Binormal.

Def^n/ $T = \alpha'$

$T \cdot T = \alpha' \cdot \alpha' = \|\alpha'\|^2 = 1$.

$\hookrightarrow T' \cdot T + T \cdot T' = 0$

$\Rightarrow 2 T \cdot T' = 0$ can use T' to obtain a $\perp$ direction to T

Def^n/ $N = \frac{T'}{\|T'\|}$ $\hookrightarrow \kappa = \|T'\|$

$T' = \frac{dT}{ds}$

$T' = \|T\|' N = \kappa N$

Frenet Serret Eq^n:

Th^{m}/ Given T, N, B for α - arclength param. (non linear $\kappa \neq 0$) there exists another fnct. τ for which:

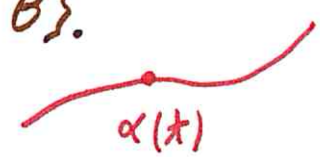
$$\frac{dT}{ds} = T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N$$

Proof/ $T = \frac{\alpha'}{\|\alpha'\|} = \alpha'$ and $T \cdot T = \alpha' \cdot \alpha' = 1 \hookrightarrow T' \cdot T + T \cdot T' = 0$

$\therefore \; T \cdot \left(\frac{T'}{\|T'\|}\right) = 0$ provided $\kappa = \|T'\| \neq 0 \; \therefore \; T \cdot N = 0$.

(where $\frac{T'}{\|T'\|} = N$)

And as $B = T \times N$ it follows that $\{T, N, B\}$.



Consider $B' = (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B$ (with $B' \cdot T \to 0$, $B' \cdot B \to 0$)

Since $B \cdot B = 1 \hookrightarrow 2B' \cdot B = 0 \; \therefore \; B' \cdot B = 0$

$B = T \times N \to B \cdot T = 0$, $B \cdot N = 0$

$B' \cdot T + B \cdot T' = 0$

$B' \cdot T = -B \cdot T' = -B \cdot (\kappa N) = -\kappa \, B \cdot N = 0$.

Let $\tau = -B' \cdot N$ then $B' = -\tau N$.

$N' = (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B = -\kappa T + \tau B$ (with $N' \cdot T \to -\kappa$, $N' \cdot N \to 0$, $N' \cdot B \to \tau$)

$N' \cdot T = -N \cdot T' = -N \cdot (\kappa N) = -\kappa$

$N \cdot T = 0 \to N' \cdot T + N \cdot T' = 0$

$N' \cdot B = -N \cdot B'$ $(N \cdot B = 0)$

$= -N \cdot (-\tau N)$

$= \tau$. //

Def^n/ $\tau = -\frac{dB}{ds} \cdot N$ for $\alpha : I \to \mathbb{R}^3$

unit-speed, nonlinear, regular curves.

TORSION

Example: $\alpha(s) = (R\cos ks,\ R\sin ks,\ mks)$: $k = \frac{1}{\sqrt{R^2+m^2}}$

$\alpha'(s) = \underbrace{-Rk\sin ks\, U_1 + Rk\cos ks\, U_2 + mk\, U_3}_{T(s)} \to \|\alpha'\| = \sqrt{k^2(R^2+m^2)} = 1.$

$T' = -Rk^2\cos ks\, U_1 - Rk^2 \sin ks\, U_2 \hookrightarrow \|T'\| = Rk^2 = \boxed{\frac{R}{R^2+m^2} = \kappa}$

curvature of circular helix

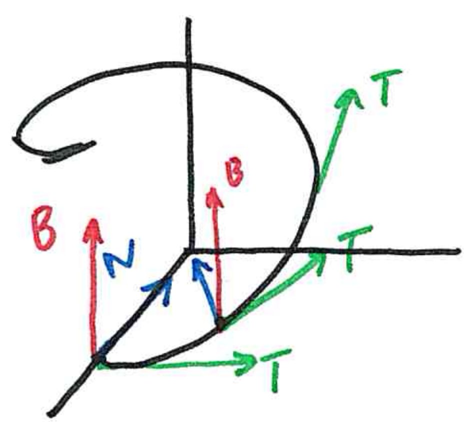
$\boxed{N = -\cos ks\, U_1 - \sin ks\, U_2}$

$B = T \times N = \underline{mk\sin ks\, U_1 - mk\cos ks\, U_2 + kR\, U_3}$

$B' = mk^2\cos ks\, U_1 + mk^2 \sin ks\, U_2 \to B' \cdot N = -mk^2 = \tau$

$\therefore \boxed{\tau = \frac{m}{R^2+m^2}}$

torsion.

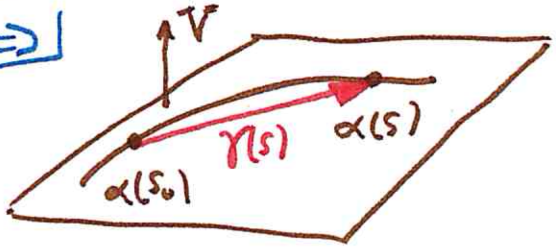


$\underline{m=0}$

$\alpha(s) = (R\cos ks,\ R\sin ks)$ $\to \tau = 0$; $\to \kappa = \frac{R}{R^2} = \frac{1}{R}.$

Th^m/ Let α be unit speed, non-linear, then α is planar curve $\iff \tau = 0$

$\Rightarrow$]



$$\gamma(s) \cdot V = 0 \qquad \gamma(s) = \alpha(s) - \alpha(s_0)$$

$$\hookrightarrow \alpha' \cdot V = 0 \qquad \gamma' = \alpha'$$

$$T \cdot V = 0$$

But diff. again, $T' \cdot V = 0 \Rightarrow \kappa N \cdot V = 0$

But, $\kappa \neq 0 \;\therefore\; N \cdot V = 0 \Rightarrow V$ colinear with $B \Rightarrow B = kV$

Then $B' = k'V + k\underbrace{V'}_{0} = \frac{k'}{k} kV = \frac{k'}{k} B$ But $B' \cdot B = 0$

$\Rightarrow \frac{k'}{k} = 0$

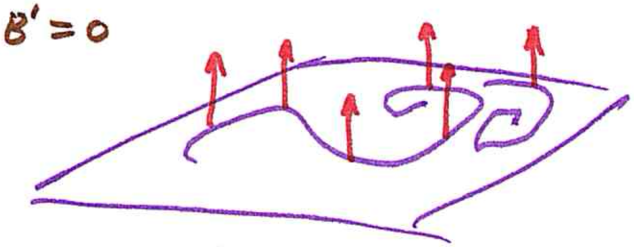
$B' \cdot B = \frac{k'}{k} B \cdot B = \frac{k'}{k}(1) = 0 \hookrightarrow k' = 0$

$\therefore \underline{k = \text{const.}} \Rightarrow \underline{\tau = 0}$

$\hookrightarrow B' = 0$

$\Leftarrow$] If $\tau = 0 \Rightarrow B' = 0 \Rightarrow \underline{B = B_0}$.

Or $B(s) = B(s_0)$ $\forall s \in \text{dom}(\alpha)$.



$$f(s) = [\alpha(s) - \alpha(s_0)] \cdot B(s_0)$$

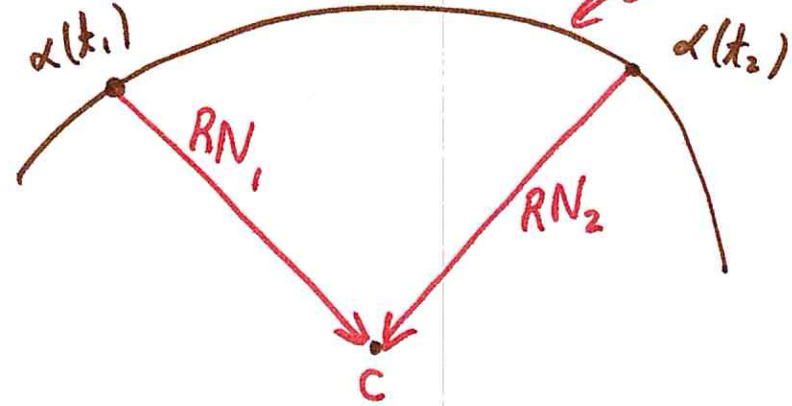
$$f'(s) = \alpha' \cdot B(s_0) = T(s) \cdot B(s_0) = 0 \quad \forall s.$$

But $f(s_0) = [\alpha(s_0) - \alpha(s_0)] \cdot B(s_0) = 0 \quad \therefore f(s) = 0 \quad \forall s \in I$

$$(\alpha(s) - \alpha(s_0)) \cdot B(s_0) = 0 \quad \forall s \Rightarrow \underline{(\alpha(s) - \alpha(s_0)) \cdot B(s) = 0}.$$

Proof (a) ($\kappa = \kappa_0$ with $\tau = 0 \Rightarrow$ curve is part of circle)

arclength $t = s$.



$$\alpha(t) + RN(t) = C$$

$$\gamma(s) = \alpha(s) + RN(s)$$

$$\gamma'(s) = \alpha'(s) + RN'(s) = T(s) + R\left(-\kappa T(s) + \cancel{\tau}^{\,0} B(s)\right)$$

$$\gamma'(s) = T(s) - R\kappa T(s) = 0$$ provided we set $R = 1/\kappa_0$

$\kappa(s) = \kappa_0 \ \forall s$.

Hence, $\gamma'(s) = 0 \quad \forall s$

$\gamma(s_0) = \gamma(s)$ Let $C = \gamma(s_0)$

$\alpha(t) = C - RN(t) \hookrightarrow \alpha(s) = C - RN(s)$

$\|\alpha(s) - C\| = R \quad \forall s.$

$\hookrightarrow \alpha$ on a circle, centered at C with radius $R = 1/\kappa_0$. //

NON UNIT SPEED CASE:

Def^n/ for $\alpha : I \to \mathbb{R}^3$ a smooth, regular curve with $\alpha'' \neq 0$ for which the arclength reparametrization $\bar{\alpha} : J \to \mathbb{R}^3$ is given by $s : I \to J$ so $\alpha(t) = \bar{\alpha}(s(t))\ \forall t \in I$ we define T, N, B, κ, τ in terms of $\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}$ of $\bar{\alpha}$ as follows:

$$T(t) = \bar{T}(s(t)) \qquad \kappa(t) = \bar{\kappa}(s(t))$$
$$N(t) = \bar{N}(s(t)) \qquad \tau(t) = \bar{\tau}(s(t))$$
$$B(t) = \bar{B}(s(t))$$

$$\alpha'(t) = \frac{d}{dt}\left(\bar{\alpha}(s(t))\right) = \frac{d\bar{\alpha}}{dt}(s(t))\frac{ds}{dt} = \bar{T}(s(t))\frac{ds}{dt} = v\,T(t) \quad \text{where } v = \frac{ds}{dt} = \|\alpha'(t)\|$$

$$\bar{\alpha}' = \bar{T}$$

$$\frac{dT}{dt} = \frac{d}{dt}\left(\bar{T}(s(t))\right) = \frac{d\bar{T}}{dt}(s(t))\frac{ds}{dt} = v\,\underbrace{\bar{T}'(s(t))}_{(\bar{\kappa}\bar{N})(s(t))} = v\,\underbrace{\bar{\kappa}(s(t))}_{\kappa(t)}\,\underbrace{\bar{N}(s(t))}_{N(t)}$$

Likewise →

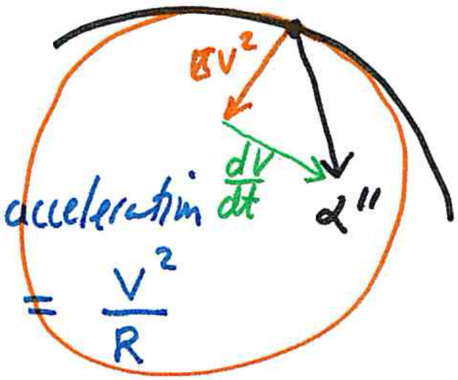
$$T' = v\kappa N$$
$$N' = -v\kappa T + v\tau B$$
$$B' = -v\tau N$$

$$\kappa = \frac{1}{v}(T' \cdot N) \quad \& \quad \tau = \frac{-1}{v} B' \cdot N.$$

Acceleration

$$\alpha'(t) = v\,T$$

$$\alpha''(t) = \frac{dv}{dt}T + v\frac{dT}{dt} = \underbrace{\frac{dv}{dt}}_{\text{tangential}}T + \underbrace{\kappa v^2}_{\text{normal acceleration}}N$$



$$a_c = \frac{v^2}{R}$$

Prop: $B = \dfrac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$ (Thm 2.4.11)

$\alpha'' = v'T + vT'$, $\alpha' = vT \hookrightarrow \alpha' \parallel T$

$$\alpha' \times \alpha'' = (vT) \times (v'T + vT') = v^2 T \times \underbrace{T'}_{\kappa v N} = v^3 \kappa \underbrace{(T \times N)}_{B}$$

(with $T \times T = 0$)

$$B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} \qquad \|\alpha' \times \alpha''\| = v^3 \kappa$$

$$\boxed{\kappa = \frac{\|\alpha' \times \alpha''\|}{v^3}}$$