LECTURE 7: COVARIANT DERIVATIVE & THE CONNECTION FORM

Def^n/ $W \in \mathfrak{X}(\mathbb{R}^3)$ and $v \in T_p\mathbb{R}^3$. THE <u>COVARIANT</u> <u>DERIVATIVE</u> of W w.r.t. v at p is the following tangent vector:

$$(\nabla_v W)(p) = W(p+tv)'(0) \in T_p\mathbb{R}^3$$

Likewise, if $V \in \mathfrak{X}(\mathbb{R}^3)$ and $V(p) = V_p$ the assignment $p \longmapsto (\nabla_{V_p} W)(p)$ defines $\nabla_V W \in \mathfrak{X}(\mathbb{R}^3)$ and we say $\nabla_V W$ is the covariant der. of W w.r.t. V.

Ex 1 $W = x U_1 + z^2 U_2 + U_3$

$p + tU_1 \longrightarrow W^1 = p^1 + t,\quad W^2 = \cancel{\ }(z^2)\big|_{z=p^3} = (p^3)^2,\quad W^3 = 1 \qquad (\nabla_{U_1} W)(p) = 1 \cdot U_1$

$p + tU_2 \longrightarrow W^1 = p^1,\quad W^2 = (p^3)^2,\quad W^3 = 1 \qquad \therefore\ (\nabla_{U_2} W)(p) = 0$

$p + tU_3 \longrightarrow W^1 = p^1,\quad W^2 = (p^3+t)^2 \longrightarrow 2(p^3+t)\big|_{t=0} \longrightarrow (\nabla_{U_3} W)(p) = 2p^3 U_2,\quad W^3 = 1$

$$\underline{\nabla_{aU_1 + bU_2 + cU_3} W = aU_1 + 2cz\,U_2}$$

$$\boxed{\textbf{Prop}\quad V, W \in \mathfrak{X}(\mathbb{R}^3) \text{ then } \quad \nabla_v W = \sum_{j=1}^{3} v[W^j]\, U_j}$$

Proof:

$$W'(p+tv)(0) = \sum_{j=1}^{3} \frac{dW^j(p+tv)}{dt}(0)\, U_j$$

$$= \sum_{j=1}^{3} (DW^j)(v)(p)\, U_j$$

$$= \underline{\sum v[W^j]\, U_j} \quad : \quad V(p) = v. \quad //$$

Ex ① again: $W = x U_1 + z^2 U_2 + U_3$

$\longrightarrow W^1 = x$, $W^2 = z^2$, $W^3 = 1$

$$\underline{\nabla_{\underset{\uparrow}{v}} W = a U_1 + 2cz\, U_2}.$$

$a U_1 + b U_2 + c U_3$

Ex ② $\quad W = f U_1 + g U_2$

$$\underline{\nabla_v W = v[f] U_1 + v[g] U_2}.$$

Prop 2.5.5.

(i.) $\nabla_{V+\mathcal{V}} W = \nabla_V W + \nabla_{\mathcal{V}} W$

(ii) $\nabla_V (V+W) = \nabla_V V + \nabla_V W$

(iii) $\nabla_{fV} W = f\, \nabla_V W$

(iv) $\nabla_V (cW) = c\, \nabla_V W$

(v) $\nabla_V (fW) = V[f]\, W + f\, \nabla_V W$

(vi.) $V[V \cdot W] = (\nabla_V V)\cdot W + V \cdot (\nabla_V W)$

linearity in the $\nabla_{\square}\, \square$

Proof (v.)

$$\nabla_V (fW) = \sum_{j=1}^{3} V[fW^j]\, U_j$$

$$= \sum_{j=1}^{3} \left(V[f]\, W^j + f\, V[W^j]\right) U_j$$

$$= \sum_j V[f]\, W^j U_j + f \underbrace{\sum_{j=1}^{3} V[W^j]\, U_j}_{\nabla_V W}$$

$$= V[f]\, W + f\, \nabla_V W. \;//$$

(4)

(vi.) $v[V \cdot W] = (\nabla_v V) \cdot W + V \cdot (\nabla_v W)$

$$\underline{\nabla_v V = \sum_{i=1}^{3} v[v^i]\, U_i}$$

Proof $v\left(\sum_i v^i w^i\right) = \sum_i v[v^i w^i]$

$$= \sum_i v[v^i] w^i + \sum_i v^i\, v[w^i]$$

$$\underline{(\nabla_v V)^i = v[V^i]}.$$

$$= \sum_i (\nabla_v V)^i w^i + \sum_i v^i (\nabla_v W)^i$$

$$= \underline{(\nabla_v V) \cdot W + V \cdot (\nabla_v W)} \,//$$

Ex) $E_i, E_j \hookrightarrow E_i \cdot E_j = \delta_{ij}$

$$0 = V[\underbrace{E_i \cdot E_j}_{\delta_{ij}}] = (\nabla_V E_i) \cdot E_j + E_i \cdot (\nabla_V E_j)$$

$$\therefore \underline{\underbrace{(\nabla_V E_i) \cdot E_j}_{\omega_{ij}(V)} = -\underbrace{(\nabla_V E_j) \cdot E_i}_{\omega_{ji}(V)}}.$$

$$\omega_{ij}(V) = -\omega_{ji}(V)$$

$$\Rightarrow \underline{\omega_{ij} = -\omega_{ji}}.$$

§2.6 frames & connection forms

Let $\{E_1, E_2, E_3\}$ be a frame for $\mathbb{R}^3$ and $V \in \mathfrak{X}(\mathbb{R}^3)$ there exist fncts $f^j = V \cdot E_j$ such that $\underline{V = f^1E_1 + f^2E_2 + f^3E_3}$.

$$\nabla_V W = \nabla_{\sum f^i E_i}\left(\sum g^j E_j\right) \qquad (g^j = W \cdot E_j)$$

$$= \sum_{i,j} f^i \, \nabla_{E_i}(g^j E_j)$$

$$= \sum_{i,j} f^i \left(E_i[g^j] E_j + g^j \nabla_{E_i} E_j \right)$$

$$= \underbrace{\sum_j V[g^j] E_j}_{\mathrm{I.}} + \underbrace{\sum_{i,j} f^i g^j \nabla_{E_i}(E_j)}_{\mathrm{II.}}$$

Def^{n}/ connection form of the E-frame, for each $V \in T_p\mathbb{R}^3$

$$\omega_{ij}(p)(V) = (\nabla_V E_i) \cdot E_j(p)$$

the assignment $p \longmapsto \omega_{ij}(p)$ is a <u>differential</u> <u>one</u>-<u>form</u> on $\mathbb{R}^3$.

<u>Observation</u>: $\omega_{ij} = -\omega_{ji} \Rightarrow \exists$ just $\underline{3}$ independent connection forms for $\mathbb{R}^3$

Prop/ $\omega_{ij} = -\omega_{ji}$ and $\nabla_V E_i = \sum_{j=1}^{3} \omega_{ij}(V) E_j$

Proof/ $\cancel{W = E_i \hookrightarrow W = \sum g^j E_j \longrightarrow g^j = \delta_{ji}}$

$$\cancel{\nabla_V E_i = \sum_{l,j} f^l \delta_{ji} \nabla_{E_l} E_j = \sum_l f^l \nabla_{E_l} E_i = \nabla_V E_i}$$

$$\omega_{ij}(V) = (\nabla_V E_i) \cdot E_j \implies$$ the j^{th} E-frame component of $\nabla_V E_i$ is $\omega_{ij}(V)$.

$$\therefore \quad \nabla_V E_i = \sum_{j=1}^{3} \omega_{ij}(V) E_j$$

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§2.6.1 on matrices of differential forms

Def^{n}/ Let A_{ik} be a p-form for $1 \le i \le m$, $1 \le k \le r$ } makes A & B "multipliable" to AB
Let B_{kj} be a q-form for $1 \le j \le n$, $1 \le h \le r$

$$(A \wedge B)_{ij} = \sum_{k=1}^{r} A_{ik} \wedge B_{kj}$$

Likewise, $(dA)_{ij} = dA_{ij}$ also $(A^T)_{ij} = A_{ji}$

Ex/
$$\begin{bmatrix} dx & dy \\ dz & dt \end{bmatrix} \wedge \begin{bmatrix} dt & dz \\ dy & dx \end{bmatrix} = \left[\begin{array}{c|c} dx\wedge dt + \cancelto{0}{dy\wedge dy} & dx\wedge dz + dy\wedge dx \\ \hline dz\wedge dt + dt\wedge dy & \cancelto{0}{dz\wedge dz} + dt\wedge dx \end{array}\right]$$

$$A \wedge A = \underbrace{\begin{bmatrix} dx & dy \\ dz & dt \end{bmatrix}}_{A} \wedge \begin{bmatrix} dx & dy \\ dz & dt \end{bmatrix} = \left[\begin{array}{c|c} dy\wedge dz & dx\wedge dy + dy\wedge dt \\ \hline dz\wedge dx + dt\wedge dz & dz\wedge dy \end{array}\right] \neq 0.$$

$$d \begin{bmatrix} x^2 dy & z\,dx \\ dz & z^3 dz \end{bmatrix} = \left[\begin{array}{c|c} 2x\, dx\wedge dy & dz\wedge dx \\ \hline 0 & 3z^2 dz\wedge dz \end{array}\right].$$

Prop 2.6.5 A an $m \times r$ matrix of p-forms and B an $r \times n$ matrix of q-forms

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$$

Proof

$$\begin{aligned}
d(A \wedge B)_{ij} &= d\Big(\sum_k A_{ik} \wedge B_{kj}\Big) \\
&= \sum_k d(A_{ik} \wedge B_{kj}) \qquad A_{ik} \text{ is in } \Lambda^p \mathbb{R}^n \\
&= \sum_k \big(dA_{ik} \wedge B_{kj} + (-1)^p A_{ik}\, dB_{kj} \\
&= \sum_k (dA)_{ik} \wedge B_{kj} + (-1)^p \sum_k A_{ik}\,(dB)_{kj} \\
&= \underline{dA \wedge B + (-1)^p A \wedge dB}. //
\end{aligned}$$

Attitude Matrix

$p \longmapsto A(p)$ where $(A(p))^T A(p) = I \quad \forall p \in \mathbb{R}^3$

$A^T A = I \longrightarrow \underbrace{A^T \wedge A = I}$

$dA^T \wedge A + A^T \wedge dA = 0 \hookrightarrow dA^T \wedge A = -A^T \wedge dA$

Right Multiply by A^T

$\underline{dA^T = -A^T \wedge dA \wedge dA}$.

left mult. by A

$\underline{dA = -A \wedge dA^T \wedge A}$.

Lemma: $\cancel{B}$

$B \wedge I = B$

Th^{m}/ Let A be the attitude of a frame E_1, E_2, E_3 and ω_{ij} the connection forms then $\omega = dA \wedge A^T$

Proof: Let $E_i = \sum_{k=1}^{3} A_{ik} U_k = A_{i1} U_1 + A_{i2} U_2 + A_{i3} U_3$

Recall, $\omega_{ij}(V) = (\nabla_V E_i) \cdot E_j$. Thus,

$$
\begin{aligned}
\omega_{ij}(V) &= \nabla_V \left(\sum_{k=1}^{3} A_{ik} U_k \right) \cdot E_j \\
&= \sum_{k=1}^{3} \nabla_V (A_{ik} U_k) \cdot E_j \\
&= \sum_k \left(V[A_{ik}] U_k + A_{ik} \cancelto{0}{\nabla_V U_k} \right) \cdot E_j \\
&= \sum_k V[A_{ik}] U_k \cdot \left(\sum_{l=1}^{3} A_{jl} U_l \right) \\
&= \sum_{k,l} V[A_{ik}] A_{jl} \cancelto{\delta_{kl}}{U_k \cdot U_l} \\
&= \sum_k V[A_{ik}] A_{jk} \\
&= \sum_k dA_{ik}(V) (A^T)_{kj} = (dA \wedge A^T)_{ij}(V)
\end{aligned}
$$

Ex) $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow dA = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} d\theta$

$$dA \wedge A^T = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} d\theta = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\theta$$

$$\therefore \quad \underline{\omega = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}.$$

Ex) <u>Spherical frame</u>

$$A = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

$$dA = \begin{bmatrix} -\sin\theta\sin\phi\, d\theta + \cos\theta\cos\phi\, d\phi & \cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi & -\sin\phi\, d\phi \\ -\sin\theta\cos\phi\, d\theta - \cos\theta\sin\phi\, d\phi & \cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi & -\cos\phi\, d\phi \\ -\cos\theta\, d\theta & -\sin\theta\, d\theta & 0 \end{bmatrix}$$

$$\underline{\omega = dA \wedge A^T = \begin{bmatrix} 0 & d\phi & \sin\phi\, d\theta \\ -d\phi & 0 & \cos\phi\, d\theta \\ -\sin\phi\, d\theta & -\cos\phi\, d\theta & 0 \end{bmatrix}}.$$