# LECTURE 8 : COFRAMES & THE STRUCTURE EQ$^{\text{ns}}$ OF CARTAN

$V$ with basis $\{v_1, v_2, \dots, v_n\}$ then $V^* = L(V, \mathbb{R})$ has basis $\{v^1, v^2, \dots, v^n\}$ where $v^i(v_j) = \delta_{ij}$. We do this at each pt. in $\mathbb{R}^3$ to take a frame $E_1, E_2, E_3$ and create a coframe $\theta^1, \theta^2, \theta^3$ $\quad p \longmapsto \theta^i(p)$

$$\underbrace{E_1(p), E_2(p), E_3(p)}_{\text{BASIS for } T_p\mathbb{R}^3} \longleftrightarrow \underbrace{\theta^1(p), \theta^2(p), \theta^3(p)}_{\text{BASIS FOR } (T_p\mathbb{R}^3)^*}$$

**Def$^{\text{n}}$/** $E_1, E_2, E_3$ a frame on $\mathbb{R}^3$ then we say a set of differential one forms $\theta^1, \theta^2, \theta^3$ on $\mathbb{R}^3$ is a <u>coframe</u> if $\theta^i(E_j) = \delta_{ij}$ for all $i, j \in \mathbb{N}_3 = \{1, 2, 3\}$.

**Ex/** $\theta^1, \theta^2, \theta^3$ are one-forms for a given frame $E_1, E_2, E_3$ then we also have $dx, dy, dz \in \Lambda^1(\mathbb{R}^3)$

$$\theta^1 = a\,dx + b\,dy + c\,dz \quad : \quad \underbrace{dx^i(U_j) = U_j[x^i] = \frac{\partial x^i}{\partial x^j} = \delta_{ij}}$$

shows $U_1, U_2, U_3$ has coframe $\left.\begin{matrix} dx, dy, dz \\ dx^1, dx^2, dx^3 \end{matrix}\right\}$ CART. COFRAME.

$$\theta^1(U_1) = a\,\underbrace{dx(U_1)}_{1} + b\,\underbrace{dy(U_1)}_{0} + c\,\underbrace{dz(U_1)}_{0}$$

$\theta^1(U_1) = a$ likewise

$\theta^1(U_2) = b$

$\theta^1(U_3) = c \hookrightarrow \underline{\theta^1 = \theta^1(U_1)\,dx + \theta^1(U_2)\,dy + \theta^1(U_3)\,dz}$.

**Prop. 2.7.3**: components of $V \in \mathfrak{X}(\mathbb{R}^3)$ and $\alpha \in \Lambda^1(\mathbb{R}^3)$ with respect to $E_1, E_2, E_3$ & $\theta^1, \theta^2, \theta^3$

$$V = \sum_{j=1}^{3} \theta^j(V) E_j \quad \& \quad \alpha = \sum_{j=1}^{3} \alpha(E_j)\theta^j$$

$$V = \sum_{j=1}^{3} (V \cdot E_j) E_j$$

**Lemma**: $V \cdot E_j = \theta^j(V)$

Proof: $E_1, E_2, E_3$ forms frame, hence basis for $T_p\mathbb{R}^3$ at each $p \in \mathbb{R}^3$ hence $\exists$ fncts. $c_1, c_2, c_3$ for which $V = c_1 E_1 + c_2 E_2 + c_3 E_3$

$$\begin{aligned} \theta^j(V) &= \theta^j(c_1 E_1 + c_2 E_2 + c_3 E_3) \\ &= c_1\, \theta^j(E_1) + c_2\, \theta^j(E_2) + c_3\, \theta^j(E_3) \qquad \theta^j(E_i) = \delta_{ij} \\ &= c_j \quad \therefore \quad \underline{V = \sum_{j=1}^{3} \theta^j(V) E_j}. \end{aligned}$$

Like $\alpha = \sum_{j=1}^{3} b_j \theta^j$ as $\theta^j$ is coframe. But,

$$\alpha(E_i) = \sum_{j=1}^{3} b_j \underbrace{\theta^j(E_i)}_{\delta_{ij}} = b_i \quad \therefore \quad \underline{\alpha = \sum_{j=1}^{3} \alpha(E_j)\theta^j}.$$

## Attitude Matrix relates to coframe how?

$$E_i = A_{i1} U_1 + A_{i2} U_2 + A_{i3} U_3 \longrightarrow A_{ij} = E_i \cdot U_j$$

$$\theta^j = c_{j1}\,dx + c_{j2}\,dy + c_{j3}\,dz$$

$$\alpha = \sum_{j=1}^{3} \alpha(E_j)\,\theta^j$$

$$\alpha = \sum_{i=1}^{3} \alpha(U_i)\,dx^i$$

$$C_{ji} = \theta^j(U_i) = U_i \cdot E_j = E_j \cdot U_i = A_{ji}$$

$$\theta^j = A_{j1}\,dx^1 + A_{j2}\,dx^2 + A_{j3}\,dx^3$$

<u>Prop</u>: If $E_1, E_2, E_3$ a frame and $\theta^1, \theta^2, \theta^3$ coframe then

$$E_i = \sum_{j=1}^{3} A_{ij} U_j \iff \theta^i = \sum_{j=1}^{3} A_{ij}\,dx^j$$

## Notation for coframe as vector of one-forms

$$\Theta = \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} \quad \& \quad d\xi = \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} \qquad \underbrace{\theta^i = \sum_{j=1}^{3} A_{ij}\, dx^j}_{\Theta = A\, d\xi}$$

$$\boxed{\begin{array}{l} \textbf{Th}^{\underline{m}}\ \textbf{(2.7.5) CARTAN's STRUCTURE EQ}^{\underline{n}}\textbf{s} \\ \text{(i.)}\quad d\theta^i = \displaystyle\sum_{j=1}^{3} \omega_{ij} \wedge \theta^j \quad \longleftrightarrow \quad \underline{d\Theta = \omega \wedge \Theta}. \\ \text{(ii)}\quad d\omega_{ij} = \displaystyle\sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} \quad \longleftrightarrow \quad \underline{d\omega = \omega \wedge \omega}. \end{array}}$$

**Proof:** Recall $\omega = dA \wedge A^T$ and $\underline{A \wedge A^T = AA^T = I}$. Thus,

$$\Theta = A\, d\xi \Rightarrow d\Theta = dA \wedge d\xi + A\, \cancel{d(d\xi)}^{\,0}$$
$$= (dA \wedge A^T)\wedge(A\, d\xi)$$
$$= \underline{\omega \wedge \Theta}.$$

$dA^T = -A^T dA\, A^T$

For (ii.)

$$d\omega = d(dA \wedge A^T) = \cancel{d(dA)}^{\,0} \wedge A^T - dA \wedge dA^T$$
$$= -dA \wedge (-A^T dA\, A^T)$$
$$= (dA \wedge A^T) \wedge (dA \wedge A^T)$$
$$= \underline{\omega \wedge \omega}. \;/\!/$$

## Examples of coframes

Ex) $E_1 = \cos\theta\, U_1 + \sin\theta\, U_2$, $E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2$, $E_3 = U_3$ } cylindrical frame

$\theta^1 = \cos\theta\, dx + \sin\theta\, dy$
$\theta^2 = -\sin\theta\, dx + \cos\theta\, dy$
$\theta^3 = dz$

often we wish to express $\theta^1, \theta^2, \theta^3$ in terms of $d\theta$, $dr$, $dz$. So

$x = r\cos\theta \rightarrow dx = \cos\theta\, dr - r\sin\theta\, d\theta$

$y = r\sin\theta \rightarrow dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$$\begin{aligned}\theta^1 &= (\cos\theta)(\cos\theta\, dr \cancel{- r\sin\theta\, d\theta}) + \sin\theta(\sin\theta\, dr + \cancel{r\cos\theta\, d\theta}) \\ &= (\cos^2\theta + \sin^2\theta)\, dr \\ &= dr\end{aligned}$$

$$\begin{aligned}\theta^2 &= -\sin\theta\, dx + \cos\theta\, dy \\ &= -\sin\theta(\cancel{\cos\theta\, dr} - r\sin\theta\, d\theta) + \cos\theta(\cancel{\sin\theta\, dr} + r\cos\theta\, d\theta) \\ &= r\, d\theta\end{aligned}$$

$$\therefore \boxed{\begin{array}{l}\theta^1 = dr, \quad \theta^2 = r\,d\theta, \quad \theta^3 = dz \\ E_1 = \dfrac{\partial}{\partial r}, \quad E_2 = \dfrac{1}{r}\dfrac{\partial}{\partial \theta}, \quad E_3 = \dfrac{\partial}{\partial z}\end{array}}$$

Ex continued

$$\Theta = \begin{bmatrix} dr \\ r\,d\theta \\ dz \end{bmatrix} \qquad A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} \neq \Theta$$

$$d\Theta = \begin{bmatrix} 0 \\ dr\wedge d\theta \\ 0 \end{bmatrix} \qquad \omega\wedge\Theta = \underbrace{\begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\omega} \wedge \begin{bmatrix} dr \\ r\,d\theta \\ dz \end{bmatrix} = \begin{bmatrix} 0 \\ -d\theta\wedge dr \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ dr\wedge d\theta \\ 0 \end{bmatrix}$$

2nd CARTAN Eqn:

$$d\omega = 0 \quad \text{as} \quad d^2\theta = 0$$

$$d\omega = \omega\wedge\omega = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

$$\left.\begin{aligned} \theta^1 &= \cos\theta\sin\phi\,dx + \sin\theta\sin\phi\,dy + \cos\phi\,dz \\ \theta^2 &= \cos\theta\cos\phi\,dx + \sin\theta\cos\phi\,dy - \sin\phi\,dz \\ \theta^3 &= -\sin\theta\,dx + \cos\theta\,dy \end{aligned}\right\}$$

to change to expressions built with $\rho, \phi, \theta$

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

take differentials

$$dx, \quad dy, \quad dz$$

plug in

Obtain:

$$\boxed{\begin{aligned} \theta^1 &= d\rho \\ \theta^2 &= \rho\,d\phi \\ \theta^3 &= \rho\sin\phi\,d\theta \end{aligned}}$$

$$r = \rho \sin\phi \qquad dr = \sin\phi\, d\rho - \rho\cos\phi\, d\phi$$

$$d\phi = \frac{1}{\rho\cos\theta}\left(dr - \sin\phi\, d\phi\right) = \frac{\rho\cos\phi\, d\phi}{\rho\cos\phi} = d\phi$$

$$\Theta^2 = \cos\theta\cos\phi\, dx + \sin\theta\cos\phi\, dy - \sin\phi\, dz$$

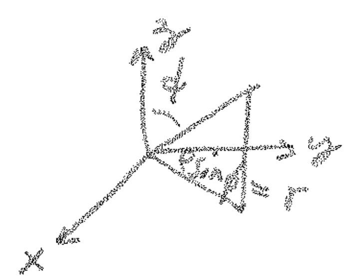
$$= \cos\theta\cos\phi\left(\cos\theta\sin\phi\, d\rho - \rho\sin\theta\sin\phi\, d\theta + \rho\cos\theta\cos\phi\, d\phi\right)$$
$$+ \sin\theta\cos\phi\left(\sin\theta\sin\phi\, d\rho + \rho\cos\theta\sin\phi\, d\theta + \rho\sin\theta\cos\phi\, d\phi\right)$$
$$- \sin\phi\left(\cos\phi\, d\rho - \rho\sin\phi\, d\phi\right)$$

$$= \cancel{\cos\phi\sin\phi\, d\rho} - \cancel{\sin\phi\cos\phi\, d\rho} + \underbrace{\cancel{(+\quad -)\, d\theta}}_{0} + \rho\left(\cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi\right) d\phi$$

$$= \rho\, d\phi \qquad \therefore \quad \boxed{\Theta^2 = \rho\, d\phi} \qquad \& \qquad E_2 = \frac{1}{\rho}\frac{\partial}{\partial\phi}.$$

$E_1 = \cos\theta \sin\phi \, U_1 + \sin\theta \sin\phi \, U_2 + \cos\phi \, U_3$ $\qquad \theta^1(E_1) = 1, \ \theta^1(E_2) = \theta^1(E_3) = 0$ (9)

$E_2 = \cos\theta \cos\phi \, U_1 + \sin\theta \cos\phi \, U_2 - \sin\phi \, U_3$




$E_3 = -\sin\theta \, U_1 + \cos\theta \, U_2$

$\theta^1 = \cos\theta \sin\phi \, dx + \sin\theta \sin\phi \, dy + \cos\phi \, dz$

$\theta^2 = \cos\theta \cos\phi \, dx + \sin\theta \cos\phi \, dy - \sin\phi \, dz$

$\theta^3 = -\sin\theta \, dx + \cos\theta \, dy$

express in terms of $d\rho, d\phi, d\theta$.

$\cancel{d\rho = \frac{\partial x}{\partial \rho} dx + \frac{\partial z}{\partial \rho} dy + \frac{\partial z}{\partial \rho} dz}$

$d\rho^2 = 2\rho \, d\rho = 2x\,dx + 2y\,dy + 2z\,dz \longrightarrow d\rho = \frac{x}{\rho} dx + \frac{y}{\rho} dy + \frac{z}{\rho} dz$

$= \cos\theta \sin\phi \, dx + \cdots + \cos\phi \, dz$

$\not\Rightarrow \boxed{\theta^1 = d\rho}$

$\tan\theta = \frac{y}{x} \qquad \sec^2\theta \, d\theta = \frac{dy}{x} - \frac{y}{x^2} dx$

$\sec^2\theta = \frac{x^2+y^2}{x^2} \qquad d\theta = \frac{x^2}{x^2+y^2}\left(\frac{dy}{x} - \frac{y}{x^2} dx\right) = \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{r}\left(\frac{x}{r} dy - \frac{y}{r} dx\right)$

$r \, d\theta = -\sin\theta \, dx + \cos\theta \, dy = \boxed{\theta^3 = \rho \sin\phi \, d\theta}$

btw $E_3 = \frac{1}{\rho \sin\phi} \frac{\partial}{\partial \theta}$

$E_2 = ?$

$\sin\phi \, d\rho + \rho \cos\phi \, d\phi = dr$

$d\phi = \frac{1}{\rho \cos\phi}\left(dr - \sin\phi \, d\rho\right)$

**Example: how to calculate $\omega$ from the coframe.**

$$\theta^1 = d\rho \qquad d\theta^1 = 0$$
$$\theta^2 = \rho\, d\phi \qquad d\theta^2 = d\rho \wedge d\phi$$
$$\theta^3 = \rho \sin\phi\, d\theta \qquad d\theta^3 = \sin\phi\, d\rho \wedge d\theta + \rho\cos\phi\, d\phi \wedge d\theta$$

However, $d\theta = \omega \wedge \theta$, $\omega_{ij} = -\omega_{ji}$

$$\left.\begin{aligned} d\theta^1 &= \cancel{\omega_{11}}^{\,0} \wedge \theta^1 + \omega_{12} \wedge \theta^2 + \omega_{13} \wedge \theta^3 \\ d\theta^2 &= \omega_{21} \wedge \theta^1 + \omega_{23} \wedge \theta^3 = -\omega_{12} \wedge \theta^1 + \omega_{23} \wedge \theta^3 \\ d\theta^3 &= -\omega_{13} \wedge \theta^1 - \omega_{23} \wedge \theta^2 \end{aligned}\right\} \text{find } \omega_{12}, \omega_{13}, \omega_{23}$$

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$d\theta^1$: $0 = \omega_{12} \wedge (\rho\, d\phi) + \omega_{13} \wedge (\rho \sin\phi\, d\theta)$ — ① $\omega_{12} \sim d\phi$, $\omega_{13} \sim d\theta$

② $\omega_{12} = \sin\phi\, d\theta$, $\omega_{13} = d\phi$

$$\boxed{d\theta^2 : \quad d\rho \wedge d\phi = -\omega_{12} \wedge (d\rho) + \omega_{23} \wedge (\rho \sin\phi\, d\theta)}$$

No.

$d\theta^3$: $\sin\phi\, d\rho \wedge d\theta + \rho\cos\phi\, d\phi \wedge d\theta = -\omega_{13} \wedge (d\rho) - \omega_{23} \wedge (\rho\, d\phi)$

$\omega_{13} = \sin\phi\, d\theta$. & $\omega_{23} = \cos\phi\, d\theta$

$$\boxed{\omega = \begin{bmatrix} 0 & d\phi & \sin\phi\, d\theta \\ -d\phi & 0 & \cos\phi\, d\theta \\ -\sin\phi\, d\theta & -\cos\phi\, d\theta & 0 \end{bmatrix}}$$