LECTURE 9 : ISOMETRY & EUCLIDEAN GEOMETRY

$(\mathbb{R}^n, d)$ where $d(p,q) = \sqrt{(q-p)\cdot(q-p)} = \| q - p \|$

Def $F : \mathbb{R}^n \to \mathbb{R}^n$ is an <u>isometry</u> $d(F(p), F(q)) = d(p,q) \;\; \forall p, q \in \mathbb{R}^n$

Prop F & G isometries then $F \circ G$ an isometry

Proof: $d((F\circ G)(p), (F\circ G)(q)) = d(F(G(p)), F(G(q)))$
$= d(G(p), G(q))$ is F is isom.
$= d(p,q)$ //

Remark: the set of isometries is closed under composition. $Id(p) = p$ is an isometry.

Ex 1 translation

$T(p) = p + a \quad \forall p \in \mathbb{R}^n$ and for some $a \in \mathbb{R}^n$

$d(T(p), T(q)) = \| (q+a) - (p+a) \| = \| q - p \| = d(p,q)$

Ex 2 orthogonal transformation

$L(p) = Rp \quad \forall p \in \mathbb{R}^n$ and $R^T R = I \qquad [L] = R$ ← orthogonal

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Continuing,

$$d(Rp, Rq)^2 = (Rq - Rp)\cdot(Rq - Rp)$$
$$= R(q-p)\cdot R(q-p)$$
$$= (R(q-p))^T R(q-p)$$
$$= (q-p)^T \underbrace{\cancel{R^T R}}_{I} (q-p) = (q-p)\cdot(q-p) = d(p,q)^2$$

$\therefore d(Rp, Rq) = d(p,q)$

~~det~~ $L = L_R$ is an isometry.

Remark: $R \in O(n) \iff R^T R = I$

$$\det(R^T R) = \det(R^T)\det(R)$$
$$= (\det(R))^2 = \det I = 1$$
$$\therefore \det(R) = \pm 1$$

$\to R \in SO(n)$ $\det(R) = 1$

$\to R \in -SO(n)$ $\det(R) = -1$

$\boxed{\text{Th}^{m}/ \text{ an isometry for which } F(0) = 0 \text{ is an orthogonal transformation}}$

Proof Assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry and $F(0) = 0$.

Notice $\underbrace{d(F(p), F(0))}_{d(F(p),\, 0) \,=\, \|F(p)\|} = d(p, 0) = \|0 - p\| = \|p\|$ $\quad\therefore\quad \|F(p)\| = \|p\|$

Observe $p \neq 0 \Rightarrow F(p) \neq 0.$

Let $p, q \in \mathbb{R}^n$,

$$d(F(p), F(q))^2 = \| F(q) - F(p) \|^2 \qquad \|v\|^2 = v \cdot v$$

$$= (F(q) - F(p)) \cdot (F(q) - F(p))$$

$$= \underbrace{F(q) \cdot F(q)}_{\|F(q)\|^2 = \|q\|^2} - 2F(p) \cdot F(q) + \underbrace{F(p) \cdot F(p)}_{\|F(p)\|^2 = \|p\|^2}$$

$$d(p, q)^2 = q \cdot q - 2p \cdot q + p \cdot p = \|q\|^2 - 2p \cdot q + \|p\|^2$$

Hence $F(p) \cdot F(q) = p \cdot q \quad \forall p, q \in \mathbb{R}^n$.

Standard basis $e_1, e_2, .., e_n$ ($e_i \cdot e_j = \delta_{ij}$) then $F(e_i) \cdot F(e_j) = e_i \cdot e_j = \delta_{ij}$.

Thus, $\{F(e_1), .., F(e_n)\}$ is an orthonormal basis for $\mathbb{R}^n$.

Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$

$$F(cx+y) \cdot F(z) = (cx+y) \cdot z$$

$$= cx \cdot z + y \cdot z$$

$$= cF(x) \cdot F(z) + F(y) \cdot F(z)$$

$$= [cF(x) + F(y)] \cdot F(z)$$

Take $F(z) = F(e_i)$ for $i = 1, 2, .., n$ $\therefore$ $F(cx+y) = cF(x) + F(y)$.

Thus F is linear transformation. $\Rightarrow F(x) = Rx$ and $F(x) \cdot F(y) = x \cdot y$ forces us to find $R^T R = I$ $\therefore$ F is isometry. //

$\boxed{\begin{array}{l}\text{Th}^{\text{m}}/\ \text{Every isometry } F:\mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ can be written uniquely} \\ \text{as } F = T\circ R \text{ where } T \text{ is a translation \& } R \text{ is an orthog. trans.} \\ \text{That is, } \exists!\, a \in \mathbb{R}^n \text{ and } M \in O(n) \text{ such that } F(x) = Mx + a \\ \hfill \text{for all } x \in \mathbb{R}^n\end{array}}$

<u>Proof</u>: $\Leftarrow$) $T \circ R$ is the comp. of isometries & hence is an isometry.

$\Rightarrow$) Let F be an isometry. Let $G(x) = F(x) - F(0)$ and observe G is an isometry & $G(0) = F(0) - F(0) = 0$. Thus $G(x) = Mx$ for some $M \in O(n)$. consequently

$$F(x) - F(0) = G(x) = Mx \Rightarrow F(x) = Mx + F(0)$$

identify $T(x) = x + F(0)$ gives together with $L_M(x) = Mx$

$$F = T \circ L_M. \text{ Check it: } F(x) = T(L_M(x)) = T(Mx) = Mx + F(0).$$

<u>Uniqueness</u>? Suppose $F(x) = M_1x + a_1$ & $F(x) = M_2x + a_2$

Notice $F(0) = a_1 = a_2$ $\therefore$ $a_1 = a_2$. But,

$$F(e_j) = F(e_j) \Rightarrow M_1e_j + \cancel{a_1} = M_2e_j + \cancel{a_2}$$

$$col_j(M_1) = col_j(M_2) \text{ for } j = 1, \dots, n$$

$$\therefore M_1 = M_2. //$$

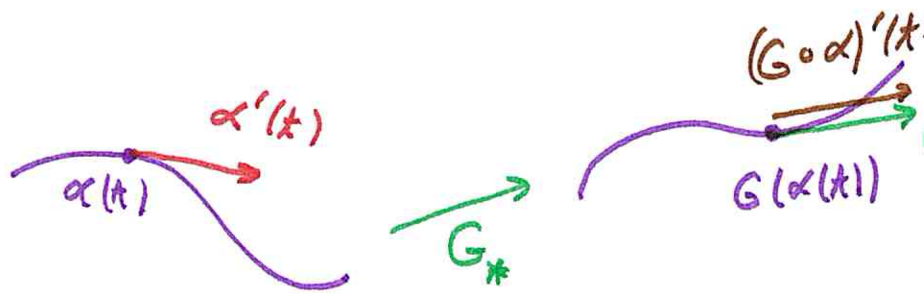
§ 3.2: HOW ISOMETRIES ACT ON VECTORS

$$\alpha : I \longrightarrow \mathbb{R}^n$$

$$\alpha'(t) = \sum_{i=1}^{n} \frac{d\alpha^i}{dt} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)} \quad \hookrightarrow \quad \alpha^{(k)}(t) = \sum_{i=1}^{n} \frac{d^k\alpha^i}{dt^k} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)}$$

Given an isometry $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ form $F \circ \alpha = \beta$ we can calculate $\beta', \beta'', \dots, \beta^{(k)}$. Consider G a map $\mathbb{R}^n \to \mathbb{R}^n$

$$(G \circ \alpha)'(t) = \sum_{i=1}^{n} \frac{d(G\circ\alpha)^i}{dt} \frac{\partial}{\partial x^i}\Big|_{G(\alpha(t))} = \underbrace{\sum_{i,j=1}^{n} \frac{\partial G^j}{\partial x^i}(\alpha(t)) \frac{d\alpha^i}{dt} \frac{\partial}{\partial x^i}\Big|_{\alpha(t)}}_{(d_{\alpha(t)} G)(\alpha'(t))}$$



$$(d_{\alpha(t)} G)(\alpha'(t)) = G_*(\alpha'(t))$$

$$\boxed{\text{Th}^{m}/\ 3.2.1 \quad F_*(\alpha'(t)) = (F \circ \alpha)'(t) \quad (F \text{ smooth } \mathbb{R}^n \longrightarrow \mathbb{R}^n)}$$

(this is not true for $\alpha'', \dots, \alpha^{(k)}$ unless we give further data about F)

$$F_*(\mathbb{X}) = F_*\Big(\sum_i \mathbb{X}^i \frac{\partial}{\partial x^i}\Big) = \sum_{i,j} \underbrace{\frac{\partial F^j}{\partial x^i}}_{(J_F)^j{}_i} \mathbb{X}^i \frac{\partial}{\partial x^j}$$

$$F_*(\mathbb{X}) = J_F \mathbb{X} \qquad \underbrace{(J_F)^j{}_i}_{(J_F)_{ji}} \mathbb{X}^i = (J_F \mathbb{X})^j$$

Suppose F is an isometry:

$$F^i(x) = \sum_{j=1}^n R^i{}_j x^j + a^i \quad (\text{for some } R^T R = I,\ R \in O(n))$$

$$\frac{\partial F^i}{\partial x^k} = \sum_{j=1}^n R^i{}_j \overbrace{\frac{\partial x^j}{\partial x^k}}^{\delta_{jk}} = R^i{}_k \qquad \therefore \quad \underline{J_F = R}.$$

$$F_*(\mathbb{X}) = R\mathbb{X}$$

$$F_*(p, v) = (F(p), Rv)$$

3.2.2

Th^{m}/ If $F(x) = Rx + a$ is an isometry for some $R \in O(n)$ and $\alpha : I \longrightarrow \mathbb{R}^n$ is a smooth parametrized curve then for $k \in \mathbb{N}$

$$F_*(\alpha') = (F \circ \alpha)', \quad F_*(\alpha'') = (F\circ\alpha)'', \ldots, F_*(\alpha^{(k)}) = (F \circ \alpha)^{(k)}$$

Moreover, $\|F_*(\alpha^{(i)}(t))\| = \|\alpha^{(i)}(t)\|$ & $F_*(\alpha^{(i)}(t)) \cdot F_*(\alpha^{(j)}(t)) = \alpha^{(i)}(t) \cdot \alpha^{(j)}(t)$.

Proof: Let $V = \alpha^{(n)}(t)$. Recall, $F_*(V) = RV$

$F_*(\alpha^{(n)}(t)) = R\alpha^{(n)}(t)$. Also calculate

$$(F \circ \alpha)(t) = R\alpha(t) + a \Rightarrow (F \circ \alpha)^{(n)}(t) = R\alpha^{(n)}(t) = F_*(\alpha^{(n)}(t)).$$

But, $F_*(V) \cdot F_*(W) = (RV)^T RW = V^T \cancel{R^T R} W = V^T W = V \cdot W$

apply this identity to the i^{th} and I j^{th} derivative along α. //

Comment: $\boxed{n=3}$ $T, N, B \leftarrow$ $T = \alpha'$

$N = \frac{1}{\kappa} T'$

$B = T \times N$

$$F_*(\alpha') \cdot F_*(\alpha'') = \alpha' \cdot \alpha''$$

$\beta = F \circ \alpha \leftarrow$ has same relations between β', β'' etc as α', α''

Newton's Mechanics: $\vec{F}_{net} = m\vec{a}$