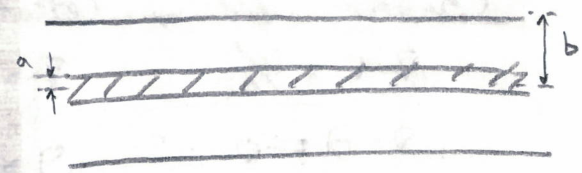
TEM wave : the coaxial transmission line

$\boxed{9.27,\ 9.28}$



$E_z = 0 \ , \ B_z = 0$

From Hank.

$$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k \overset{0}{\cancel{\frac{\partial E_z}{\partial x}}} + \omega \overset{0}{\cancel{\frac{\partial B_z}{\partial y}}}\right)$$

for TEM wave we must have $\frac{\omega}{c^2} = k^2$

No dispersion just one wave.

If we charge up the inside wire,

$$\vec{E}_0(s, \phi) = \frac{A}{s}\hat{s} \implies \vec{E} = \vec{E}_0 e^{i(kz - \omega t)}$$

from $\nabla \times \vec{E} = i\omega\vec{B}$

thus $\frac{\partial E_s}{\partial z} - \underset{0}{\cancel{\frac{\partial E_z}{\partial s}}} = i\omega B_\varphi$ taking the φ component

$$\therefore i\frac{A}{s}k = i\omega B_\varphi \implies B_\varphi = \frac{A}{s}\frac{k}{\omega}e^{i(kz - \omega t)}$$

$$\vec{B}_0 = \frac{Ak}{s\omega}\hat{\varphi} \ , \ B^{\perp} = 0 \text{ satisfied.}$$

$$\vec{E}_0 = \frac{A}{s}\hat{s} \ , \ E^{\parallel} = 0.$$

In purely real notation

$$\vec{E}(s, \varphi, z) = \frac{A\cos(kz - \omega t)}{s}\hat{s}$$

$$\vec{B}(s, \varphi, z) = \frac{A\cos(kz - \omega t)}{cs}\hat{\varphi}$$

Chapter 10, Potential and Fields

$$\nabla \cdot E = \rho/\epsilon_0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Given $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$, what are the $\vec{E}$ and $\vec{B}$

- In static case,

$\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$ but (iii) contradict

$\nabla \cdot \vec{B} \Rightarrow \vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \qquad \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\Rightarrow \boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$$

Ex/ If $V = 0$ $\quad \vec{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2 \hat{z} & |x| < ct \\ 0 & |x| > ct \end{cases}$

Find charge and current distribution

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = \frac{-\partial \mu_0 k}{A}(ct - |x|)\hat{z} \quad |x| < ct$$

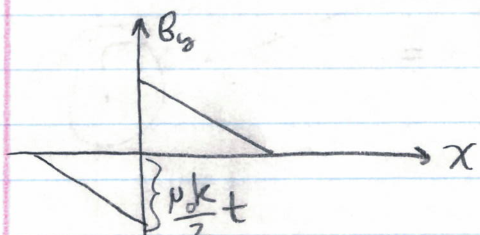
$$\vec{E} = 0$$

$$\vec{B} = \nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + (\quad)\hat{z}$$

$$\vec{B} = -\frac{\partial A_z}{\partial x}\hat{y}$$

$$B_y = -\frac{\partial A_z}{\partial x} = \frac{\partial \mu_0 k}{4c}(ct - |x|)\frac{d|x|}{dx} \qquad \frac{d|x|}{dx} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$$B_y = \pm \frac{\mu_0 k}{\partial c}(ct - |x|) \quad \begin{matrix} x > 0 \\ x < 0 \end{matrix}$$



Satisfy Maxwell Eq?

$\nabla \cdot \vec{E} = 0$ as only component E_z has no z comp.

$\nabla \cdot \vec{B} = 0$ as only component B_y has no y dependence

$$\nabla \times \vec{E} = -\frac{\partial E_z}{\partial x}\hat{y} = -\frac{\mu_0 k}{2}\frac{\partial |x|}{\partial x}\hat{y} = \mp \frac{\mu_0 k}{2}\hat{y}$$

$$\therefore \frac{\partial B_y}{\partial t} = \pm \frac{\mu_0 k}{2} \quad \left(\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\right.$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \frac{\partial \vec{E}}{\partial t} = -\frac{\mu_0 k c}{2}\hat{z} \qquad \rho = 0.$$

$$(\nabla \times \vec{B})_z = \left(\frac{\partial B_y}{\partial x}\right)\hat{z} = -\frac{\mu_0 k}{\partial c} \qquad \therefore \vec{J} = 0.$$

surface current on y-z plane in z direction, grows with t

$$\vec{B} = \frac{\mu_0 \vec{K} \times \hat{n}}{\partial} \qquad \hat{n} = \hat{x} \qquad B_y \hat{y} = \frac{\mu_0 \vec{K} \times \hat{x}}{\partial} \qquad B_y \text{ at } x = 0$$

(10.1, 10.2)

B_y at $x=0$ (y,z plane is $\pm \frac{\mu_0 k t}{2}$)

— // —

If $V=0$

$$\vec{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct-|x|)^2 \hat{z} & |x| \leq ct \\ 0 & |x| > ct \end{cases}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = \frac{-2\mu_0 k}{4}(ct-|x|)\hat{z}$$

$$\vec{B} = \frac{\pm \mu_0 k}{2c}(ct-|x|)\hat{y}$$

$\nabla \cdot \vec{E} = 0$, $\rho = 0$. $\therefore$ $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Use step function $\Theta(x)$, for better analysis

$$\vec{B} = (2\Theta(x)-1)\frac{\mu_0 k}{2c}(ct-|x|)\hat{y} \quad \text{replacing} \pm \text{sign.}$$

$$\nabla \times \vec{B} = \frac{\partial B_y}{\partial x}\hat{z} = \left[2\delta(x)\frac{\mu_0 k}{2c}(ct-|x|) + (2\Theta(x)-1)\frac{\mu_0 k}{2c}(-)(2\Theta(x)-1)\right]$$

$$= \left[\delta \mu_0 k t - \frac{2\mu_0 k}{2c}\right]\hat{z} = \quad ?$$

$$\mu_0 \vec{J} = \delta(x)\mu_0 k t$$
$$\vec{J} = \delta(x) k t$$

10.3, 10.5, 10.4

Gauge Transformation

too ugly
$$\left(\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}$$
$$\nabla^2 V + \frac{\partial(\nabla\cdot\vec{A})}{\partial t} = -\rho/\epsilon_0$$

Note that $\begin{cases}\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla\times\vec{A}\end{cases}$ replace $\vec{A}$ by $\vec{A}' = \vec{A} + \nabla f$; replace V by $V' = V - \frac{\partial f}{\partial t}$

$$\vec{E} = -\nabla V' - \frac{\partial A'}{\partial t} = -\nabla V + \frac{\partial \nabla f}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \nabla f}{\partial t}$$

thus the primed potentials yield identical $\vec{E}$
also likewise as $\nabla\times\vec{A} = \nabla\times\vec{A}'$ so identical $\vec{B}$.
Note f is any well behaved function.
We have an ∞ set of solⁿ for f.
Hence let us choose a particular f, such that the pair of $\vec{A}$ and V satisifies,

$$\boxed{\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0}$$ Lorentz Guage

$$\begin{cases}\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \\ \nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0\end{cases}$$ these are solⁿ in Lorentz Guage.

☐ Other Customary Gauges
- We could choose an f such that $\boxed{\nabla\cdot\vec{A} = 0}$ Coulomb Gauge.

$$\begin{cases}\nabla^2 V = -\rho/\epsilon_0 \\ \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \epsilon_0\mu_0\frac{\partial V}{\partial t} = -\mu_0\vec{J}\end{cases}$$

- in this case
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r},t)}{\mathscr{r}}\,d\tau'$$ → action at a distance

For Lorentz Guage, $\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$, $\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t} = -\rho/\epsilon_0$ | $\Box^2\vec{A} = \mu_0\vec{J}$, $\Box^2 V = -\rho/\epsilon_0$

Defⁿ $\Box^2 = \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}$ (d-Alembertian)

Problem 10.6

$$V = 0 \;, \quad \vec{A} = \begin{cases} \frac{\mu_0 k}{4c}(ct - |x|)^2 \hat{z} & |x| < ct \\ 0 & |x| > ct \end{cases}$$

Ex 10.1

$$\nabla \cdot \vec{A} = 0 \rightarrow \text{Coulomb}$$

$$\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0$$

$$V(\vec{r}, t) = 0 \qquad \vec{A}(\vec{r}, t) = \frac{-1}{4\pi\epsilon_0} \frac{qt}{r^2} \hat{r}$$

$$\nabla \cdot \vec{A} = \frac{-qt}{4\pi\epsilon_0} \nabla \left(\frac{\hat{r}}{r^2}\right) = \frac{-qt}{4\pi\epsilon_0}\left(-4\pi\, \delta(\vec{r})\right) = \frac{q\,\delta(\vec{r})\,t}{\epsilon_0}$$

$$\frac{\partial V}{\partial t} = 0 \quad \Big| \quad \begin{array}{l} V = 0 \\ \vec{A} = A_0 \sin(kx - \omega t)\hat{y} \end{array} \qquad \text{Both}$$

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$\Box^2 V = -\rho/\epsilon_0 \quad , \quad \Box^2 \vec{A} = -\mu_0 \vec{J}$$

The sol^{n} will be

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{\mathscr{r}} d\tau'$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{\mathscr{r}} d\tau'$$

$$t_r = t - \mathscr{r}/c$$

Continuous Distribution, Retarded Potential

In static case, $\nabla^2 V = -\rho/\epsilon_0$, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

sol^n/ $\quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\,d\tau'}{\mathscr{r}}$ $\quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\,d\tau'}{\mathscr{r}}$

$$\mathscr{r} = |\vec{r} - \vec{r}'|$$

Lorentz Guage,

$\Box^2 V = -\rho/\epsilon_0$

$\Box^2 \vec{A} = -\mu_0 \vec{J}$

Given Sol^n/

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau' \qquad \boxed{t_r = t - \frac{\mathscr{r}}{c}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

Pf//

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\nabla\rho}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right) d\tau'$$

$$\nabla\rho = \dot{\rho}\,\nabla t_r \qquad\qquad \nabla t_r = \nabla(t - \mathscr{r}/c) = \nabla\left(\frac{-\mathscr{r}}{c}\right) = \frac{-\vec{\mathscr{r}}}{\mathscr{r}c}$$

$$= -\dot{\rho}\frac{\vec{\mathscr{r}}}{\mathscr{r}c}$$

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{\mathscr{r}}\nabla^2\rho + 2\nabla\rho\,\nabla\frac{1}{\mathscr{r}} + \rho\nabla^2\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$\nabla^2\rho = -\nabla\left(\dot{\rho}\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right) = \ddot{\rho}\left(\frac{\mathscr{r}}{\mathscr{r}c}\right)^2 - \dot{\rho}\,\nabla\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right)$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \quad : \quad \nabla\cdot(\vec{\mathscr{r}}) = 3 \quad : \quad \nabla\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}}\right) = \frac{2}{\mathscr{r}}$$

$$\nabla^2\rho = \nabla\left(\dot{\rho}\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right) = \ddot{\rho}\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right)^2 - \dot{\rho}\left(\nabla\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right)\right) = \ddot{\rho}/c^2 - 2\dot{\rho}/c\mathscr{r}$$

$$2\nabla(\rho)\nabla\left(\frac{1}{\mathscr{r}}\right) = 2\left(-\dot{\rho}\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right)\cdot\left(-\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) = \frac{2\dot{\rho}}{c\mathscr{r}^2}$$

← cancel each other.

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\ddot{\rho}\frac{1}{c^2\mathscr{r}} + \rho\nabla^2\left(\frac{1}{\mathscr{r}}\right)\right]d\tau' = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{4\pi\epsilon_0}\int\frac{\rho\,d\tau'}{\mathscr{r}} + \frac{1}{4\pi\epsilon_0}\int\rho d\tau'$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{4\pi\epsilon_0}\int\frac{\rho\,d\tau'}{\mathscr{r}} + \underbrace{\frac{1}{4\pi\epsilon_0}\int\rho\,d\tau'(-4\pi\delta(\vec{r}-\vec{r}')\,d\tau'}_{-\rho/\epsilon_0}$$

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}V = -\rho/\epsilon_0$$, QED, similarily for $\vec{A}(\vec{r},t)$.

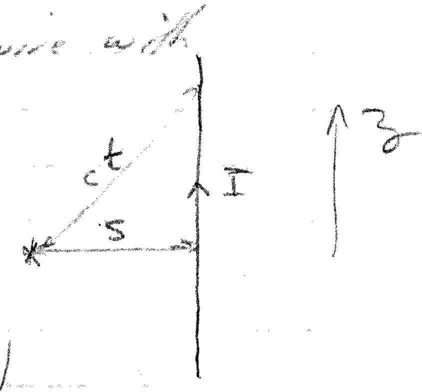
$\Box^2 V = -\rho/\epsilon_0$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$ must satisfy Lorentz' Gauge.

Note

$$V(\vec{r},t) = \int \frac{\rho(\vec{r}, t_a)}{\mathscr{r}}\, d\tau' \qquad \text{where} \quad t_a = t + \mathscr{r}/c$$

? t_a also satisfies $\Box^2 V = -\rho/\epsilon_0$, but it is not a physical reality, we must have causality our solution must also adhere to Lorentz Gauge

Example (10.2) Suppose we have an infinite wire with current $I = I(t) = \left\{ \begin{matrix} 0 & \text{at} & t \le 0 \\ I_0 & \text{at} & t > 0 \end{matrix} \right\}$



Find $\vec{E}(s,t)$, $\vec{B}(s,t)$

As $\rho = 0 \Rightarrow V = 0$.

$J\,d\tau' \longrightarrow \hat{z}\, I\, dz'$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I\,dz'}{\mathscr{r}}\hat{z} = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I(t_r)}{\mathscr{r}}\,dz\,(\hat{z})$$

we only need consider points within ct from source at time t, thus

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}} \frac{I_0}{\mathscr{r}}\,dz = \frac{\mu_0}{4\pi}\hat{z}\cdot 2\int_0^{\sqrt{(ct)^2-s^2}} \frac{I_0}{\sqrt{s^2+z^2}}\,dz$$

$$A(\vec{r},t) = \frac{\mu_0 I_0}{2\pi}\hat{z}\ln\left(\sqrt{s^2+z^2}+z\right)\Big|_0^{\sqrt{(ct)^2-s^2}}$$

$$A(\vec{r},t) = \frac{\mu_0 I_0 \hat{z}}{2\pi}\ln\frac{\sqrt{(ct)^2-s^2}+ct}{s}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 I_0}{2\pi}\,\frac{\frac{2c^2t}{2\sqrt{c^2t^2-s^2}}+c}{\sqrt{(ct)^2-s^2}+ct} = \boxed{-\frac{\mu_0 I_0 c}{2\pi}\frac{1}{\sqrt{(ct)^2-s^2}}\hat{z} = \vec{E}}$$

$$\vec{B} = -\nabla\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \boxed{\frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}\hat{\phi} = \vec{B}}$$

$t \to \infty \Rightarrow \vec{E} \longrightarrow 0$ and

$t \to \infty \Rightarrow \vec{B} \longrightarrow \dfrac{\mu_0 I}{2\pi s}$

Now Let us return to the issue of $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)\,d\tau'}{\mathscr{r}}$ and $V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)\,d\tau'}{\mathscr{r}}$ satisfy the Lorentz Guage.

$\boxed{10.9,\ 10.10}$

☆ Need to prove $\nabla\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t} = 0$

☆ Rember $t_r = t - \mathscr{r}/c$

Recall $\underbrace{\nabla}_{r \text{ only}}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$ but $\underbrace{\nabla}_{r \text{ only}}\cdot J(\vec{r},t_r) + \frac{\partial\rho(\vec{r},t_r)}{\partial t} \neq 0$

Pf/ for $\nabla\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0$

$$\nabla\cdot\vec{A} = \frac{\mu_0}{4\pi}\int \nabla\cdot\frac{\vec{J}(\vec{r},t_r)}{\mathscr{r}}\,d\tau'$$

$$\nabla\cdot\left(\frac{\vec{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\nabla\cdot\vec{J} + \vec{J}\,\nabla\left(\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\nabla\cdot\vec{J} - \vec{J}\,\nabla'\frac{1}{\mathscr{r}}$$

$$= \frac{1}{\mathscr{r}}\nabla\cdot\vec{J} - \nabla'\left(\frac{\vec{J}}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla'\cdot\vec{J}$$

Then

$$\nabla\cdot\vec{J}(\vec{r},t_r) = \cancel{\frac{\partial\vec{J}}{\partial t_r}\cdot\nabla t_r}$$

$$\nabla'\cdot\vec{J}(\vec{r}',t_r) = \nabla'\cdot\vec{J}(\vec{r}',t_r) + \cancel{\frac{\partial\vec{J}}{\partial t_r}\nabla' t_r}$$

Continuity Eq,

$$\nabla'\cdot\vec{J}(\vec{r}',t_r) + \frac{\partial}{\partial t_r}\left(\rho(\vec{r}',t_r)\right) = 0$$

Therefore,

$$\nabla\cdot\vec{A} = \frac{\mu_0}{4\pi}\int\left[-\frac{\partial\rho(\vec{r}',t_r)}{\partial t_r} + \nabla'\cdot\frac{\vec{J}}{\mathscr{r}}\right]d\tau'$$

Note,

$$\mu_0\epsilon_0\frac{\partial V}{\partial t} = \frac{\mu_0}{4\pi}\int\frac{\partial\rho(\vec{r}',t_r)}{\partial t}\,d\tau'$$

Then

$$\int\nabla'\cdot\frac{\vec{J}}{\mathscr{r}}\,d\tau' = \oint\frac{\vec{J}}{\mathscr{r}}\cdot da'$$

$\boxed{10.11\ ,\ 10.14}$

Jefimenko's Equation

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left(\frac{1}{\mathscr{r}}\nabla\rho + \rho\,\nabla\frac{1}{\mathscr{r}}\right)d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{\mathscr{r}}\left(\dot{\rho}\frac{\vec{\mathscr{r}}}{\mathscr{r}c}\right) - \rho\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right]d\tau'$$

$$\nabla t_r = -\frac{\vec{\mathscr{r}}}{\mathscr{r}c}$$

$$= \frac{-1}{4\pi\epsilon_0}\int\left(\frac{\dot{\rho}}{c\mathscr{r}} + \frac{\rho}{\mathscr{r}^2}\right)\hat{\mathscr{r}}\,d\tau'$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\int\frac{\dot{\vec{J}}(\vec{r}, t_r)\,d\tau'}{\mathscr{r}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}\hat{\mathscr{r}}}{c\mathscr{r}} - \frac{\dot{\vec{J}}}{c^2\mathscr{r}}\right]d\tau'$$

$$\vec{B} = \nabla\times\vec{A} = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}\nabla\times\vec{J} - \vec{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$(\nabla\times\vec{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z} = -\frac{\dot{J}_z}{c}\frac{\partial \mathscr{r}}{\partial y}$$

$$\frac{\partial}{\partial y}J_z(\vec{r}', t_r) = \dot{J}_z\frac{\partial t_r}{\partial y}$$

$$\frac{\partial J_y}{\partial z} = \dot{J}_y\frac{\partial \mathscr{r}}{\partial z}$$

$$\nabla\mathscr{r} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \hat{\mathscr{r}}$$

$$(\nabla\times\vec{J})_x = \frac{1}{c}\left[\dot{\vec{J}}\times\hat{\mathscr{r}}\right]_x$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}}{\mathscr{r}^2} + \frac{\dot{\vec{J}}}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'$$

§ 10.3 Point Charge

: Find $V(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ of point charge q at $\vec{w}(t) \leftarrow$ location of source point.

$$\mathscr{r} = |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

Static Case :

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\,d\tau'}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\int \frac{q\,\delta(\vec{r}' - \vec{w})}{|\vec{r} - \vec{r}'|}\,d\tau'$$

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r} - \vec{w}|} \quad \text{(Chapter 2 stuff)}$$

Junk ↓

Not so static case,

$$\vec{w} = \vec{w}(t_r) = \vec{w}\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right)$$

the integral is complicated.

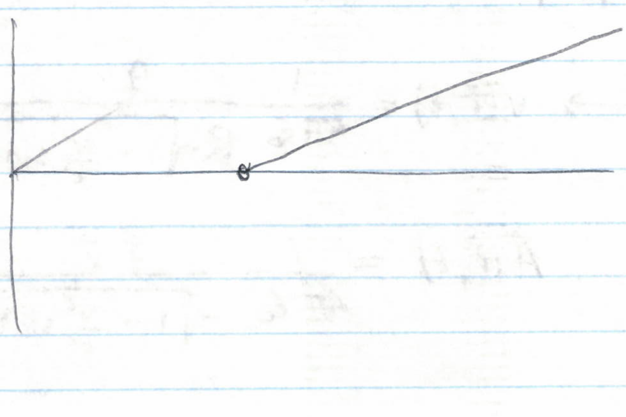
$$\nu_o = \frac{\nu_s}{1 \pm \vec{\nu}_s/\vec{\nu}_o}$$

$$\nu_o = \frac{\nu_s}{1 - \frac{\vec{\nu}_s \cdot \hat{\mathscr{r}}}{\nu_o}} \qquad q_{effective} = \frac{q}{1 - \frac{\vec{v}\cdot\hat{\mathscr{r}}}{c}}$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q_{eff}}{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r} - \frac{\vec{v}\cdot\hat{\mathscr{r}}}{c}} = \frac{c\,q}{4\pi\epsilon_0(\mathscr{r}c - \vec{v}\cdot\vec{\mathscr{r}})}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\left(\frac{qc\,\vec{v}}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})}\right) = \frac{\vec{v}}{c^2}V$$

Lienard - Wiechert Potential is retarded potential of point charge.



Lienard Wiechert Potential

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{cq}{(\mathscr{r}c - \vec{v}\cdot\vec{\mathscr{r}})}\Bigg|_{t=t_r}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t)$$

If $\vec{w}=0$ at $t=0$ and $\vec{v}=$ constant $\qquad$ Example 10.3

$\vec{w} = \vec{v}t \longleftarrow$ that is $\vec{v}=$ constant!

$|\vec{r}-\vec{w}(t_r)| = c(t-t_r)$

Thus

$$r^2 - 2\vec{r}\cdot\vec{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$$

$$(c^2-v^2)t_r^2 - (2c^2t - 2\vec{r}\cdot\vec{v}) + c^2t^2 - r^2 = 0 \quad \text{—} ?$$

$$t_r = (c^2t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 - (c^2t^2-r^2)(c^2-v^2)}$$

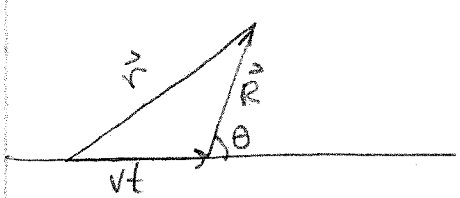
set $v=0 \Rightarrow -$ should be taken

$$\begin{aligned}\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v} &= c^2(t-t_r) - (\vec{r}-\vec{v}t_r)\cdot\vec{v}\\ &= c^2t - \vec{r}\cdot\vec{v} - c^2t_r + v^2t_r\\ &= \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 - (c^2-v^2)(c^2t^2-r^2)}\end{aligned}$$

hence

$$V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}$$

$$\vec{A}(\vec{r},t) = \frac{\vec{v}qc\mu_0}{4\pi\sqrt{\qquad ''\qquad}}$$



$$\longrightarrow V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$

$$\vec{A}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q\vec{v}}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$

§ 10.3.2 Field of Moving Charge

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^2} \nabla(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})$$

note $\nabla \mathscr{r} = \nabla(c(t - t_r))$ note ∇ and t are independent but ∇, t_r not.

$\therefore \nabla \mathscr{r} = -c\nabla(t_r)$

$$-\nabla(\vec{\mathscr{r}}\cdot\vec{v}) = \vec{\mathscr{r}} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{\mathscr{r}}) + (\vec{\mathscr{r}}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{\mathscr{r}}$$ prod. #4 cover text!

$$(\vec{\mathscr{r}}\cdot\nabla)\vec{v} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)(\vec{v}(t_r)) = \mathscr{r}_x \frac{d\vec{v}}{dt_r}\frac{dt_r}{dx} + \mathscr{r}_y \frac{d\vec{v}}{dt_r}\frac{dt_r}{dy} + \mathscr{r}_z \frac{d\vec{v}}{dt_r}\frac{dt_r}{dz}$$

$$= \vec{a}(\vec{\mathscr{r}}\cdot\nabla t_r)$$

After a bunch of calculus: Not supplied by Master Chung,

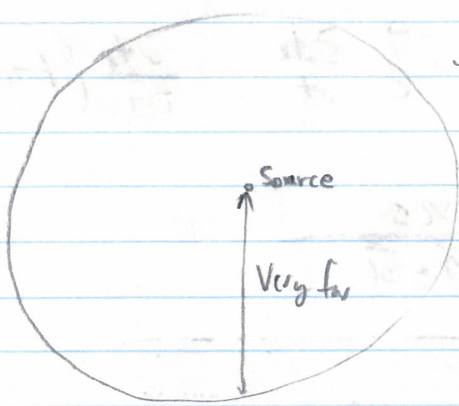
$$\nabla V = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3}\left[(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})\vec{v} - (c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{\mathscr{r}}\right]$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})^3}\left[(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})(-\vec{v} + \mathscr{r}\vec{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \vec{\mathscr{r}}\cdot\vec{a})\vec{v}\right]$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})\right]$$

$$\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}} \times \vec{E}(\vec{r},t)$$



the only term which cause radiation to get to us is $\vec{\mathscr{r}} \times (\vec{u} \times \vec{a})$
it is only because of $\vec{a}$ that we get radiation

$\boxed{10.18\ ,\ 10.19}$

In $\vec{E}(\vec{r},t)$ the 1st term $O(\frac{1}{r^2})$, the 2nd term $O(\frac{1}{r})$

$$\vec{E}\times\vec{B} \longrightarrow O\frac{1}{r^4} \text{ for the 1st term}$$

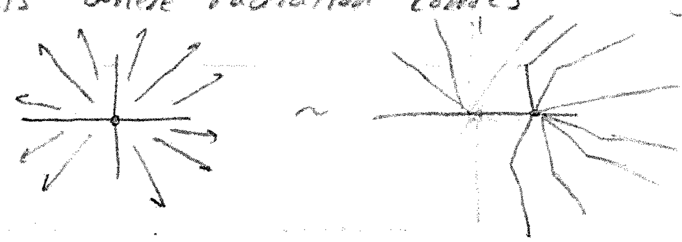
$$\longrightarrow O\frac{1}{r^3} \text{ for the mixed term}$$

$$\rightarrow O\frac{1}{r^2} \text{ for the 2nd term}$$

only 2nd term contributes to $\vec{E}\times\vec{B}$ for large r.
If we let $\vec{v}=0$ then

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0 \eta^2}\left[\hat{\eta} + \frac{\vec{\eta}\times(\hat{\eta}\times\vec{a})}{c^2}\right]$$

first term is generalized Coulomb Field. (velocity field)
second term is radiation Field, thats where radiation comes from aka acceleration field.



$\boxed{10.17}$

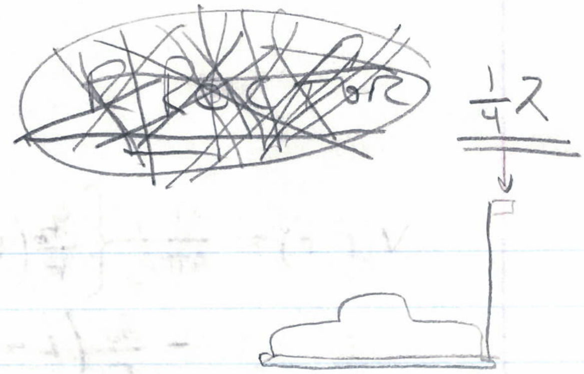
Prove $\frac{\partial t_r}{\partial t} = \frac{\eta c}{\vec{\eta}\cdot\vec{u}}$

$$\frac{\partial t_r}{\partial t} = \frac{\partial}{\partial t}\left(t - \frac{\eta}{c}\right) = 1 - \frac{\partial \eta}{\partial \vec{w}}\frac{\partial \vec{w}}{\partial t_r}\frac{\partial t_r}{\partial t}\frac{1}{c}$$

$$\frac{\partial \eta}{\partial \vec{w}} = \frac{-\vec{\eta}}{\eta} \qquad \frac{\partial t_r}{\partial t} = 1 + \frac{\vec{\eta}}{\eta}\cdot\frac{\vec{v}}{c}\frac{\partial t_r}{\partial t} \qquad \frac{\partial t_r}{\partial t}\left(1 - \frac{\vec{\eta}\cdot\vec{v}}{\eta c}\right) = 1$$

$$\frac{\partial t_r}{\partial t} = \frac{\eta c}{\eta c - \vec{\eta}\cdot\vec{v}} = \frac{\eta c}{\vec{\eta}\cdot\vec{u}}$$

Chapter 11 - Radiation



§11.1 Dipole Radiation

$$P(r) = \oint \vec{S} \cdot d\vec{a} = \frac{1}{\mu_0}\oint (\vec{E} \times \vec{B}) \cdot d\vec{a}$$

The power radiated , $P_{radiated} \equiv \lim_{r\to\infty} P(r)$

use the field of a point charge

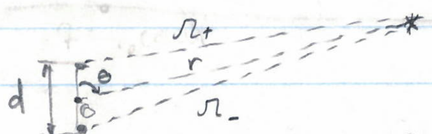
$$\vec{E} = O\left(\frac{1}{r^2}\right) + O\left(\frac{1}{r}\right)$$

at different r, $P(r)$ will be different but,

$$P(r) \xrightarrow[r\to\infty]{} \underline{\text{constant}}$$

- Consider Oscillating Dipole

$$P = qd = P_0 \cos\omega t \Rightarrow P_0 = q_0 d$$



$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q_0 \cos\omega(t - \frac{r_+}{c})}{r_+} - \frac{q_0 \cos\omega(t - \frac{r_-}{c})}{r_-} \right\}$$

$$r_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}$$

1st Approximation : $d \ll r$ in this case simplifies greatly.

$$\frac{1}{(r^2 \mp rd\cos\theta + \underbrace{(d/2)^2}_{0})^{1/2}} \cong \frac{1}{r}\left(1 \pm \frac{d\cos\theta}{r}\right)$$

$$(r^2 \mp rd\cos\theta + (d/2)^2)^{1/2} = r\left(1 \mp \frac{d\cos\theta}{2r}\right)$$

$$\omega\left(t - \frac{r_+}{c}\right) = \omega\left(t - \frac{r}{c} \pm \frac{d\cos\theta}{2c}\right)$$

2nd Approximation : $d \ll \frac{c}{\omega} = \frac{\lambda}{2\pi}$

$$\cos\omega\left(t - \frac{r_\pm}{c}\right) = \cos\omega\left(t - \frac{r}{c}\right)\cos\left(\frac{\omega d\cos\theta}{2c}\right) \mp \sin\omega\left(t - \frac{r}{c}\right)\sin\left(\frac{\omega d\cos\theta}{2c}\right)$$

$$\cong \cos\omega\left(t - \frac{r}{c}\right) \mp \frac{\omega d\cos\theta}{2c}\sin\omega\left(t - \frac{r}{c}\right)$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0}{r}\left(1+\frac{1}{2}\frac{d\cos\theta}{r}\right)\left(\cos\omega(t-\tfrac{r}{c}) - \frac{\omega d\cos\theta}{2c}\sin\omega(t-\tfrac{r}{c})\right)\right.$$

$$\left. - \frac{q_0}{r}\left(1-\frac{1}{2}\frac{d\cos\theta}{r}\right)\left(\cos\omega(t-\tfrac{r}{c}) + \frac{\omega d\cos\theta}{2c}\sin\omega(t-r/c)\right)\right.$$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q_0\, d\cos\theta}{r}\left\{\frac{\cos\omega(t-\frac{r}{c})}{r} - \frac{\sin\omega(t-\frac{r}{c})\,\omega}{c}\right\}$$

$\omega \longrightarrow 0$ $\quad V(r,t) = \frac{q_0 d}{4\pi\epsilon_0 r^2}$ recover static dipole formula

3^{rd} Approximation, $r \gg \frac{c}{\omega} = \frac{\lambda}{2\pi}$ (radiation zone)

$\frac{\cos\omega(t-\frac{r}{c})}{r^2}$ term $\longrightarrow 0$ $\quad \therefore \boxed{V(\vec{r},\theta,t) = \frac{-P_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\sin\omega(t-r/c)}$

Now we seek the vector ~~potential~~ $\vec{A}$

$$\vec{I} = \frac{dq}{dt}\hat{z} = -q_0\omega\sin(\omega t)\hat{z}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{I}\,dz}{\mathscr{r}} = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q_0\omega\sin\omega(t-r/c)}{\mathscr{r}}\hat{z}\,dz$$

$$\vec{A} = \frac{-\mu_0}{4\pi}\frac{q_0 d\omega\sin\omega(t-r/c)}{r}\hat{z}$$

From the potentials we get $\vec{E}$.

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} \cong \frac{-P_0\omega^2\cos\theta}{4\pi\epsilon_0 c^2 r}\cos\omega\left(t-\frac{r}{c}\right)\hat{r}$$

$$-\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 P_0\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)\hat{z}, \quad \hat{z} = \cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$$

$\hat{r}$ exactly cancel $\left(\frac{1}{\epsilon_0 c^2} = \mu_0\right)$

$$-\nabla V - \frac{\partial\vec{A}}{\partial t} = \boxed{\vec{E} = \frac{-\mu_0}{4\pi}P_0\omega^2\frac{\sin\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta}}$$

11.2, 11.3

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A}{\partial\theta}\right]\hat{\varphi} + \frac{1}{r\sin\theta}\left[\cancel{\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)} - \frac{\partial A_\theta}{\partial\varphi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \cancel{\frac{\partial}{\partial r}(rA_\varphi)}\right]\hat{\theta} \qquad A_\varphi = 0$$

others go away as $\frac{1}{r} \to 0$

$$\boxed{\vec{B} \cong \frac{\mu_0 P_0 \omega^2}{4\pi c r}\sin\theta\cos\omega(t - \tfrac{r}{c})\hat{\varphi}}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0 P_0^2\omega^4}{16\pi^2 c r^2}\sin^2\theta\cos^2\omega(t - \tfrac{r}{c})\hat{r}$$

$$\langle\vec{S}\rangle = \frac{\mu_0}{2c}\left(\frac{P_0\omega}{4\pi r}\sin\theta\right)^2\hat{r}$$

$$P = \int\langle\vec{S}\rangle\cdot d\vec{a} = \int\langle\vec{S}\rangle r^2 d\Omega = \frac{\mu_0}{2c}\left(\frac{P_0\omega}{4\pi}\right)^2\times\frac{8\pi}{3} = \boxed{\frac{\mu_0 P_0^2\omega^4}{12\pi c} = P}$$

Magnetic Dipole Radiation

$$\vec{m} = I\vec{a}$$
$$I(t) = I_0\cos\omega t$$
$$\vec{a} = \pi b^2\hat{z}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{I_0\cos\omega(t - ᴫ/c)}{ᴫ'}d\vec{l}' \qquad d\vec{l} = d\vec{r}'$$

$$\vec{r}' = b(i\cos\phi' + j\sin\phi')$$

$$\vec{A}(r,t) = \frac{\mu_0 b}{4\pi}\int I_0\frac{\cos\omega(t - ᴫ/c)}{ᴫ}(j\cos\phi' - i\sin\phi')d\phi' \qquad d\vec{r}' = b(j\cos\phi' - i\sin\phi')d\phi'$$

$$ᴫ = |\vec{r} - \vec{r}'| = \sqrt{r^2 + b^2 - 2\vec{r}\cdot\vec{r}'} = \sqrt{r^2 + b^2 - 2rb\beta}$$

$$\vec{r}\cdot\vec{r}' = r(i\sin\theta\cos\phi + j\sin\theta\sin\phi)\cdot(i\cos\phi' + j\sin\phi')$$
$$= rb\sin\theta\cos\phi\cos\phi' + rb\sin\theta\sin\phi\sin\phi'$$
$$= b\sin\theta\cos\theta\cos(\phi - \phi') \equiv rb\beta$$

$$ᴫ = r\sqrt{1 - \frac{2b\beta}{r}} \cong r\left(1 - \frac{b\beta}{r}\right) = r - b\beta \qquad \text{also} \quad \frac{1}{ᴫ} = \frac{1}{r}\frac{1}{(1 - b\beta/r)}$$

$$\frac{1}{ᴫ} = \frac{1}{r}\left(1 + \frac{b\beta}{r}\right)$$

$\boxed{11.4,\ 11.6}$

$$\cos\omega(t - \mathscr{r}/c) = \cos\omega\left(t - \frac{r}{c} + \frac{b\beta}{c}\right) = \cos\omega(t - r/c)\cos\frac{\omega\beta b}{c} - \sin\omega(t - r/c)\sin\left(\frac{\omega\beta b}{c}\right)$$

$$\frac{b2\pi}{\lambda} \ll 1 \Rightarrow \cos\frac{\omega b\beta}{c} \to 1 \text{ and } \cancel{\sin\frac{\omega b\beta}{c}} \to 1$$

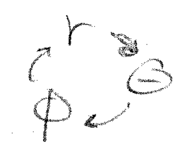
thus $= \cos\omega(t - r/c) - \dfrac{b\beta\omega}{c}\sin\omega(t - r/c)$

$$\therefore \vec{A} = \frac{\mu_0 b I_0}{4\pi r}\int\left[\cancel{\cos\omega(t - r/c)} - \frac{b\beta\omega}{c}\sin\omega(t - r/c)\right]\left[1 + \cancel{\frac{b\beta}{r}}\right]\left[j\cos\phi' - i\sin\phi'\right]d\phi'$$

why it go away

$$\beta = \sin\theta\cos\phi\cos\phi' + \sin\theta\sin\phi\sin\phi'$$

$$\int\beta(j\cos\phi' - i\sin\phi')d\phi' = j\int(\sin\theta\cos\phi\cos^2\phi'\,d\phi')$$



$$-i\int\sin\theta\sin\phi\sin^2\phi'\,d\phi' = \sin\theta(j\cos\phi - i\sin\phi)\pi$$

$$= \pi\sin\theta\,\hat{\phi}$$

$$\vec{A} = \frac{-\mu_0\pi b^2 I_0}{4\pi c}\frac{\sin\theta}{r}\sin\omega(t - r/c)\,\hat{\phi} = \boxed{\frac{-\mu_0 M_0\omega}{4\pi c}\frac{\sin\theta}{r}\sin\omega(t - r/c) = \vec{A}}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \boxed{\frac{\mu_0 M_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\omega(t - r/c)\hat{\phi} = \vec{E}}$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} = \frac{-\mu_0 M_0\omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos\omega(t - r/c)\hat{\theta}$$

$$\vec{B} = \frac{\hat{r}\times\vec{E}}{c}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left\{\frac{M_0\omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega t - \frac{r\omega}{c}\right]\right\}^2\hat{r}$$

$$\langle\vec{S}\rangle = \frac{\mu_0 M_0^2\omega^4}{32\pi^2 c^3}\frac{\sin^2\theta}{r^2}\hat{r}$$

$$\int\langle\vec{S}\rangle\cdot d\vec{a} = \frac{\mu_0 m_0^2\omega^4}{32\pi^2 c^3}\cdot\frac{8\pi}{3} = \boxed{\frac{\mu_0 m_0^2\omega^4}{12\pi c^3}}$$

magnetic

Power ratio $= \dfrac{P_{magnetic}}{P_{electric}} = \dfrac{\frac{\mu_0 m_0^2\omega^4}{12\pi c^3}}{\frac{\mu_0 P_0^2\omega^4}{12\pi c}} = \left(\dfrac{m_0}{P_0 c}\right)^2 \quad : \quad m_0 = \pi b^2 I_0,\ P_0 = q_0 d$

if $I_0 = q_0\omega_0$ then

$$\frac{P_{mag}}{P_{elect}} = \left(\frac{\pi b^2 q_0\omega_0}{q_0 d c}\right)^2 \overset{b\approx d}{=} \left(\frac{b\omega_0}{c}\right)^2 = \left(\frac{v_0}{c}\right)^2 \ll 1.$$