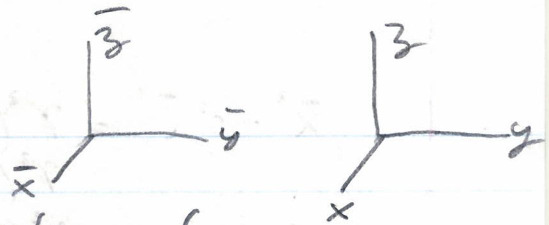
Chapter 12 : Electrodynamics and Relativity

§12.1 Lorentz Transformation

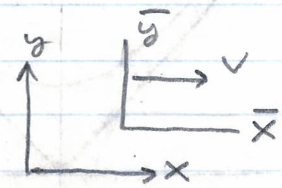


Einstein's 2 Postulates:

① The principle of relativity,
Physical laws are invariant in all intertial reference frames

② The universal speed of light
The speed of light in vacuum is the same in all intertial reference frames.

GALILEAN Transformation (Newtonian Mech's Base)

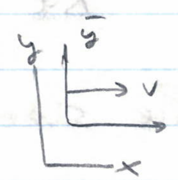


$$\bar{x} = x - vt$$
$$\bar{y} = y$$
$$\bar{z} = z$$
$$\bar{t} = t \qquad \text{Note: } \frac{d\bar{x}}{d\bar{t}} = \frac{dx - v\,dt}{d\bar{t}} = \frac{dx}{dt} - v$$

Then consider light in $\bar{S}$, $\frac{dx}{dt} = \frac{d\bar{x}}{d\bar{t}} + v = c + v$, we must give up GALILEAN.

Lorentz Transformation

Def^n/ $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = v/c$



$$\left.\begin{aligned} \bar{x} &= \gamma(x - vt) \\ \bar{y} &= y \\ \bar{z} &= z \\ \bar{t} &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}\right\} \text{Lorentz Transforms}$$

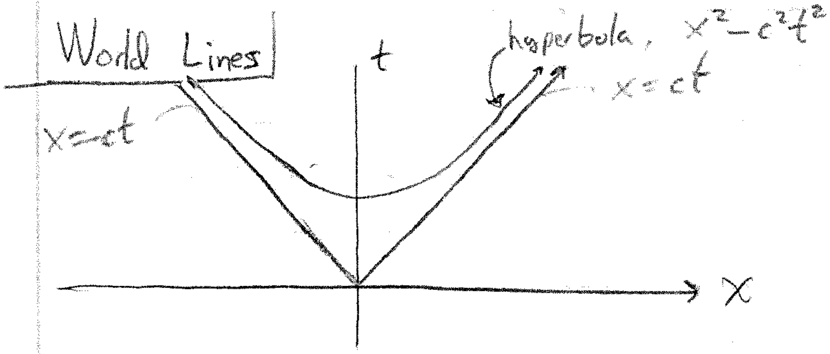
Derive inverse Lorentz transform,

$$\bar{x} = \gamma\left[x - v\left(\frac{\bar{t}}{\gamma} + \frac{vx}{c^2}\right)\right] = \gamma\left[x - \frac{v^2}{c^2}x\right] - v\bar{t}$$
$$\bar{x} + v\bar{t} = \gamma\left(1 - \frac{v^2}{c^2}\right)x = \frac{\gamma}{\gamma^2}x$$

$$\left.\begin{aligned} x &= \gamma(\bar{x} + v\bar{t}) \\ y &= \bar{y} \\ z &= \bar{z} \\ t &= \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right) \end{aligned}\right\} \text{Inverse Lorentz Transforms}$$

$$\bar{x}^2 - c^2\bar{t}^2 = \gamma^2(x - vt)^2 - c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2$$
$$= \gamma^2\left(x^2 + v^2t^2 - c^2t^2 - \frac{v^2x^2}{c^2}\right)$$
$$= \gamma^2\left(1 - \frac{v^2}{c^2}\right)x^2 - c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right)t^2$$
$$= x^2 - c^2t^2$$

This is an invariant!



any point on hyperbola can be transformed to it's corresponding point on the hyperbola by a Lorentz Transform.

If we have 2 points (x_1, t_1) and (x_2, t_2)
def^{n} $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$

$$\Delta\bar{x} = \gamma(\Delta x - v\Delta t)$$
$$\Delta\bar{t} = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$$

By the same Algebra as above,

$$(\Delta\bar{x})^2 - c^2(\Delta\bar{t})^2 = \boxed{(\Delta x)^2 - c^2(\Delta t)^2}$$ ← This is the distance between 2 points in the 4 dimensional space

Def^{n} $I = (\Delta x)^2 - c^2(\Delta t)^2$ = Interval

Note:
$I > 0$, we can always find a coord. system with $\Delta t = 0$ there is 1 system in which $\Delta t = 0$, the event is simultaneous in that system

$I < 0$, then its possible to find $\Delta x = 0$, that is there is some system where these events occur at the same place.

$\boxed{12.32,\ 12.33}$

Due wednesday.

Relativistic Velocity Addition

$$\bar{x} = \gamma(x - vt) \Rightarrow d\bar{x} = \gamma(dx - v\,dt)$$
$$\bar{t} = \gamma(t - vx/c^2) \Rightarrow d\bar{t} = \gamma\left(dt - \frac{v\,dx}{c^2}\right)$$

$$\frac{d\bar{x}}{d\bar{t}} = \frac{dx - v\,dt}{dt - \frac{v\,dx}{c^2}} = \frac{\frac{dx}{dt} - v}{1 - v\frac{dx}{dt}\frac{1}{c^2}}$$

$$\frac{dx}{dt} = \frac{\frac{d\bar{x}}{d\bar{t}} + v}{1 + v\frac{d\bar{x}}{d\bar{t}}/c^2} = \frac{u + v}{1 + vu/c^2}$$

, note if $u = c \Rightarrow \frac{dx}{dt} = c$.

§ 12.2 RELATIVISTIC MECHANICS

$$\bar{t}_1 = \gamma\left(t_1 - \frac{vx_1}{c^2}\right) \quad , \; dx = 0 \Rightarrow \quad d\bar{t} = \gamma dt \quad \Big\| \; dt = \sqrt{1-\beta^2}\, dt$$
$$\bar{t}_2 = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \qquad dt < d\bar{t} = \text{proper time}$$
$$d\bar{t} = \gamma\left(dt - \frac{v\,dx}{c^2}\right)$$

time dialation , $d\bar{t} > dt = d\tau \equiv$ proper time
$d\tau < d\bar{t} \Rightarrow$ time dialated.

ordinary velocity,
$$\vec{u} = \frac{d\vec{r}}{dt} \quad \text{— not a vector under Lorentz Trans.}$$

proper velocity,
$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{\sqrt{1-\beta^2}\,dt} = \frac{\vec{u}}{\sqrt{1-\beta^2}} = \gamma\vec{u}$$

Thus we define
$$\eta^\mu = \frac{dx^\mu}{d\tau} \qquad \mu = 1, 2, 3$$
$$\eta^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{d\tau} = c/\sqrt{1-\beta^2} = c\gamma$$
$$\eta^\mu = (c\gamma, u^1, u^2, u^3)$$

$$\bar{\eta}^0 = \gamma(\eta^0 - \beta\eta^1)$$
$$\bar{\eta}^1 = \gamma(\eta^1 - \beta\eta^0)$$
$$\bar{\eta}^2 = \eta^2$$
$$\bar{\eta}^3 = \eta^3$$

contrast with $\vec{u}$ transformation from $\bar{S} \to S$.
$$\bar{u}_x = (u_x - v) / \left(1 - \frac{v u_x}{c^2}\right)$$
$$\bar{u}_y = (u_y) / \left(1 - \frac{v u_x}{c^2}\right)$$
$$\bar{u}_z = (u_z) / \left(1 - \frac{v u_x}{c^2}\right)$$

Length Contraction

$$\bar{x} = \gamma(x - vt)$$
$$\Delta\bar{x} = \gamma(\Delta x - v\Delta t)$$

$$\Delta t = 0 \Rightarrow \Delta\bar{x} = \gamma\Delta x \Rightarrow \Delta x = \sqrt{1-\beta^2}\,\Delta\bar{x}$$

space contraction.

Relativistic Momentum

$$\vec{P} = m\vec{\eta} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

$$p^\mu = m\eta^\mu \quad , \mu = 1,2,3$$

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-\beta^2}}$$

$$m_{relativistic} \equiv \frac{m}{\sqrt{1-u^2/c^2}} \qquad m \text{ is the rest mass}$$

If we define $p^0 c$ as the energy, $E = p^0 c$ then

$$E = \frac{mc^2}{\sqrt{1-\beta^2}} \cdot \qquad \text{then if } \beta = 0 \Rightarrow E = mc^2$$

$$\cong mc^2\left[1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \ldots\right]$$

$$mc^2 \times \tfrac{1}{2}\beta^2 = \tfrac{1}{2}mv^2$$

3 term $\quad mc^2 \times \frac{3}{8}\beta^4 = \frac{3}{8}m\frac{v^4}{c^2} = \frac{1}{2}mv^2\left(\frac{3}{4}\frac{v^2}{c^2}\right)$, shows extreme closeness of Newtonian Mechanics to relativity for $v \ll c$.

Note that

$$p^\mu p_\mu = -p^{0^2} + \vec{p}\cdot\vec{p} = -\frac{m^2c^2}{1-\beta^2} + \frac{m^2u^2}{1-\beta^2} = \frac{m^2(u^2-c^2)}{1-\beta^2}$$

$$= \frac{m^2c^2(\beta^2-1)}{1-\beta^2} = -m^2c^2$$

$$\Rightarrow -\frac{E^2}{c^2} + p^2 = -m^2c^2 \Rightarrow \boxed{E^2 - p^2c^2 = m^2c^4}$$

Ex// Pion at rest decays,

$$\pi \longrightarrow \mu + \nu$$

find the energy and velocity of muon(μ) in terms of m_π and m_μ ($m_\nu = 0$)

$$E_{Before} = m_\pi c^2 \qquad \vec{P}_{Before} = 0$$

$$E_{after} = E_\mu + E_\nu \qquad \vec{P}_{after} = \vec{P}_\mu + \vec{P}_\nu$$

$$\vec{P}_\mu + \vec{P}_\nu = 0$$

$$E_\nu + E_\mu = m_\pi c^2 \Rightarrow E_\nu = m_\pi c^2 - E_\mu$$

$$E_\nu^2 - P_\nu^2 c^2 = 0 \Rightarrow E_\nu = P_\nu c$$

$$P_\nu = \frac{E_\nu}{c} = \frac{m_\pi c^2 - E_\mu}{c}$$

$$E_\mu^2 - P_\mu^2 c^2 = m_\mu^2 c^4$$

$$E_\mu^2 - (m_\pi c^2 - E_\mu)^2 = m_\mu^2 c^4$$

$$\Rightarrow 2 E_\mu m_\pi c^2 = m_\pi^2 c^4 + m_\mu^2 c^4$$

$$\Rightarrow E_\mu = \left(\frac{m_\pi^2 c^2 + m_\mu^2 c^2}{2 m_\pi}\right) c^2$$

$$P_\mu = \left[m_\pi c^2 - \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} c^2\right] / c = \left(\frac{m_\pi^2 - m_\mu^2}{2 m_\pi}\right) c$$

$$P_\mu = \frac{m_\mu V}{\sqrt{1-\beta^2}} = \left(\frac{m_\pi^2 - m_\mu^2}{2 m_\pi}\right) c$$

$$P_\mu = \frac{\beta}{\sqrt{1-\beta^2}} = \left(\frac{m_\pi^2 - m_\mu^2}{2 m_\pi m_\mu}\right) c = \alpha$$

$$\frac{\beta^2}{1-\beta^2} = \alpha^2 \Rightarrow \beta^2 = \alpha^2 - \alpha^2\beta^2 \Rightarrow \beta^2 = \frac{\alpha^2}{1+\alpha^2}$$

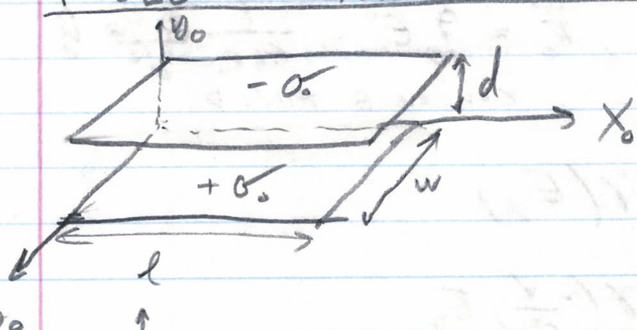
$$\beta = \sqrt{\frac{\alpha^2}{1+\alpha^2}} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

$$\boxed{V = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right) c}$$

Alex's Lecture

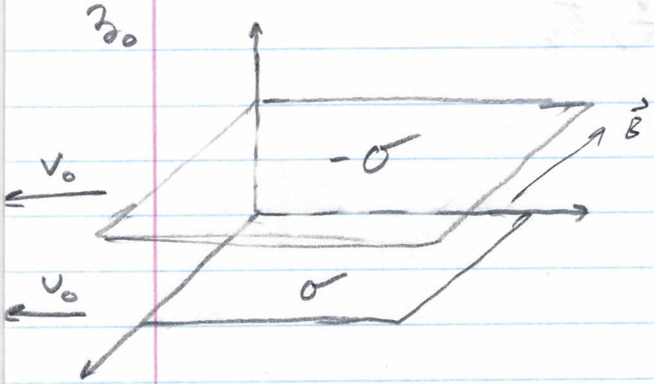
① $x^0 = \ldots \; x^3 = z$, Lorentz trans. $\Rightarrow \Lambda$

Field Transformations



$$\vec{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{y}$$

$$\vec{B} = 0$$



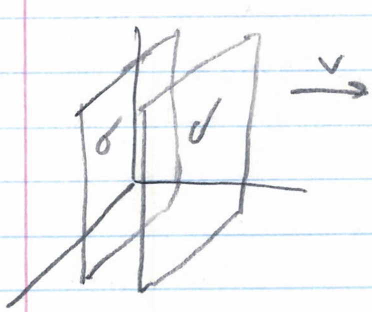
$$l = l_0\sqrt{1 - v_0^2/c^2}$$

$$\sigma_0 (l_0 w) = \sigma l w$$

$$\sigma = \gamma_0 \sigma_0 \Rightarrow \vec{E} = \frac{\gamma\sigma_0}{\epsilon_0}\hat{y}$$

$$\vec{E} = \gamma_0 \vec{E}_0$$

$$\boxed{E^\perp = \gamma_0 E_0^\perp}$$



$$\Rightarrow \boxed{E_0^\parallel = E^\parallel}$$

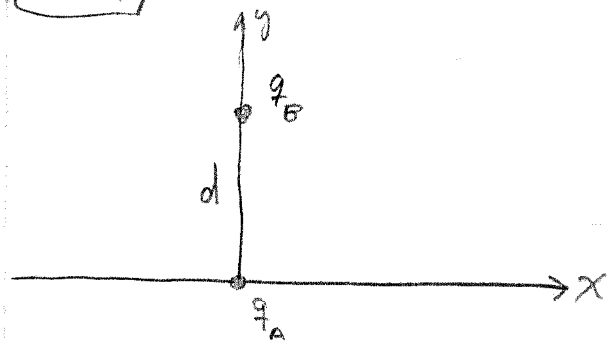
$$\vec{K}_+ = -\sigma_0 v_0 \hat{x} \Rightarrow \boxed{\vec{B} = -\mu_0 \sigma v_0 \hat{z}}$$

We see by just Δ to a moving frame we convert a purely $\vec{E}$ to an $\vec{E}$ and $\vec{B}$ field.

FIELD TRANSFORMS

$$\bar{E}_x = E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y$$
$$\bar{B}_x = B_x \qquad \bar{B}_y = \gamma\left(B_y - \frac{v}{c^2}E_z\right) \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right.$$

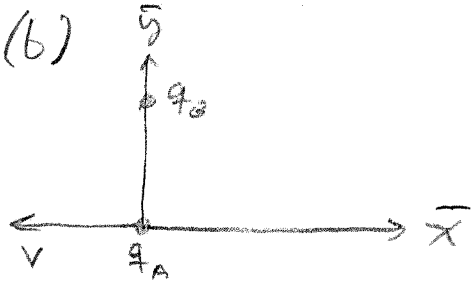
$\boxed{12.44}$



(a) $E_A = \frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{y}$, $\vec{F}_B = q_B\vec{E}_A = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{y}$

COULOMB FIELD.

(b)



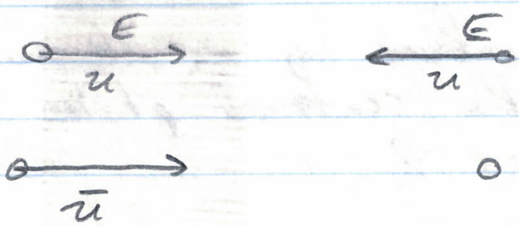
$$\bar{E}_y = \gamma(E_y)$$
$$\bar{B}_z = \gamma\left(-\frac{v}{c^2}\right)E_y$$

$$\vec{\bar{F}} = q_B\left(\vec{\bar{E}} + \cancel{\bar{v}_0 \times \overset{0}{\vec{\bar{B}}}}\right)$$
$$= q_B\gamma E_y$$
$$= \gamma\vec{F}$$

EXAMPLE. ACCELERATOR - EXPERIMENT

CLASSICALLY

RELATIVISTICALLY



$$\bar{u} = \frac{u+u}{1+u^2/c^2} \qquad \beta = u/c \qquad E = \frac{mc^2}{\sqrt{1-\beta^2}}$$

$$\bar{\beta} = \frac{2\beta}{1+\beta^2} \qquad \bar{E} = \frac{mc^2}{\sqrt{1-\bar{\beta}^2}} = \frac{mc^2}{1-\beta^2}(1+\beta^2)$$

$$1-\bar{\beta}^2 = 1 - \frac{4\beta^2}{(1+\beta^2)^2} = \frac{(1+\beta)^2 - 4\beta^2}{(1+\beta^2)^2} = \frac{1-2\beta^2+\beta^4}{(1+\beta^2)^2} = \frac{(1-\beta^2)^2}{(1+\beta^2)^2}$$

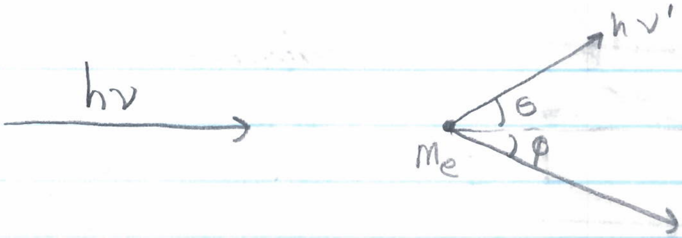
$$\bar{E} = \frac{mc^2}{1-\beta^2}(2+\beta^2-1) = \frac{2mc^2}{1-\beta^2} - mc^2 =$$

$$\boxed{\bar{E} = \frac{2E^2}{mc^2} - m^2}$$

For proton $mc^2 \cong 1\,\text{GeV}$ if $E \approx 30\,\text{GeV}$

$$\bar{E} = \frac{2\cdot(30)^2}{1} - 1 = 1799\ \text{GeV}$$

Example // Compton Scattering



A photon is scattered by an electron at rest. Find the energy of the scattered photon as a function of θ, m_e, and ν

$$P_e \sin\varphi = P_{\nu'} \sin\theta \qquad \text{verticle } \vec{P} \text{ cons.}$$
$$P_\nu = P_{\nu'} \cos\theta + P_e \cos\phi \qquad \text{horiz } \vec{P} \text{ cons.}$$
$$m_e c^2 + E_\nu = E_{\nu'} + E_e \qquad \text{energy conservation}$$

- $E_\nu = h\nu$ also $E^2 - c^2p^2 = m^2c^4 \Rightarrow P_\nu = \frac{h\nu}{c}$

$$P_e^2 \sin^2\phi = P_{\nu'}^2 \sin^2\theta$$
$$P_e^2 \cos^2\phi = (P_\nu - P_{\nu'}\cos\theta)^2$$
$$P_e^2 = P_{\nu'}^2 + P_\nu^2 - 2P_\nu P_{\nu'}\cos\theta$$

$$E_e^2 = (E_\nu - E_{\nu'} + m_e c^2)^2$$
$$E_e^2 - P_e^2c^2 = m_e^2c^4$$

$$m_e^2c^4 = (E_\nu - E_{\nu'})^2 + 2m_ec^2(E_\nu - E_{\nu'}) + m_e^2c^4 - (P_{\nu'}^2 + P_\nu^2 - 2P_\nu P_{\nu'}\cos\theta)c^2$$
$$-2E_\nu E_{\nu'} + 2P_\nu P_{\nu'}\cos\theta\, c^2 + 2m_ec^2(E_\nu - E_{\nu'}) = 0$$

$$\therefore \quad E_\nu E_{\nu'} - E_\nu E_{\nu'}\cos\theta + m_ec^2E_{\nu'} = m_ec^2E_\nu$$

$$E_{\nu'} = \frac{m_ec^2 E_\nu}{E_\nu(1-\cos\theta) + m_ec^2} = \frac{1}{(1-\cos\theta)/mc^2 + 1/E_\nu} = E_{\nu'} = \frac{hc}{\lambda'}$$

$$E_\nu = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)}$$