ELECTROMAGNETISM I

1.1 VECTOR ALGEBRA

- vector - magnitude and direction
  - location doesn't matter

- LAWS OF VECTOR ARITHMATIC

① $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

② $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

③ $\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$

④ $a(\vec{A}+\vec{B}) = a\vec{A}+a\vec{B}$

⑤ $\vec{A}\cdot\vec{B} = AB\cos\theta$

$\vec{A}\cdot\vec{A} = A^2$ = length squared

⑥ $\vec{A}\times\vec{B} = (AB\sin\theta)\hat{n}$, start from $\vec{A}$ then curl fingers from a into $\vec{B}$ this gives direction of $\hat{n}$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

- LAWS IN COMPONENT FORM

Defⁿ $\vec{A} = A_x i + A_y j + A_z k$

② $\vec{A}+\vec{B} = (A_x+B_x)i + (A_y+B_y)j + (A_z+B_z)k$

④ $a\vec{A} = aA_x i + aA_y j + aA_z k$

$i\cdot i = j\cdot j = k\cdot k = 1$

$i\cdot j = j\cdot k = k\cdot i = 0$

⑤ $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 \implies A = \sqrt{A_x^2+A_y^2+A_z^2}$

$i\times j = k$

$j\times k = i$

$k\times i = j$

⑥ $\vec{A}\times\vec{B} = (A_x i + A_y j + A_z k)\times(B_x i + B_y j + B_z k)$

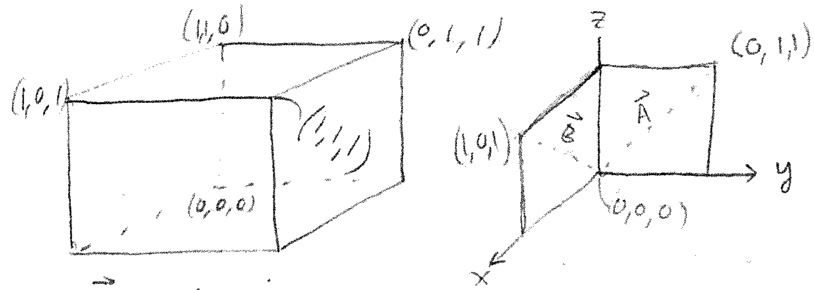
$= A_xB_y k - A_xB_z j - A_yB_x k + A_zB_z i + A_zB_x j - A_zB_y i$

$= (A_yB_y - A_zB_y)i + (A_zB_x - A_xB_z)j + (A_xB_y - A_yB_x)k$

$= \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ - determinant mneumonic for cross product

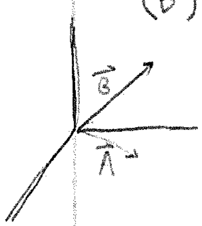
Ex

(a) Find Angle between the diagonal for the neighboring faces of a cubic box



$$\vec{A} = i + j$$
$$\vec{B} = j + k$$
$$\vec{A} \cdot \vec{B} = AB \cos\theta$$
$$\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

(b) Find angle between body and face diagonal



$$\vec{A} = i + j + k$$
$$\vec{B} = j + k$$
$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{1+1}{\sqrt{3}\sqrt{2}} = \frac{2}{\sqrt{6}} \Rightarrow \theta = 35.3^\circ$$

- TRIPLE PRODUCTS

$$\vec{A} \cdot (\vec{B} \times \vec{C})$$

$$\vec{B} \times \vec{C} = \begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

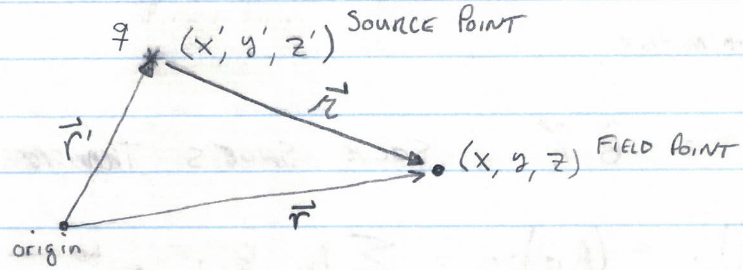
$$A \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = - \begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$A \cdot (\vec{B} \cdot \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = C \cdot (\vec{A} \times \vec{B})$$
$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$
$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \vec{D} \cdot \quad (\vec{A} \times \vec{B} =$$
$$= \vec{C} \cdot (\vec{D} \times (\vec{A} \times \vec{B}))$$

ch1 #'s 2, 4, 6

POSITION : DISPLACEMENT : AND SEPERATION VECTORS

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$
$$|\vec{r}\,'| = \sqrt{x'^2 + y'^2 + z'^2}$$



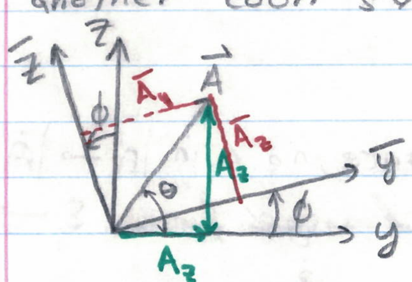
$$\hat{\vec{r}} = \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$\hat{\vec{r}} = \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \quad , \quad |\vec{\mathscr{r}}| = |\vec{r} - \vec{r}\,'| \; , \left\{(x-x')i + (y-y')j + (z-z')k = \vec{\mathscr{r}}\right\}$$

$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|} = \frac{(x-x')i + (y-y')j + (z-z')k}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

How do vectors transform from one coördinate system to another coör. SYSTEM



$$\vec{A} = A_y j + A_z k$$
$$A_y = A\cos\theta \qquad A_z = A\sin\theta$$
$$\bar{A}_y = A\cos(\theta - \phi)$$
$$\bar{A}_z = A\sin(\theta - \phi)$$

$$\bar{A}_y = A\cos(\theta - \phi) = A(\cos\theta\cos\phi - \sin\theta\sin\phi) = A_y\cos\phi + A_z\sin\phi$$
$$\bar{A}_z = A\sin(\theta - \phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi) = -A_y\sin\phi + A_z\cos\phi$$

$$\begin{bmatrix}\bar{A}_y \\ \bar{A}_z\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}A_y \\ A_z\end{bmatrix}$$ the same in matrix form

ye old 2D rotational matrix about the x-axis

$$R(90^\circ) = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} \quad \text{then} \quad \begin{bmatrix}\bar{A}_y \\ \bar{A}_z\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix}A_y \\ A_z\end{bmatrix} = \begin{bmatrix}A_z \\ -A_y\end{bmatrix}$$

$$\widetilde{\begin{bmatrix} \bar{A}_y \\ \bar{A}_z \end{bmatrix}} = \begin{bmatrix} \bar{A}_y & \bar{A}_z \end{bmatrix}$$

collumn matrix — row matrix

$\widetilde{AB} = \tilde{B}\tilde{A}$ SOCK SHOES TRANSPOSE PRODUCT RULE

Proof

$(\widetilde{AB})_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$ ← using $\sum$ def^n for matrix mult.

— same $\therefore$ $\widetilde{AB} = \tilde{B}\tilde{A}$

$(\tilde{B}\tilde{A})_{ij} = \sum_k \hat{B}_{ik} \hat{A}_{kj} = \sum_k B_{ki} A_{jk}$

Using $\widetilde{AB} = \tilde{B}\tilde{A}$ and matrix mult. to show invariance of length

$$\begin{bmatrix} \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} A_y \\ A_z \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{A}_y & \bar{A}_z \end{bmatrix} = \begin{bmatrix} A_y & A_z \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\begin{pmatrix} \bar{A}_y & \bar{A}_z \end{pmatrix} \begin{bmatrix} \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} A_y A_z \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \bar{A}_y & \bar{A}_z \end{bmatrix}$$

$$\bar{A}_y^2 + \bar{A}_z^2 = \begin{bmatrix} A_y A_z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{A}_y & \bar{A}_z \end{bmatrix}$$

$\bar{A}_y^2 + \bar{A}_z^2 = A_y^2 + A_z^2$ invariance, no Δ in $|A| \rightarrow |\bar{A}|$

length of A does not change from $S \rightarrow \bar{S}$

$\tilde{R}R = I$

in 3D $\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ $\quad \bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j$

vector is a tensor of rank one.

scalar is a tensor of rank ZERO

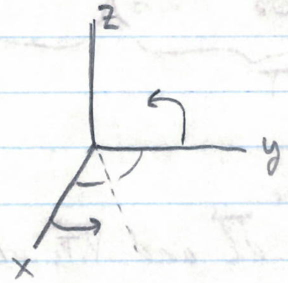
HW. PROBLEMS 1.8, 1.9

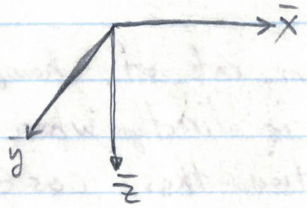
www.physics.ncsu.edu/courses/PY414/index.html

Tensor of Second Rank

$$\bar{T}_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ij} R_{jl} T_{kl}$$

EXAMPLE OF SPECIFIC 3D rotation matrix: what is R if rotate about an axis on x-y plane? Which makes 45° with x-y AXIS'S such that x-y axis is interchanged.





$$\bar{A}_x = A_y$$
$$\bar{A}_y = A_x$$
$$\bar{A}_z = -A_z$$

$$\bar{A}_x^2 + \bar{A}_y^2 + \bar{A}_z^2 = A_y^2 + A_z^2 + A_x^2$$

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Longrightarrow R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ by "observation"

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad R = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

rotation about x-axis

rotation about y-AXIS

§ 1.2 THE DIFFERENTIAL CALCULUS

$\frac{df(x)}{dx}$ tells us how fast $f(x)$ is changing in the x- dimension $\quad df(x) = \frac{df}{dx}dx$

$T = T(x,y,z)$, (A SCALAR FUNCTION)

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = \underbrace{\left(i\frac{\partial T}{\partial x} + j\frac{\partial T}{\partial y} + k\frac{\partial T}{\partial z}\right)}_{\nabla T}\underbrace{\left(i\,dx + j\,dy + k\,dz\right)}_{d\vec{l}}$$

Now rewriting in vector form:

$$dT = |\nabla T|\, dl \cos\theta$$

$\frac{dT}{dl} = |\nabla T| \cos\theta$, maximum rate of change in T is directly where $d\vec{l}$ and $\vec{\nabla T}$ are the same direction thus $\cos\theta = \cos 0 = 1$

Ex/ find the gradiant of

(a) $f(x,y,z) = x^2 + y^3 + z^4 \qquad \nabla f = (2x)i + (3y^2)j + (4z^3)k$

(b) $f(x,y,z) = x^2y^3z^4 \qquad \nabla f = (2xy^3z^4)i + (3x^2y^2z^4)k + (4x^2y^3z^3)k$

(c) $f(x,y,z) = e^x \sin y \ln z$

$$\nabla f = (e^x \sin y \ln z)i + (e^x \cos y \ln z)j + \left(\frac{e^x \sin y}{z}\right)k$$

DIVERGENCE

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, $\vec{V} = V_x i + V_y j + V_z k = \vec{V}(x,y,z)$

$$\nabla \cdot \vec{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(V_x i + V_y j + V_z k\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \cdot \vec{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(ix + jy + kz) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\nabla \cdot (x^2 i + y^2 j + z^2 k) = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2x + 2y + 2z$$

$\nabla \cdot \vec{B}$ where $\vec{B}$ is constant in 3-D $\quad \nabla \cdot \vec{B} = 0$

$\vec{A} = y\vec{k}$ or $\vec{A} = xj$ where a dimension doesn't effect itself then

$\nabla \cdot \vec{A} = 0$

HW Problems $\boxed{1.12,\ 1.15,\ 1.17}$

Curl

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\nabla \times \vec{V} = i\left(\frac{\partial V_y}{\partial y} - \frac{\partial V_y}{\partial z}\right) + j\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + k\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Examples

$$\nabla \times \vec{r} = i(0) + j(0) + k(0) = \vec{0} \neq 0\ !!!$$

$$\vec{A} = y\hat{k} \qquad \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y \end{vmatrix} = i$$

$\vec{V}_1 = -yi + xj$

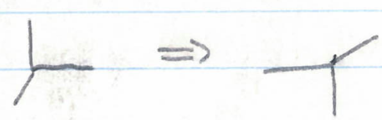
$\vec{V}_2 = xj$ find curls $\nabla \times \vec{V}_1 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = k + k = 2k$

$$\nabla \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = k$$

$\boxed{1\text{-}10}$ how does a vector transform under translation

(a) suppose $\vec{A} = A_x i + A_y j + A_z k$ then $\vec{A}' = A_x i + A_y j + A_z k$

(b) how about inversion where



$x \to -x$
$y \to -y$
$z \to -z$

then $\bar{A}_x = -A_x$
$\bar{A}_y = -A_y$
$\bar{A}_z = -A_z$

(c) $(\vec{A}\cdot\vec{B})$ and $(\vec{A}\times\vec{B})$ under inversion

$\bar{\vec{A}} \cdot \bar{\vec{B}} = \vec{A}\cdot\vec{B}$

$\bar{\vec{A}} \times \bar{\vec{B}} = \begin{vmatrix} i & j & k \\ -A_x & -A_y & -A_z \\ -B_x & -B_y & -B_z \end{vmatrix} = \vec{A}\times\vec{B}$ since $\bar{\vec{A}} = -\vec{A}$, $\bar{\vec{B}} = -\vec{B}$

$\bar{\vec{A}} \times \bar{\vec{B}} = (-1)\vec{A} \times (-1)\vec{B} = (-1)(-1)\,\vec{A}\times\vec{B}$

since $\vec{A}\times\vec{B}$ doesn't change sign under inversion like normal ordinary vectors we call this vector a psuedo vector.

$\boxed{1\text{-}10}$

(d) $D = \vec{A} \cdot (\vec{B} \times \vec{C})$ Does $D = \vec{A} \cdot (\vec{B} \times \vec{C})$

under inversion

$\vec{A} = -\vec{A}$

$\vec{B} = -\vec{B}$

$\vec{C} = -\vec{C}$ then $D = \vec{A} \cdot (\vec{B} \times \vec{C})$

$= -\vec{A} \cdot (-\vec{B} \times -\vec{C})$

D is psuedo scalar $= -\vec{A} \cdot (\vec{B} \times \vec{C})$

$=$

• Monday 2^{nd} Week •

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

∇(scalar function) = vector this is gradiant operation

$\nabla \cdot$(vector funtion) = scalar this " the divergence function

$\nabla \times \vec{V}$ = vector this is the curl of $\vec{V}$

PRODUCT RULES FOR ∇ OPERATOR.

(1) $\nabla(fg) = f\nabla g + g\nabla f$

(2) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$

(3) $\nabla(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$

(4) $\nabla(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(5) $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times\nabla f$

(6) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

- Proof for (3)

$$\nabla\cdot(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) = f\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$+ A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z} = f(\nabla\cdot\vec{A}) + (\vec{A}\cdot\nabla f)$$

- Proof for (5)

$$[\nabla\times(f\vec{A})]_i = i\left(\frac{\partial fA_z}{\partial y} - \frac{\partial fA_y}{\partial z}\right) = i\left[\underbrace{f\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)}_{f\nabla\times\vec{A}}\underbrace{-\left(A_y\frac{\partial f}{\partial z} + A_z\frac{\partial f}{\partial y}\right)}_{-(\vec{A}\times\nabla f)}\right]$$

$$= f(\nabla\times\vec{A}) - \vec{A}\times\nabla f$$

ARITHMATIC FOR ∇ CONCERNING SECOND DERIVATIVES

(1) $\nabla\cdot(\nabla T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \nabla^2 T$ Laplacian on T.

(2) $\nabla\times(\nabla T) = \emptyset$

(3) $\nabla(\nabla\cdot\vec{V}) = \nabla\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) =$

(4) $\nabla\cdot(\nabla\times\vec{V}) = 0$

(5) $\nabla\times(\nabla\times\vec{V}) = \vec{V}(\nabla\cdot\vec{V}) - \nabla^2\vec{V}$

where $\vec{V}$ is a vector

∇ is del

T is any scalar function.

HW. 1.23, 1.27

Proof (A)

$$\nabla\cdot(\nabla\times\vec{V}) = \nabla\cdot\left(i\left(\frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y}\right) + j\left(\frac{\partial V_x}{\partial z} - \frac{\partial z}{\partial x}\right) + k\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\right)$$

$$= \frac{\partial^2 V_z}{\partial x\partial y} - \frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_x}{\partial y\partial z} - \frac{\partial^2 V_z}{\partial y\partial x} + \frac{\partial^2 V_y}{\partial x\partial z} - \frac{\partial^2 V_x}{\partial y\partial z}$$

$$= \emptyset .$$

Example

calculate Laplacian of...

(a) $T_a = x^2 + 2xy + 3z + 4$ $\qquad \nabla^2 T_a = 2.$

(b) $T_b = \sin x \sin y \sin z$ $\qquad \nabla^2 T_\theta = -3T_\theta$

(c) $T_c = e^{-5x}\sin 4y \cos 3z$ $\qquad \nabla^2 T_c = 0$

(d)

$\vec{V} = x^2 i + 3xz j - 2xz k \qquad \nabla^2\vec{V} = 2i - 6xj$

$\nabla^2\vec{V} = (\nabla^2 x^2)i + \nabla^2(3xz)j - \nabla^2(2xz)k = 2i - 6xj.$

§ 1.3 The Integral Calculus

- Line, Surface and Volume Integrals.
- LINE Integral

$$\int_a^b \vec{V}\cdot\vec{dl} \quad \text{or for complete closed path} \quad \oint \vec{V}\cdot\vec{dl}$$

Ex ① if $\vec{V} = iy^2 + 2x(y+1)j$ for $\vec{a} = (2,1) \longrightarrow (2,2)$ and $\vec{b} = (2,2)$

$$\int_a^b \vec{V}\cdot\vec{dl} = \int_1^2 \vec{V}(y=1)\cdot i\,dx + \int_1^2 \vec{V}(x=2)\cdot j\,dy$$

$$= \int_1^2 1\,dx + \int_1^2 4(y+1)\,dy = x\Big|_1^2 + (2y^2 + 4y)\Big|_1^2 = 11$$

② same $\vec{V}$ except $(1,1) \longrightarrow (2,2)$

$$\int \vec{V}\cdot\vec{dl} = \int y^2\,dx + \int 2x(y+1)\,dy$$

$dx = dy$
$x = y$

$$= \int x^2\,dx + \int (2x^2 + 2x)\,dx$$

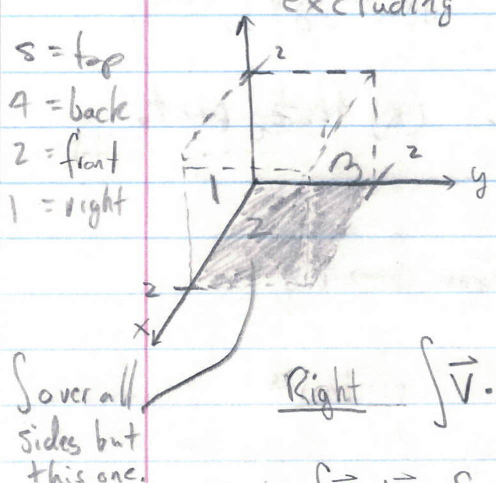
$$= \frac{x^3}{3} + \frac{1}{3}x^3 + \frac{2x^2}{2}\Bigg]_1^2 =$$

(b) Surface Integrals

$$\int_S \vec{v} \cdot \vec{dA} \qquad \text{for a closed surface} \qquad \oint \vec{v} \cdot \vec{dA}$$

Ex/ $\vec{v} = 2xz\,i + (x+2)j + y(z^3-3)k$

Integrate along the 5 sides of a cube of length 2 excluding the bottom



<u>Top</u> $\int \vec{V} \cdot \vec{dA} = \int \vec{V} \cdot (dx\,dy\,k) = \int y(z^2-3)\,dx\,dy$

$= \int y(4-3)\,dx\,dy = x\Big|_0^2 \frac{1}{2}y^2\Big|_0^2 = 4$

<u>Right</u> $\int \vec{V} \cdot \vec{dA} = \int \vec{V} \cdot (dy\,dz(-j)) = \int -(x+2)\,dx\,dz = -z\Big|_0^2 \left(\frac{1}{2}x^2 + 2x\right)\Big|_0^2 = -12$

<u>Front</u> $\int \vec{V} \cdot \vec{dA} = \int \vec{V} \cdot (dy\,dz\,i) = \int \underbrace{2xz}_{x=2}\,dy\,dz = \int 4z\,dz\,dy = 2z^2\Big|_0^2 y\Big|_0^2 = 16$

<u>Left</u> $\int \vec{V} \cdot \vec{dA} = \int \vec{V} \cdot (dx\,dz\,j) = \int (x+2)\,dx\,dz = 12$

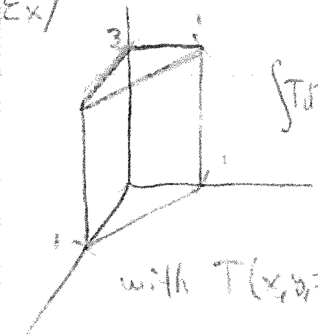
<u>Top</u> $\int \vec{V} \cdot \vec{dA} = \int \vec{V} \cdot (dy\,dz(-i)) = \int -2\underbrace{x}_{z}z\,dy\,dz = 0$ (?) <u>TOTAL</u> = 20.

(c) <u>Volume Integral</u>

Let T be a scalar function. And let $d\tau = dx\,dy\,dz$

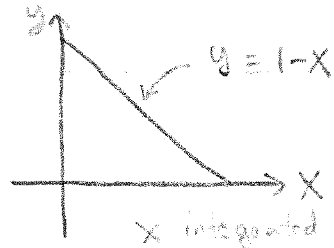
$$\int T\,d\tau = \int T\,dx\,dy\,dz$$

Ex/



$$\int T d\tau = \int T\,dx\,dy\,dz = \int_0^3 \left(z^2 dz \int_0^{1-y} xy\,dx\,dy \right) = \int_0^3 \left(z^2 dz \int_0^1 \left(y\,dy \int_0^{1-y} x\,dx \right)\right)$$

with $T(x,y,z) = xyz^2$



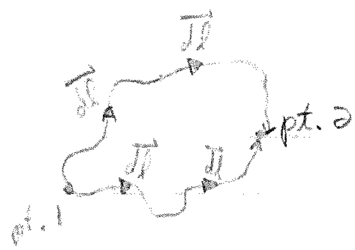
$y = 1-x$. (*) tricky part.

x integrated from $0 \to 1-y$

$$\int T d\tau = \frac{1}{3} z^3 \Big|_0^3 \int_0^1 y\,dy\, \frac{1}{2}(1-y)^2$$

$$= \left(9 \cdot \frac{1}{2}\right)\left(\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_0^1 = 9/24 = 3/8.$$

<u>Fundamental Theorem for Gradient.</u>



$$dT = \nabla T \cdot \vec{dl}$$

$$\int_1^2 dT = \int_1^2 (\nabla T) \cdot \vec{dl}$$

$$T(2) - T(1) = \int_1^2 (\nabla T) \cdot \vec{dl} =$$

if vector function comes from gradiant of a scalar function then it has path independence in integration. Path Independence if $\vec{F} = \nabla T$.

Clearly $\oint (\nabla T) \cdot \vec{dl} = 0$

Note could use curl to see if $\vec{F} = \nabla T$ because $\nabla \times \nabla T = 0$ thus if $\nabla \times \vec{F} = 0$ then $\vec{F} = \nabla T$.

1.28, 1.30

Fundamental Theorem For Divergence

$$\int_\tau (\nabla\cdot\vec{V})\,d\tau = \oint_{\text{Surface}} \vec{V}\cdot\vec{dA} \quad ; \text{ Gauss Theorem.}$$

Fundamental Theorem For Curl

$$\int(\nabla\times\vec{V})\cdot\vec{dA} = \oint_{\substack{\text{line}\\ \text{(Closed Curve)}}} \vec{V}\cdot\vec{dl} \quad ; \text{ Stokes Theorem.}$$

over boundary of surface that is covered by $\vec{dA}$

Some Fundamental Relations

nothing new just rearranged some.

$$\int \nabla T\cdot d\vec{l}$$

$$\int(\nabla\cdot\vec{V})\,d\tau = \oint \vec{V}\,d\vec{a}$$

$$\int(\nabla\times\vec{V})\cdot d\vec{a} = \oint \vec{V}\cdot d\vec{l}$$

$$\int(\nabla\times\vec{V})\cdot d\vec{a} = \int \nabla\times(\nabla\times\vec{V})\cdot da$$ by gauss Theorem! :)

Integration by parts

Since $\dfrac{d(fg)}{dx} = f g_x + g f_x$

$$\int (fg)' = fg = \int f\frac{dg}{dx} - g\frac{df}{dx} \Rightarrow \boxed{\int f g' dx = fg - \int g f' dx.}$$

Example

$$\int_a^b x^2\sin x = -x^2\cos x + \int 2x\cos x\,dx = -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x\Big]_a^b.$$

Example

Prove $\int f(\nabla \cdot \vec{A})\,d\tau = \oint f\vec{A} \cdot d\vec{a} - \int_{\tau} \vec{A} \cdot \nabla f\, d\tau$

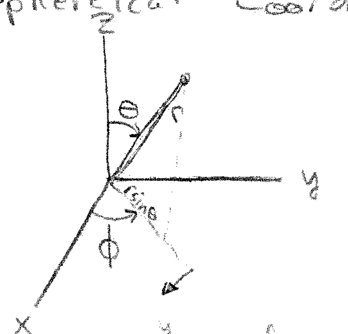
divergence theorem $\int \nabla \cdot (f\vec{A})\,d\tau = \oint f\vec{A} \cdot d\vec{a}$

$$\int \nabla \cdot f\vec{A}\,d\tau = \int \left[(\nabla f) \cdot \vec{A} + f(\nabla \cdot \vec{A})\right] d\tau = \oint f\vec{A} \cdot d\vec{a}$$

or $\int f(\nabla \cdot \vec{A})\,d\tau = \oint f\vec{A} \cdot d\vec{a} - \int (\nabla f) \cdot \vec{A}\, d\tau$

Curvelinear Coordinates — $\boxed{\S\, 1.4}$

Sphereical Coordinates.

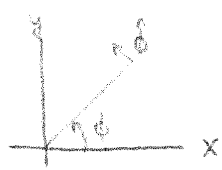


$z = r\cos\theta$

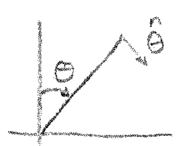
$x = (r\sin\theta)(\cos\phi)$

$y = (r\sin\theta)(\sin\phi)$

$\hat{r} = \dfrac{xi + yj + zk}{r} = (\sin\theta\cos\phi)i + (\sin\theta\sin\phi)j + (\cos\theta)k.$



$\hat{\phi} = k \times (i\cos\phi + j\sin\phi) = k \times i\cos\phi + k \times j\sin\phi$
$= j\cos\phi - i\sin\phi.$

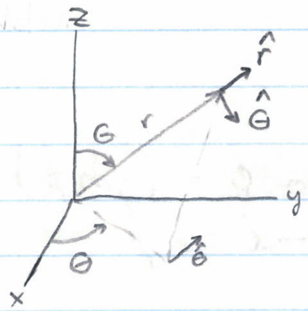


$\hat{\theta} = \hat{\phi} \times \hat{r} = (j\cos\phi - i\sin\phi) \times [(\sin\theta\cos\phi)i + \sin\theta\sin\phi j + \cos$
$= i\cos\phi\cos\theta + j\sin\phi\cos\theta - k\sin\theta.$

- Note That i, j, k constant through $\mathbb{R}^3$ while $\hat{r}, \hat{\theta}, \hat{\phi}$ are unique to each $(x, y, z) \in \mathbb{R}^3$.

1.38, 1.39

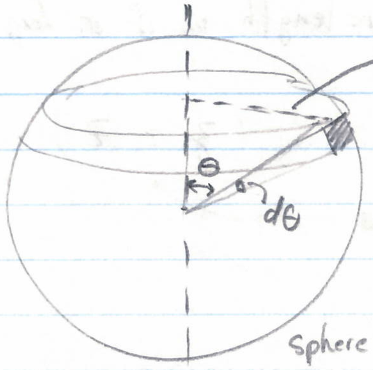
Sphereical coordinate integration considerations



Volume element : $(dr)(r\,d\theta)(r\sin\theta\,d\phi)$
or $r^2\,dr\,d\theta\,d\phi$.

line element $d\vec{l} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$.

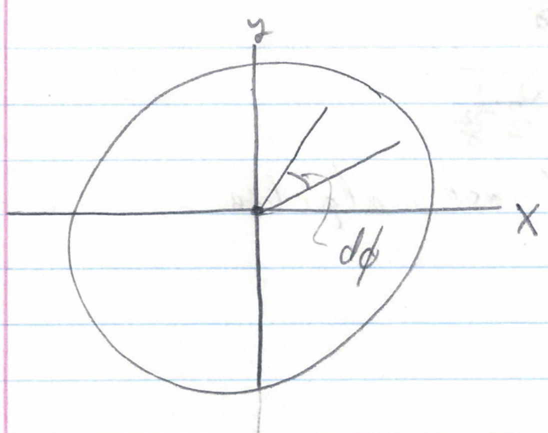
Surface Element : depends on Surface.



$d\vec{a} = \hat{r}\,r\sin\theta\,d\phi\,r\,d\theta = \hat{r}\,r^2\,d\Omega$

So called solid angle.

sphere to integrate over surface of.



on xy plane
$\theta = \frac{\pi}{2}$

$d\vec{a} = \hat{\theta} \cdot r\,d\phi\,dr$

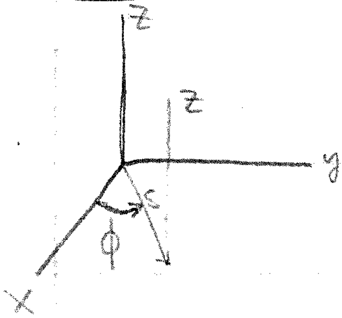
Gradiant in Sphereical's

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

$$\nabla \cdot \vec{V}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} .$$

will be given if needed.

Cylindrical Coordinates



unit vectors

$\hat{s} = i\cos\phi + j\sin\phi$

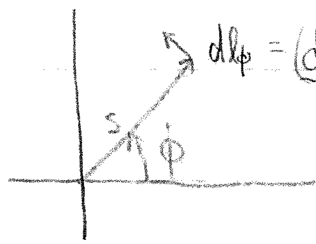
$\hat{z} = k$

$\hat{\phi} = \hat{z} \times \hat{s} = k \times (i\cos\phi + j\sin\phi)$

$= j\cos\phi - i\sin\phi = \hat{\phi}$

$dl_s = ds$

$dl_\phi = s\,d\phi$



$dl_\phi = \underbrace{(d\phi)\,s}$ (arc length, which is forgotten)

$\theta = \frac{s}{r} \Rightarrow s = r$
$= s$

$(z = z)$

$$\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

$$\nabla \cdot \vec{V} = \frac{1}{s}\frac{\partial}{\partial s}(s V_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

$\nabla \times \vec{V}$ = yucky in text if need. (just algebra.

§1.5 Dirac Delta Function

Consider $\vec{v} = \frac{1}{r^2}\hat{r}$ then $\text{div}\,\vec{v} = \nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = \frac{1}{r^2}(\emptyset)$
but the divergence theorem tells us
that

$$\oint_S \vec{v}\cdot d\vec{a} = \oint \frac{1}{R^2} R^2 \sin\theta\, d\theta\, d\phi = 4\pi.$$

$\neq$ why?

but $\int_V (\nabla\cdot\vec{v})\,d\tau = \int_V 0\,d\tau = 0.$

$\nabla\cdot\vec{v}$ blows up at origin, as $\vec{v}$ blows up at origin. A function that equals zero everywhere except at one point ($\rightarrow\infty$) is called a dirac-delta function.

$\int \delta = 1(?)$

$$\boxed{\text{Def}^{n} \quad \delta(x) = \begin{cases} 0 & \text{if } x\neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{but} \int_{-\infty}^{\infty}\delta(x)\,dx = 1}$$

Hence $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$ because $f(x)\delta(x) = f(0)\,\delta(x)$

or more generally $\delta(x-a) = 0 \quad x\neq a$
$= \infty \quad x = a$

$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$ because $f(x)\delta(x-a) = f(a)\,\delta(x)$.

ex/ $f = \frac{1}{L} \quad \frac{-L}{2} < x < \frac{L}{2} \qquad f = 0$ everywhere

$$\int_{-\infty}^{\infty} f\cdot dx = \frac{1}{L}x\Big|_{-\frac{1}{2}L}^{\frac{1}{2}L} = \frac{1}{L}\cdot L = 1$$

Let $L\rightarrow 0 \quad f\rightarrow\delta(x)$.

1.44, 1.45

Properties of δ function

$$\delta(kx) = \frac{1}{|k|}\delta(x)$$

Pf/ $\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx = \int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)\,\frac{dy}{|k|} = \frac{f(0)}{|k|}$

$y = kx$, $dy = k\,dx$

$$\int_{-\infty}^{\infty} \frac{1}{|k|}\,\delta(x) f(x)\,dx = \frac{1}{|k|} f(0) \implies \delta(kx) = \frac{1}{|k|}\delta(x)$$

$$\int A\,c(x)\,dx = \int B\,c(x)\,dx$$

Ex/ $\int_{2}^{6} (3x^2 - 2x - 1)\,\delta(x-3)\,dx = f(3) = 20.$

$\int_{0}^{5} (\cos x)\,\delta(x-\pi) = -1$

$\int_{0}^{3} x^3\,\delta(x+1) = 0.$

$\int_{-\infty}^{\infty} \ln(x+3)\,\delta(x+2) = \ln(1) = 0.$

In 3 dimensions we have

$$\delta^3(\vec{r}) = \delta(\vec{r}) \equiv \delta(x)\,\delta(y)\,\delta(z)$$

$$\int_{\mathbb{R}^3} \delta(\vec{r})\,d\tau = \int_{-\infty}^{\infty}\delta(x)\,dx \int_{-\infty}^{\infty}\delta(y)\,dy \int_{-\infty}^{\infty}\delta(z)\,dz = (1)(1)(1) = 1.$$

Assignment

$\boxed{1.44,\ 1.45}$

$\delta(x)$ is even

$$\delta(a-x) = \delta(x-a)$$

$$I = \int_{-\infty}^{\infty} f(x)\,\delta(a-x)\,dx$$

let $\quad y = a-x \qquad dy = -dx$

$$I = \int_{+\infty}^{-\infty} f(a-y)\,\delta(y)\,(-dy) = \int_{-\infty}^{\infty} f(a-y)\,\delta(y)\,dy = f(a)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int_{-\infty}^{\infty} f(x)\,\delta(a-x)\,dx \iff \delta(x-a) = \delta(a-x)$$

3-D δ FUNCTION

Def:

$$\boxed{\begin{aligned} &\delta(\vec{R}) = \delta(x)\,\delta(y)\,\delta(z) \\ &\int f(\vec{R})\,\delta(\vec{R}-\vec{a})\,d\tau = f(\vec{a}) \end{aligned}}$$

$\int_V \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d\tau = \oint_S \frac{\hat{r}}{r^2} \cdot d\vec{a}$ then assume divergence theorem holds

$\oint_S \frac{d\vec{a}}{r^2} = 4\pi$ then $\int_V \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d\tau = \int 4\pi\,\delta(\vec{r})$ because $\nabla \cdot \frac{\hat{r}}{r^2} = 0$ every where except origin.

So we must choose $\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi\,\delta(\vec{r})$ if we are to get $\int \nabla \cdot \vec{v}\,d\tau = 4\pi$. So we must Accept

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r})$$

if the singularity is at $\vec{r}\,'$ then $\vec{v} = \frac{\hat{\imath}}{r^2}$ where $\vec{\imath} = \vec{r} - \vec{r}\,'$

$$\nabla \cdot \vec{v} = \nabla \cdot \left(\frac{\hat{\imath}}{r^2}\right) = 4\pi\,\delta(\vec{\imath}) = 4\pi\,\delta(\vec{r}-\vec{r}\,').$$

from calculus we have $\nabla \cdot \frac{\hat{\imath}}{r^2} = -\nabla' \cdot \frac{\hat{\imath}}{\imath^2}$

If $f = \frac{1}{\imath} = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \frac{(-\frac{1}{2})(2)\left((x-x')i + (y-y')j + (z-z')k\right)}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} = \frac{-\vec{\imath}}{\imath^3} = \frac{-\hat{\imath}}{\imath^2}$$

So we have shown $\nabla \cdot \left(\underbrace{\nabla \frac{1}{\imath}}_{\nabla f}\right) = -4\pi\,\delta(\vec{\imath})$

thus $\nabla^2\left(\frac{1}{\imath}\right) = -4\pi\,\delta(\vec{\imath})$.

Assignment = $\boxed{1.46,\ 1.48}$

$$\delta(\vec{r}-\vec{r}\,') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

say $\vec{r}$ and $\vec{r}\,'$ possess sphereical coordinates
then we must construct . . .

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')}{r^2\sin\theta}$$

$\leftarrow$ to cancel out $d\tau$ in $\int$
so $\int_{-\infty}^{\infty}\delta(\vec{r}-\vec{r}\,') = 1$

Some Examples

$$\int\left(r^2+\vec{r}\cdot\vec{a}+a^2\right)\delta(r^2-a)\,d\tau = 3a^2$$

$$\int|\vec{r}-\vec{b}|^2\,\delta(5\vec{r})\,d\tau = \frac{|-\vec{b}|^2}{5}$$

τ = cubic of side 2 centered at origin

$$\int\left(r^4+r^2(\vec{r}\cdot\vec{c})+c^4\right)\delta(\vec{r}-\vec{c})\,d\tau = 0.$$

$\vec{c} = 5i + 3j + 2k$
τ = sphere of radius 6. $\quad |\vec{c}| = \sqrt{38} > \sqrt{36} > 6$

$$\int\vec{r}\cdot(\vec{d}-\vec{r})\,\delta(\vec{e}-\vec{r})\,d\tau$$

$\vec{d} = (1,2,3)\quad \vec{e} = (3,2,1) \qquad |\vec{e}| = \sqrt{(3-2)^2+(2-2)^2+(3-2)^2} = \sqrt{2}$

τ = radius of sphere 1.5 centered at $(2,2,2)$
$\sqrt{2} < 1.5$ so singularity is inside volume.

$$\int\vec{r}\cdot(\vec{d}-\vec{r})\,\delta(\vec{e}-\vec{r})\,dt = \vec{e}\cdot(\vec{d}-\vec{e}) = 4.$$

$$\int Q\,\delta(|\vec{r}|-R)\, r^2 \sin\theta \, dr\, d\phi\, d\theta$$

$$\rho = \frac{dq}{d\tau}$$

$$\rho(\vec{r}) = \underline{Q\,\delta(|\vec{r}|-R)} = \frac{dq}{d\tau}$$

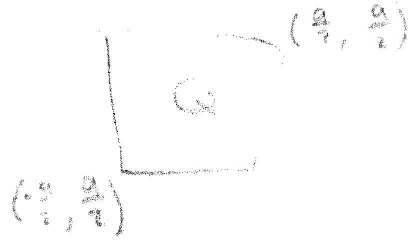
$$\int Q\,\delta(|\vec{r}|-R)\,d\tau = \int dq = Q$$

If a line segment l of charge Q along x-axis from $-l/2$ to $l/2$.

$$\int \rho(\vec{r})\,d\tau = Q \quad \text{and every where but } l \;\; \rho(\vec{r}) = 0.$$

$$\rho(\vec{r}) = \left[\Theta\left(x+\tfrac{l}{2}\right) - \Theta\left(x-\tfrac{l}{2}\right)\right]\frac{Q}{l}\,\delta(y)\,\delta(z).$$

$$\int \rho(\vec{r}) = \frac{Q}{l}\, l = Q$$



$$\rho = \left[\Theta\left(x+\tfrac{a}{2}\right) - \Theta\left(x-\tfrac{a}{2}\right)\right]\left[\Theta\left(y+\tfrac{a}{2}\right) - \Theta\left(y-\tfrac{a}{2}\right)\right]\delta(z)\,\frac{Q}{a^2}$$

DIVERGENCE OF A CURL IS EQUAL TO ZERO

1.49
1.55

§ 1.6 VECTOR FIELDS

$\vec{E}$: Electric FIELDS
$\vec{B}$: Magnetic FIELDS

} VECTOR FIELDS

SCALAR POTENTIAL:

If $\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$ where V is the scalar potential

WE CALL SUCH A FIELD IRROTATIONAL.

THEOREM: IRROTATIONAL OR CURLESS FIELDS

(a) $\nabla \times \vec{F} = 0$ every where

(b) $\int_a^b \vec{F} \cdot d\vec{l}$ independent of path

(c) $\oint \vec{F} \cdot d\vec{l} = 0$

(d) $\vec{F} = -\nabla V$ where V is scalar function

VECTOR POTENTIAL:

If $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$

THEOREM: DIVERGENCELESS FIELDS THEN

(a) $\nabla \cdot \vec{F} = 0$ every where

(b) $\int \vec{F} \cdot d\vec{a}$ independent of the surface for a fixed loop

(c) $\oint \vec{F} \cdot d\vec{a} = 0$

(d) $\vec{F} = \nabla \times \vec{A}$ where $\vec{A}$ is vector potential