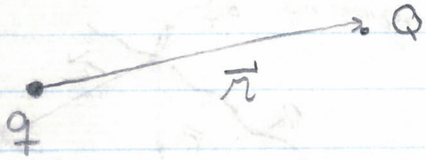
Chapter 2 Electrostatics

§2.1 The Electric Field

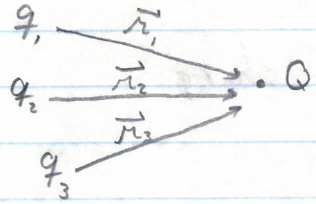
- Coulomb's Law: by experiment,

$$\boxed{\vec{F}(\text{on } Q) = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}}$$



where $\epsilon_0 = 8.89 \times 10^{-12}\ C^2/Nm^2$

- Multiple charge interaction



$$\vec{F} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \frac{q_3 Q}{\mathscr{r}_3^2}\hat{\mathscr{r}}_3\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1\hat{\mathscr{r}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\mathscr{r}}_2}{\mathscr{r}_2^2} + \frac{q_3\hat{\mathscr{r}}_3}{\mathscr{r}_3^2}\right)Q$$

$$= \vec{E}(\vec{r})Q$$

charges q_1, q_2, q_3 produce electric fields which produce force at Q's locality.

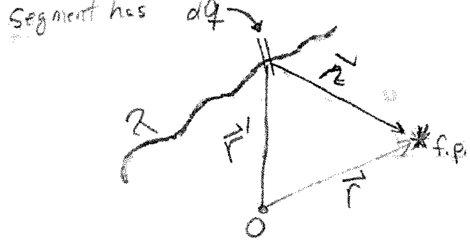
- In general we may say

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i\hat{\mathscr{r}}_i}{\mathscr{r}_i^2} \quad \text{and} \quad \vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i{}' \quad \begin{cases}\vec{r} = \text{field point} \\ \vec{r}_i{}' = \text{source point}\end{cases}$$

• Continuous Charge Distribution $\quad * =$ field point

$\lambda =$ linear charge density $= \frac{dq}{dl'}$

and $dq = \lambda dl'$



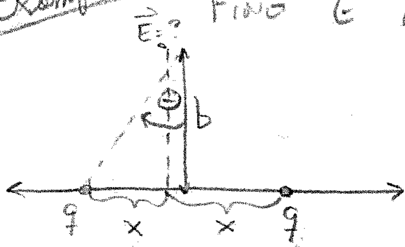
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dl'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

• $\vec{E}(\vec{r}) = \int d\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda dl'}{\mathscr{r}^2}\hat{\mathscr{r}}$ for a line of charge

• $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma da'}{\mathscr{r}^2}\hat{\mathscr{r}}$ for surface with charge

• $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau$ for a volume with charge in it.

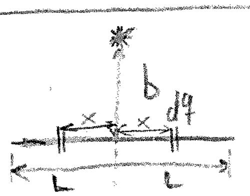
Example Find $\vec{E}$ due to 2 charges q



$\sum E_H = 0$ by symmetry

$E_y = E\cos\theta$ (from one charge)

$= \frac{1}{4\pi\epsilon_0}\frac{q}{(\sqrt{x^2+b^2})^2}\left(\frac{b}{\sqrt{x^2+b^2}}\right) \Rightarrow \vec{E}(\vec{r}) = \frac{1}{2\pi\epsilon_0}\frac{bq}{(x^2+b^2)^{3/2}}\,\hat{j}$



we have that $\vec{E}(\vec{r}) = \frac{1}{2\pi\epsilon_0}\frac{bq}{(x^2+b^2)^{3/2}}\,\hat{j}$

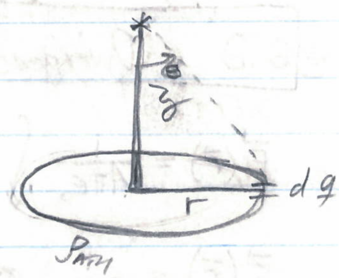
$$dE_y = \frac{1}{2\pi\epsilon_0}\frac{b\,dq}{(x^2+b^2)^{3/2}} = \frac{1}{2\pi\epsilon_0}\frac{b\lambda\, dx}{(x^2+b^2)^{3/2}}$$

$$E_y = \int dE_y = \frac{b\lambda}{2\pi\epsilon_0}\int_0^L \frac{dx}{(b^2+x^2)^{3/2}} = \frac{b\lambda}{2\pi\epsilon_0}\left(\frac{x}{b^2\sqrt{b^2+x^2}}\Bigg|_0^L\right) = \frac{\lambda}{2\pi\epsilon_0}\frac{L}{b\sqrt{b^2+L^2}}$$

as $b \gg L$ we have the function reduce to that of a point charge.

$E = \frac{\lambda}{2\pi\epsilon_0}\,\frac{\frac{q}{2}}{\quad}$

Example Ring of Charge Q of radius r

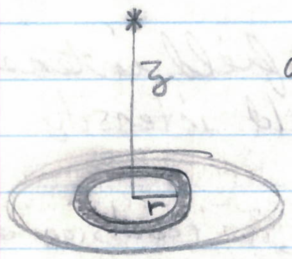


$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2+z^2}\cdot\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2+z^2}\frac{z}{(r^2+z^2)^{1/2}}$$

$$E_z = \int dE_z = \frac{1}{4\pi\epsilon_0}\frac{zQ}{(r^2+z^2)^{3/2}}$$ because $z, r,$ are constant over Path.

Ex/ Uniform disk (density σ, total charge Q, radius R.



$$dE_z = \frac{dq}{4\pi\epsilon_0}\cdot\frac{z}{(r^2+z^2)^{3/2}}$$ and $dq = \sigma(2\pi r\,dr)$

$$= \frac{\sigma(2\pi r\,dr\,z)}{4\pi\epsilon_0(r^2+z^2)^{3/2}} = dE_z$$

$$E_z = \frac{\sigma z}{2\pi\epsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} = \frac{\sigma z}{2\pi\epsilon_0}\left(-\frac{1}{(r^2+z^2)^{1/2}}\Bigg|_0^R\right) = \frac{\sigma z}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{(R^2+z^2)^{1/2}}\right)$$

$$E_z = \frac{Q}{2\epsilon_0\pi R^2}\left(1 - \frac{z}{(z^2+R^2)^{1/2}}\right)$$ if $R \gg z$ then $E = \frac{\sigma}{2\epsilon_0}$

but if $z \gg R$ then by binomial expansion $(1+x)^{-n} = 1 - nx + \frac{(-n)(-n-1)x^2}{2!} + \ldots$

$$\frac{1}{(R^2+z^2)^{1/2}} = \frac{1}{z(1+R^2/z^2)^{1/2}} \cong \frac{1}{z}\left(1 - \frac{1}{2}\frac{R^2}{z^2}\right)$$

$$\Rightarrow E_z = \frac{Q}{4\pi\epsilon_0 z^2}$$ (like a point charge)

$$\frac{d}{dx}(x\,\delta(x)) = 0 = \delta(x) + x\frac{d\delta(x)}{dx}$$

$$x\frac{d\delta(x)}{dx} = -\delta(x).$$

$F_g = G\frac{M\, m_{enc}}{r^2}$ $\quad \boxed{\Phi_g = 4\pi G}$

$F_e = \frac{1}{4\pi\epsilon_0}\frac{qQ}{R^2}$ $\quad \Phi_g = \int_S \vec{Z}\cdot d$

where

$Z = \frac{F_g}{M}$

§2.2 Divergence and Curl of Electric Field

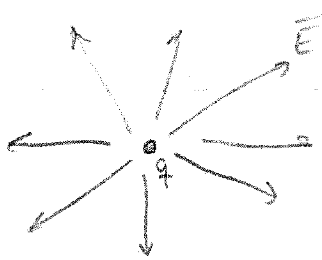
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl' \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da'$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau$$

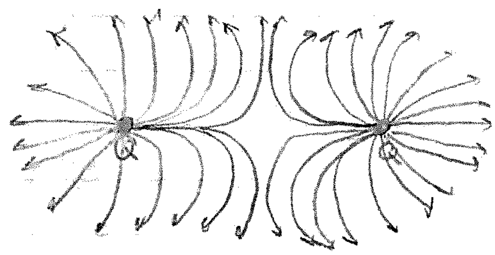
These integrals can be yucky, but Mr. Gauss has provided us some tr

$\vec{E}$ flows out of q



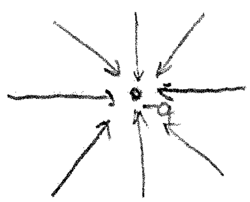
line density $\Leftrightarrow$ to electric field intensity

flux density $\Longleftrightarrow$ $\vec{E}$ field intensity



repulsion effect.

$\vec{E}$ flows into $-q$



We Define $\Phi_E = \int_S \vec{E}\cdot d\vec{a}$

$$\boxed{\text{Gauss' Law} \quad \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}}$$

this is a property of all inverse-square force law.

$$\oint \vec{E}\cdot d\vec{a} = \int_\tau (\nabla\cdot\vec{E})\,d\tau$$

$$Q_{enc} = \int \rho\, d\tau$$

$$\int(\nabla\cdot\vec{E})\,d\tau = \int\frac{\rho\, d\tau}{\epsilon_0} \quad \text{for any volume}$$

$$\therefore \boxed{\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$ guass law in differential form

1 of maxwells Equations.

2.9, 2.10

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} \frac{\rho(\vec{r}\,')}{\scriptstyle{r}^2}\hat{\scriptstyle{r}}\, d\tau'$$

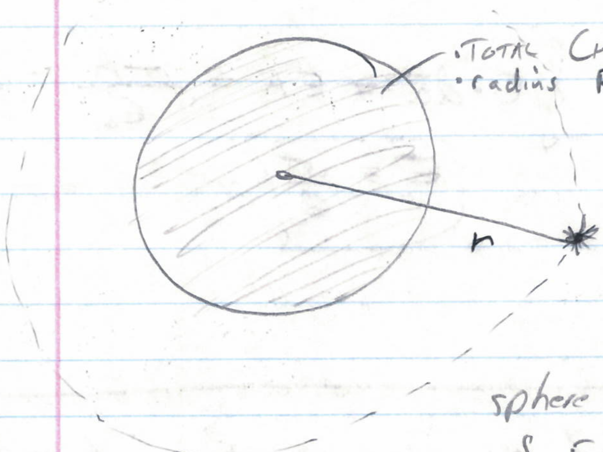
take divergence,

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\scriptstyle{r}^2}\underbrace{\left(\nabla \cdot \frac{\hat{\scriptstyle{r}}}{\scriptstyle{r}^2}\right)}_{4\pi\delta(\vec{\scriptstyle{r}}) = 4\pi\delta(\vec{r}-\vec{r}\,')} d\tau$$

$$= \frac{1}{4\pi\epsilon_0} 4\pi\rho(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

Application of Gauss' Law.

Inside and outside of a uniformly charged sphere



by symetry all $\vec{E}$ fields are going radially outward.

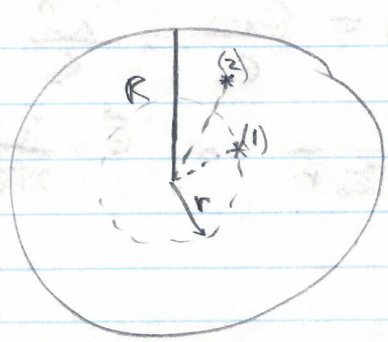
$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \oint E\, da$$

and sine E is the same around guassian sphere we have it's constant over all da

$$\oint E\, da = E\oint da = E(4\pi R^2) = \frac{Q_{enc}}{\epsilon_0}$$

Then we have $E = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{R^2}$ for outside sphere

Case 2: Inside Sphere



$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

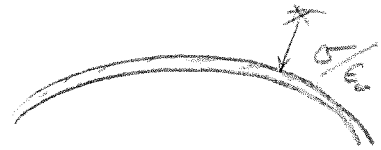
$$\int E\cdot da = 4\pi r^2 E \quad , \quad Q_{enc} = \rho \cdot V_{enc} \, , \quad \rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$4\pi r^2 E = \frac{Q}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi r^3\right) = \frac{Q r^2}{R^3} = 4\pi r^2 E$$

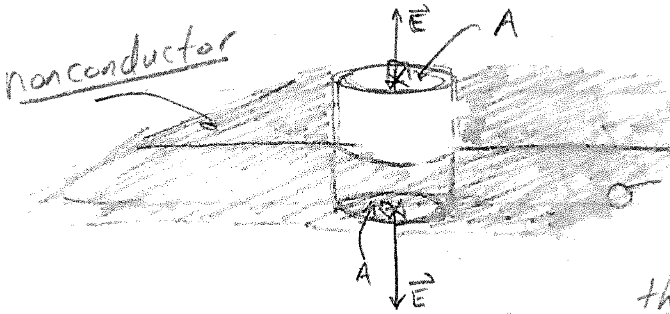
$$\boxed{E = \frac{Q r}{4\pi\epsilon_0 R^3}}$$

E depends linearly on r as you go into a sphere.

for Paul



E next to a thin surface, uniform charge density σ



E is vertical, or perpendicular to a plane by symetry.

construct guassian surface note that flux only cuts through tops of can since E vertical

$$\oint \vec{E}\cdot d\vec{a} = 2EA = \frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$2E = \frac{\sigma}{\epsilon_0} \implies E = \frac{\sigma}{2\epsilon_0}$$

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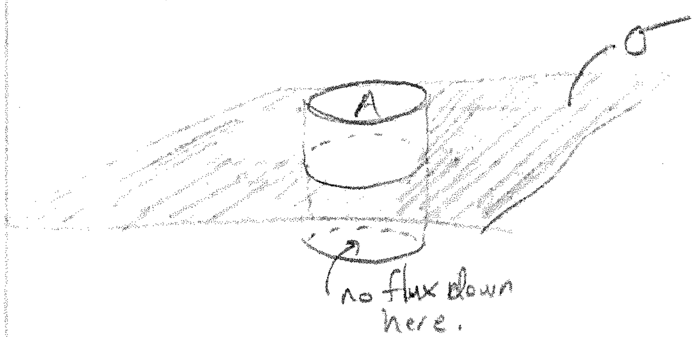
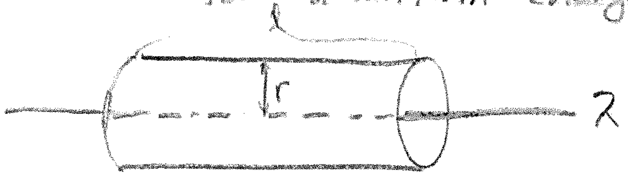
for conductor



$$\oint E\cdot d\vec{a} = EA = \frac{\sigma A}{\epsilon_0} = \frac{Q_{enc}}{\epsilon_0}$$

$$\boxed{E = \frac{\sigma}{\epsilon_0}}$$

E field for a uniform charge on long wire, note $\vec{E}$ points radially outward and is // to $d\vec{a}$

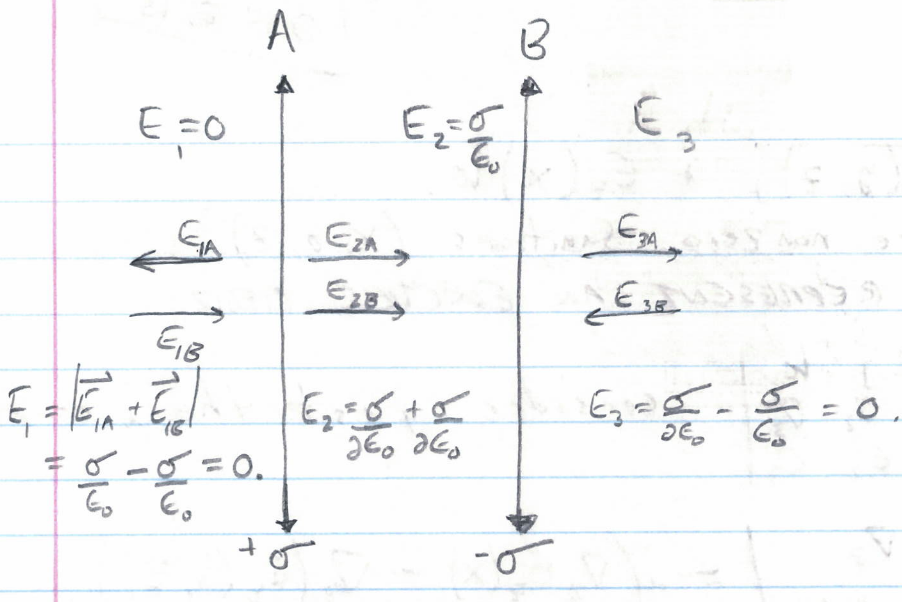


$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \int_{\text{no ends}} E\,da = E\int da = E(2\pi r l) = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

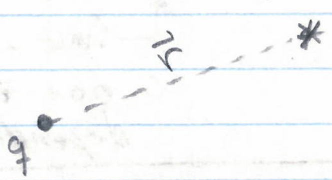
$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad \text{or} \quad \vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{r}}{r},$$

for paul

INFINITE PLANES OF UNIFORM CHARGE σ and $-\sigma$



The curl of $\vec{E}$



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q\,\hat{r}}{r^2}$$

$$\int_a^b \vec{E}\cdot d\vec{l}\,, \quad d\vec{l} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi.$$

$$\frac{q}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Big|_{r_a}^{r_b} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right).$$

THEN FOR A CLOSE LOOP

$\oint_P E\cdot d\vec{l} = 0$ which is an important property of $\vec{E}$ fields

by stokes theorem $\oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow \int_S (\nabla\times\vec{E})\cdot d\vec{a} = 0$

which infers that $\nabla\times\vec{E} = 0$

$\vec{E}$ is a curless, irrotational field.

for paul

$\boxed{\text{Hwk } 2.14, 2.15}$

Example

$$\vec{E} = E_x(x,y)\,i + E_y(y,z)\,j + E_z(x)\,k$$

E_x, E_y, E_z are non zero functions (x, y, z)

DOES THIS $\vec{E}$ REPRESENT AN ELECTRIC FIELD

$$\nabla \times \vec{E} = \begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ E_1 & E_2 & E_3 \end{vmatrix}$$ consider just the i component

$$\{\nabla \times \vec{E}\}_{(i)} = i \begin{vmatrix} \nabla_2 & \nabla_3 \\ E_y(y,z) & E_z(x) \end{vmatrix} = i\left(\nabla_2 E_z(x) - \nabla_3\left(E_y(y,z)\right)\right).$$

$$= i\left(0 - \frac{\partial}{\partial z} E_y(y,z)\right) \neq 0$$

since it has a non-zero z dependence.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots$$

$$\nabla \times \vec{E} = \nabla \times \vec{E}_1 + \nabla \times \vec{E}_2 + \nabla \times \vec{E}_3 + \ldots$$

for paul

$\boxed{\S 2.3 \text{ ELECTRIC POTENTIAL}}$

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l}.$$

$\vec{r}_0 \longleftarrow$ reference point.

- IRRELEVANCE OF ref. point when consider potential between two points,

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{r}_0}^{\vec{b}} \vec{E} \cdot d\vec{l} + \int_{\vec{r}_0}^{\vec{a}} \vec{E} \cdot d\vec{l} = \int_{\vec{b}}^{\vec{r}_0} \vec{E} \cdot d\vec{l} + \int_{\vec{r}_0}^{\vec{a}} \vec{E} \cdot d\vec{l} = \int_{\vec{b}}^{\vec{a}} \vec{E} \cdot d\vec{l}$$

$$\boxed{V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}}$$

but $$V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} \nabla V \cdot d\vec{l} = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l} \Longrightarrow \boxed{-\nabla V = \vec{E}}$$

this is from the fact $\nabla \times \vec{E} = 0 \Longrightarrow \vec{E} = \nabla T$ where T is some scalar function.

~~FRIDAY~~
MONDAY

- NOTE POTENTIAL $\neq$ POTENTIAL ENERGY
- ALSO NOTE $\vec{E}$ is A VECTOR WHILE V is a scalar since $\vec{E} = -\nabla V$ it is often easier to solve for V to find $\vec{E}$ or something.
- NOTE That potential is defined between two points $v(\vec{b}) - v(\vec{a}) - \int_{\vec{a}}^{b} \vec{E} \cdot d\vec{l}$ before we often spoke of potentials with a reference point of ∞, but there are cases where this doesn't work for instance the ∞ plane of charge has $|\vec{E}| = \dfrac{\sigma}{2\epsilon_0}$

$$V = -\int_{\infty}^{d} \frac{\sigma}{2\epsilon_0} dz = \frac{-d}{2\epsilon_0} + \infty = \infty.$$

So it's not wise to take ∞ as ref. pt. for this situation

We may only take ∞ as reference when we have a localized charge distribution

Superposition Principle

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots$$

$$\int \vec{E} \cdot d\vec{l} = \int \vec{E}_1 \cdot d\vec{l} + \int \vec{E}_2 \cdot dl + \int \vec{E}_3 \cdot dl + \dots$$

$$V = V_1 + V_2 + V_3 + \dots$$

Units

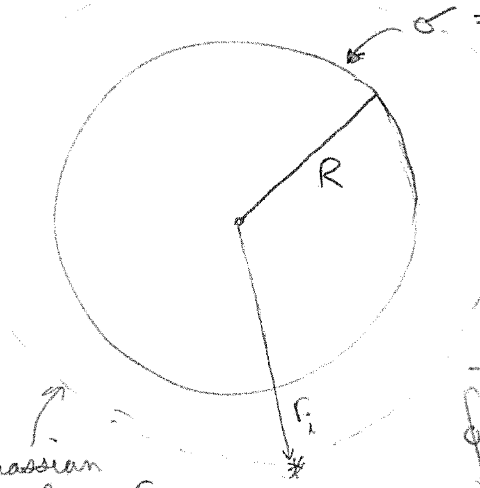
$\vec{F}$ in Newtons

$\vec{E}$ in Newtons/coulomb

V in Nm/C = Volt. Since $\{E \cdot dl\} = \frac{N}{C} m = \frac{Nm}{C} = \{\vec{E}\}$.

Example

$\sigma = \frac{Q}{4\pi R^2}$



quassian surface. S_i

If Def$^{\underline{n}}$ $V=0$ at ∞ find V at r ($r > R$ (i), $r < R$ (ii)).

$$\oint_{S_i} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = E \oint_{S_i} da = E\, 4\pi r_i^2 = \frac{Q}{\epsilon_0}$$

Then $E_i = \frac{Q}{4\pi\epsilon_0 r_i^2}\hat{r}$ for case i) $r > R$

$$\oint_{S_{ii}} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = 0 \Rightarrow \vec{E}_{ii} = 0$$

$$V_i = -\int_{+\infty}^{\vec{r}} \vec{E} \cdot d\vec{l} = \int_{\vec{r}}^{\infty} \vec{E} \cdot d\vec{l} = \int_r^{\infty} \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{-Q}{4\pi\epsilon_0 r}\Big|_r^{\infty} = \boxed{\frac{Q}{4\pi\epsilon_0 r} = V_{outs}}$$

$V_{ii} = \frac{Q}{4\pi\epsilon_0 R}$ this means there is no Δ in Voltage from outside shell inword with the sphere.

What if sphere has uniform ρ throughout sphere,

$$-\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{R} \vec{E}_{out} \, d\vec{l} - \int_{R}^{\vec{r}} \vec{E}_{in} \cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0 R} - \int_{R}^{\vec{r}} \vec{E}_{in} \cdot d\vec{l}$$

$$= \frac{Q}{4\pi\epsilon_0 R} - \int_{R}^{r} \frac{Q \, r \, dr}{4\pi\epsilon_0 R^3}$$

$$= \frac{Q}{4\pi\epsilon_0 R^3} \left[\frac{1}{2} \left(R^2 - r^2 \right) \right]$$

$$V_{inside} = \frac{Q}{4\pi\epsilon_0 R} + \frac{Q}{8\pi\epsilon_0 R^3} \left(R^2 - r^2 \right)$$

Voltage inside sphere (?)
Sorry got lost here.

WEDNESDAY

2.20 2.23

Poisson's Equation and Laplace's Equation

$$\oint \vec{E} \cdot d\vec{a} = \int_{v} (\nabla \cdot \vec{E})\, d\tau = \frac{Q_{enc}}{\epsilon_0}$$

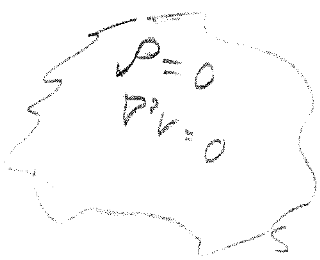
for very small v $\quad (\nabla \cdot \vec{E})\, d\tau = \frac{dQ_{enc}}{\epsilon_0} \Rightarrow (\nabla \cdot \vec{E}) = \frac{dQ_{enc}}{d\tau\, \epsilon_0} = \frac{\rho}{\epsilon_0}$.

$$\vec{E} = -\nabla V, \quad \nabla^2 V = -\rho/\epsilon_0 = \nabla \cdot (\nabla V); \text{ (Def}^{n} \text{ of } \nabla^2 V)$$

$$\boxed{\text{Poisson's Eq} \quad \nabla^2 V = \frac{-\rho}{\epsilon_0}}$$

In the region $\rho = 0$ we have laplace's equation

$\boxed{\nabla^2 V = 0 \quad \text{Laplace's Equation}}$, special case of poisson



$\nabla^2 V = 0$ is easy to solve charge outside S comes into play by the boundary condition on S. Solving Electrostatic problem becomes a boundary value problem

- Potential of a *localized* charge distribution.

set $V = 0$ at $r = \infty$ $\qquad q_1 \cdot \xrightarrow{\vec{r}_1}$

$$V = -\int \vec{E} \cdot d\vec{l} = -\int_{\infty}^{\vec{r}_1} E_r\, dr = \frac{-q_1}{4\pi\epsilon_0 r}\Big|_{\infty}^{r_1} = \frac{q_1}{4\pi\epsilon_0 r}$$

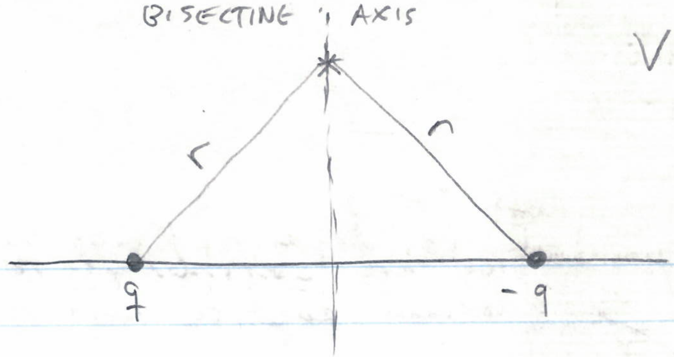
if $q_1, q_2, q_3, \ldots q_n$ then

$$V = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0 r_i}$$

if the charge is continuously distributed

$$V = \int \frac{dq}{4\pi\epsilon_0 r} = \int \frac{\rho(\vec{r}')\, d\tau'}{4\pi\epsilon_0 r} \text{ or } \int \frac{\sigma(\vec{r}')\, d\vec{a}}{4\pi\epsilon_0 r} \text{ or } \int \frac{\lambda(\vec{r}')\, dl}{4\pi\epsilon_0 r}$$

these can be used if $V = 0$ @ $r = \infty$ and the charge is localized! Be careful with these.

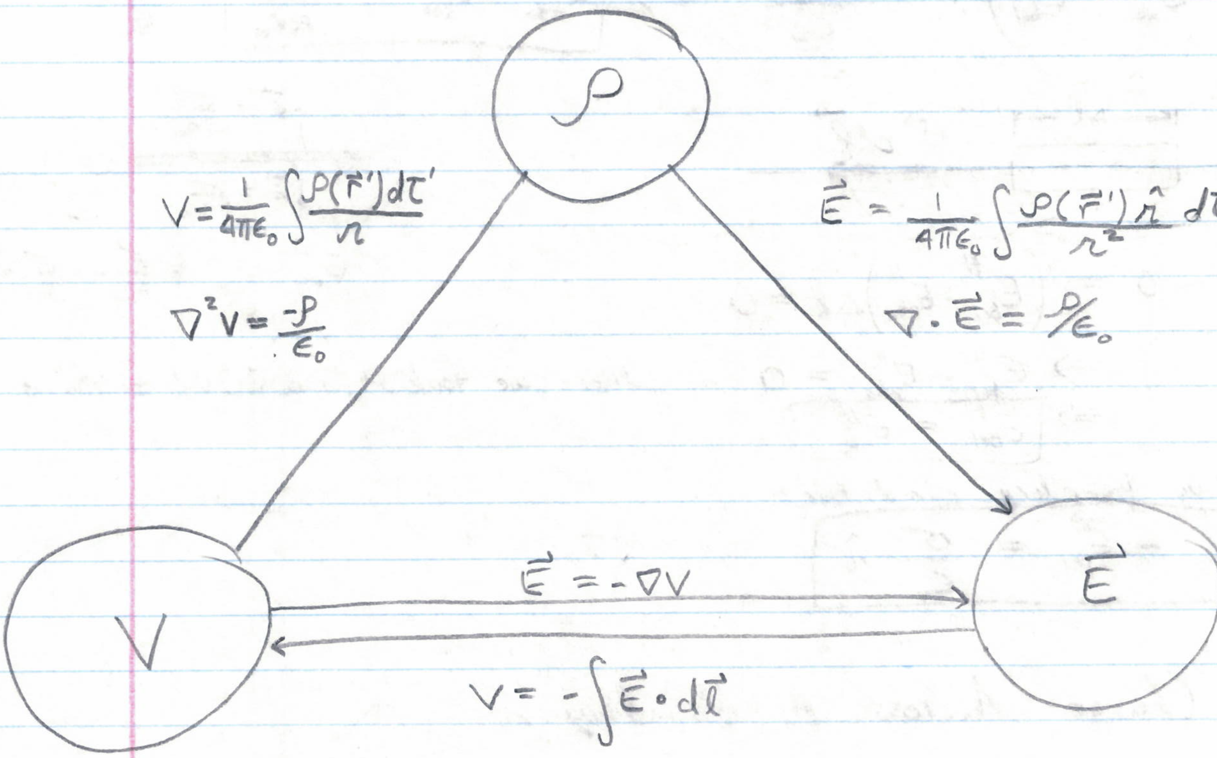


$$V = \frac{q}{4\pi\epsilon_0 r} \quad \frac{-q}{4\pi\epsilon_0 r} = 0.$$

$\nabla V = 0$?

<u>No</u> to find gradiant we must consider functional dependence.

Electrostatic Boundary Conditions



$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

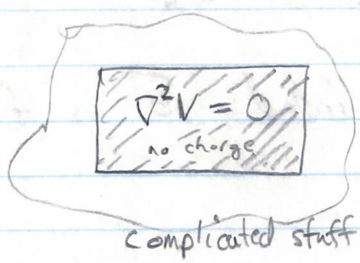
$$\nabla^2 V = \frac{-\rho}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{E} = -\nabla V$$

$$V = -\int \vec{E} \cdot d\vec{l}$$

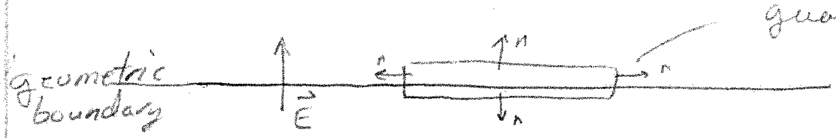


$\nabla^2 V = 0$ gives a solution with a large number of integration constants.
we use boundary conditions to specify the constants.

$\boxed{\text{Surface Boundary } \vec{E}}$

$\vec{E}$ is drawn vertically but we are considering general case

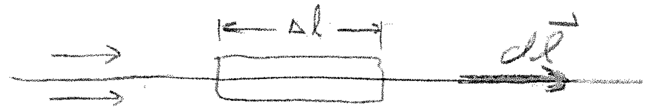
gaussian surface with area A.

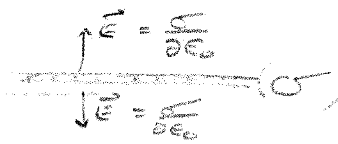


but side area $\to 0$
by construction

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{a} = \left(E^{\perp}_{abo} - E^{\perp}_{bel}\right) A = \frac{Q_{enc}}{\epsilon_0} \implies \boxed{E^{\perp}_{abo} - E^{\perp}_{bel} = \frac{Q_{enc}}{A\epsilon_0}} = \frac{\sigma}{\epsilon_0}$$





$$E^{\perp}_{abo} - E^{\perp}_{bel} = \frac{\sigma}{2\epsilon_0} - -\frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

$$\oint_{\mathcal{P}} \vec{E} \cdot d\vec{l} = 0 \implies \left(E^{\parallel}_{bel} - E^{\parallel}_{abo}\right)\Delta l = 0$$

$$\implies E^{\parallel}_{bel} - E^{\parallel}_{abo} = 0$$

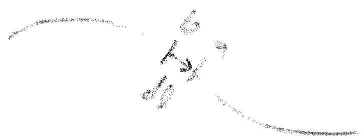
$$\implies \boxed{E^{\parallel}_{bel} = E^{\parallel}_{abo}}$$

where we took $\mathcal{P}$ in a CCW fashion

Combining two boundary conditions,

$$\boxed{\vec{E}_{abo} - \vec{E}_{bel} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

Potential Change Accross A Boundary



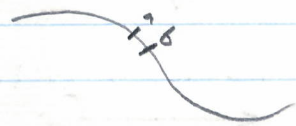
$$V = -\int_a^b \vec{E} \cdot d\vec{l} = 0 \text{ for}$$

a boundary provided $\vec{E}$ is finite

2.31, 2.32

Potential Change Across a Boundary

$$V = -\int_{\vec a}^{\vec b} \vec E \cdot d\vec l$$

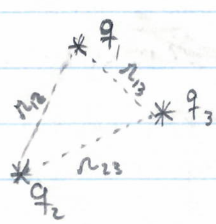


$\Delta V = V_{abo} - V_{bel} = -\int \vec E \cdot d\vec l = 0$ provided $\vec E$ is bounded.

§2.4 Work and Energy

$$W = \int_{\vec a}^{\vec b} \vec F \cdot d\vec l = \int_{\vec a}^{\vec b} -(q\vec E)\cdot d\vec l = q\left(V(\vec b) - V(\vec a)\right)$$

$V(\vec b) - V(\vec a) = W/q$ take $\vec a$ to be at ∞ then $W = q\,V(\vec r)$.



$$\Delta W_2 = q_2 V(\vec r_2) = q_2 \frac{q_1}{4\pi\epsilon_0 r_{12}}$$

$$\Delta W_3 = q_3 V(\vec r_3) = q_3\left(\frac{q_1}{4\pi\epsilon_0 r_{13}} + \frac{q_2}{4\pi\epsilon_0 r_{23}}\right)$$

$\Delta W_1 = 0$, trivially. and ΔW_i is work done on q_i to bring it from ∞ to $\vec r_i$

$$\text{Total Work} = \Delta W_1 + \Delta W_2 + \Delta W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

In general if we have n charges then

$$W_{TOTAL} = \sum_{i=1}^{n} \Delta W_i = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n}\sum_{i>j}^{n} \frac{q_i q_j}{r_{ij}} = \frac{\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}}}{2}$$

$$= \frac{1}{2}\sum_i q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum q_i V(\vec r_i)$$

for continuous charge distributions,

$$dW = \frac{1}{2}\,dq\,V(\vec r) = \frac{1}{2}\rho(\vec r)\,V(\vec r)\,d\tau \qquad , \quad \rho = \epsilon_0 \nabla\cdot\vec E$$

$$W = \frac{\epsilon_0}{2}\int(\nabla\cdot\vec E)V\,d\tau \qquad \text{but as } \nabla\cdot(V\vec E) = V\nabla\cdot\vec E + \vec E\cdot\nabla V = V\nabla\cdot\vec E - \vec E\cdot\vec E$$

$\vec E \to \frac{1}{r^2}$

$V \to \frac{1}{r}$

$$= \frac{\epsilon_0}{2}\int\nabla\cdot(V\vec E)\,d\tau - \frac{\epsilon_0}{2}\int\vec E\cdot(-\vec E)\,d\tau$$

$$= \frac{\epsilon_0}{2}\oint(V\vec E)\cdot d\vec a \quad \ldots$$ we start again monday

2.36, 2.38

$$W = \frac{1}{8\pi\epsilon_0}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{r_{ij}} \longleftarrow \text{for point charge distribution}$$

$$W = \frac{\epsilon_0}{2}\int_{\mathbb{R}} |E|^2 \, d\tau \longleftarrow \text{for continuous distribution}$$

if you try on point charge $W \to \infty$

$$|\vec{E}|^2 = \frac{q^2}{4\pi\epsilon_0 r^4} \quad \text{if } \int_{\mathbb{R}^3} \text{ it blows up at origin}$$

self energy is energy it would take to make a charge of infinite density

Conductors § 2.5

charge is free to move about conductor

(1) $\vec{E} = 0$ inside

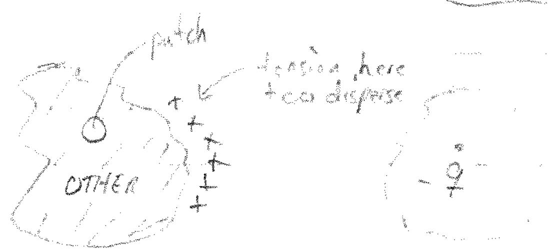
(2) $\rho = 0$ inside

(3) Any net charge must reside on surface

(4) A conductor is an equal potential

(5) $\vec{E}$ is perpendicular to the surface

induction



$\vec{E} = \sigma/\epsilon_0$

surface charge and force on a conductor

$\vec{E}_{abo}(\text{patch}) = \frac{\sigma}{2\epsilon_0}\hat{n}$ $\qquad \vec{E}_{abo}(\text{total}) = \sigma/\epsilon_0 \, \hat{n} \Rightarrow E_{other} = \frac{\sigma}{2\epsilon_0}\hat{n}$

$\vec{E}_{bel}(\text{patch}) = -\sigma/2\epsilon_0 \, \hat{n}$ $\qquad \vec{E}_{bel}(\text{total}) = 0.$

Force on patch

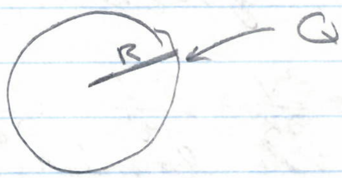
$= q\vec{E}_{other} = \sigma \cdot A \cdot \frac{\sigma}{2\epsilon_0}\hat{n} \Rightarrow \vec{P} = \frac{\vec{F}}{A} = \frac{\sigma^2}{2\epsilon_0}\hat{n}$

test next

CAPACITORS

— MAGIC SPHERE CAPACITOR —

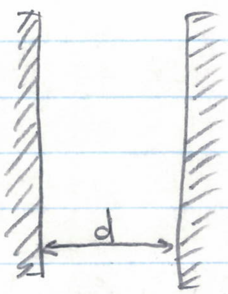
$$C = \frac{Q}{V}$$



$$V = \frac{Q}{4\pi\epsilon_0 R}$$ is potential for conducting sphere

$$C = \frac{Q}{V} = \underline{4\pi\epsilon_0 R = C}$$

— PARALLEL PLATE —



$$V = Ed = \frac{\sigma}{\epsilon_0} d = \frac{Qd}{\epsilon_0 A}$$

$$\frac{Q}{V} = \boxed{\frac{\epsilon_0 A}{d} = C}$$

A = Area of Plate.

— Energy ; $dW = V\,dq = \frac{q}{c}\,dq$

$$W = \int dW = \int_0^Q \frac{q}{c}\,dq = \boxed{\frac{1}{2c}Q^2 = \frac{1}{2}cV^2 = \frac{1}{2}QV}$$

3.2, 3.3

Chapter 3 – Special Techniques

§3.1 Laplace's Eq and Mean-Value Theorem

We may know ρ in some region but not in all space then we can try to solve

$$\nabla^2 V = \frac{1}{\epsilon_0}\rho$$

especially if $\rho = 0 \Longrightarrow \nabla^2 V = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

in those cases we need to know the boundary conditions so that we may choose correct sol^{n} out of a family of sol^{n}'s

- Laplace Eq in one dimension

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$V = mx + b \qquad \frac{\partial V}{\partial x} = m \Rightarrow V(x+a) - V(x) = V(x) - V(x-a)$

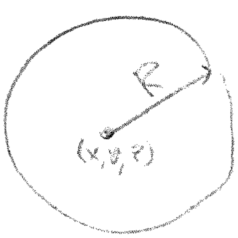
$\Rightarrow V(x) = \frac{1}{2}\left(V(x+a) + V(x-a)\right)$

$\Rightarrow V(x)$ can have no max or min

- 3 & 2 D // Laplace Equations

$2D: \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad 3D: \nabla_x^2(V) + \nabla_y^2(V) + \nabla_z^2(V)$

A mean value theorem for potential in 3-D space



if no charge inside spherical in space

then $V(x,y,z) = \frac{1}{4\pi R^2}\oint V\,dA = V_{AVG}$ on sphere.

$\Rightarrow$ no max or min V in space (boundary)