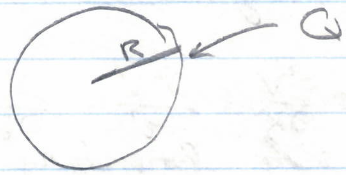
test next

CAPACITORS

— MAGIC SPHERE CAPACITOR —

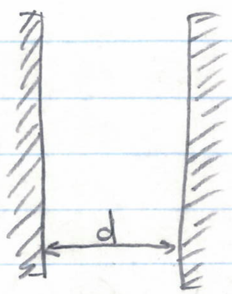
$$C = \frac{Q}{V}$$



$$V = \frac{Q}{4\pi\epsilon_0 R}$$ is potential for conducting sphere

$$C = \frac{Q}{V} = \underline{4\pi\epsilon_0 R = C}$$

— PARALLEL PLATE —



$$V = Ed = \frac{\sigma}{\epsilon_0} d = \frac{Qd}{\epsilon_0 A}$$

$$\frac{Q}{V} = \boxed{\frac{\epsilon_0 A}{d} = C}$$

A = Area of Plate.

— Energy ; $dW = V\,dq = \frac{q}{c}\,dq$

$$W = \int dW = \int_0^Q \frac{q}{c}\,dq = \boxed{\frac{1}{2c}Q^2 = \frac{1}{2}cV^2 = \frac{1}{2}QV}$$

3.2, 3.3

Chapter 3 – SPECIAL TECHNIQUES

§3.1 Laplace's Eq and Mean-Value Theorem

We may know ρ in some region but not in all space then we can try to solve

$$\nabla^2 V = \frac{1}{\epsilon_0}\rho$$

especially if $\rho = 0 \implies \nabla^2 V = 0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2}$

in these cases we need to know the boundary conditions so that we may choose correct sol$^{\underline{n}}$ out of a family of sol$^{\underline{n}}$'s

- LAPLACE Eq in ONE DIMENSION

$$\frac{\partial^2 V}{\partial x^2} = 0$$

$V = mx + b \qquad \frac{\partial V}{\partial x} = m \implies V(x+a) - V(x) = V(x) - V(x-a)$

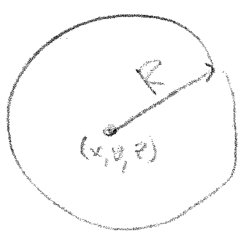
$\implies V(x) = \frac{1}{2}\left(V(x+a) + V(x-a)\right)$

$\implies V(x)$ can have no max or min

- 3&2D LAPLACE EQUATIONS

2D: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad$ 3D: $\nabla_x^2(V) + \nabla_y^2(V) + \nabla_z^2(V)$

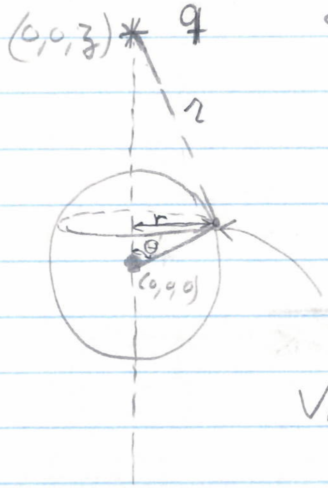
A mean value theorem for potential in 3-D space



if no charge inside spherical surface then $V(x,y,z) = \frac{1}{4\pi R^2}\oint V\,dA = V_{AVG}$ on sphere.

$\implies$ no max or min V in sphere (imaginary)

Let us prove the 3D/2D Mean Value observation for pt. charge



So q will produce $V_{surface}$. We will show that $\int V_{surface}\, dA = V_{AVG} = V_m$.

$$V_{AVG} = \frac{1}{4\pi R^2}\int V dA = \frac{1}{4\pi R^2}\oint \frac{q\, da}{4\pi\epsilon_0 r}$$

$$da = 2\pi r (R\, d\theta) = 2\pi \sin\theta\, R\, d\theta$$

$C^2 = A^2 + B^2 - 2AB\cos\theta$

$r^2 = z^2 + R^2 + 2zR\cos\theta$

$r = \sqrt{z^2 + R^2 - 2AB\cos\theta}$

$$V_{AVG} = \frac{1}{4\pi R^2}\oint \frac{q\, 2\pi R \sin\theta\, d\theta}{4\pi\epsilon_0 (z^2 + R^2 - 2zR\cos\theta)^{1/2}}$$

$$= \frac{1}{2}\frac{q}{4\pi\epsilon_0}\frac{(z^2+R^2-2zR\cos\theta)^{1/2}}{zR}\Bigg|_0^{\pi}$$

$$= \frac{1}{2}\frac{q}{4\pi\epsilon_0}\left(\frac{(z^2+R^2+2zR)^{1/2}}{zR} - \frac{(z^2+R^2-2zR)^{1/2}}{zR}\right)$$

$$= \frac{1}{2}\frac{q}{4\pi\epsilon_0 zR}\left[R + z - \begin{cases}(z-R) & \text{if } z>R\\ (R-z) & \text{if } z<R\end{cases}\right]$$

$$= \frac{1}{2}\frac{q}{4\pi\epsilon_0 zR}(2R) = \boxed{\frac{q}{4\pi\epsilon_0 z} = V_{AVE} = V_{middle}}$$

$\left(V_{middle} = \frac{1}{4\pi\epsilon_0}\frac{q}{z}\right.$ AS ALLWAYS for point charge distance z

if we move * with q to inside mathmaticall sphere the math stays same

$V_{AVE} = \frac{q}{4\pi\epsilon_0 R}$ nomatter where q is inside

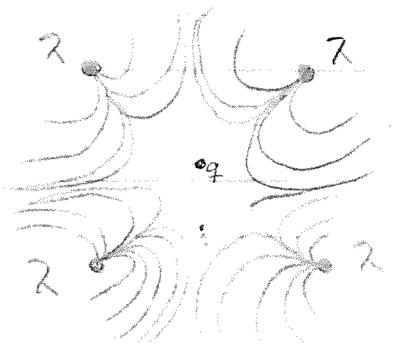
$$V_{AVE}(\text{on surface}) = \frac{1}{4\pi R^2}\oint V^{(out)}\,da = V(x,y,z) = V_c^{(out)}$$

<u>*If q is inside then*</u>

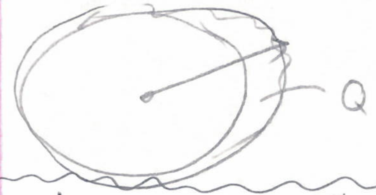
$$V_{AVE}^{(q)} = \frac{1}{4\pi R^2}\oint V^{(q)}\cdot da = \frac{q}{4\pi\epsilon_0 R}$$

<u>*In general*</u>

$$V_{AVE}(\text{on surface}) = V_c^{(out)} + \frac{q_{in}}{4\pi\epsilon_0 R}$$



3.4
3.5



constant bound. $\Longrightarrow$ const. V in space.

$V = \frac{Q}{4\pi\epsilon_0 R}$ if no q inside

Uniqueness Theorem

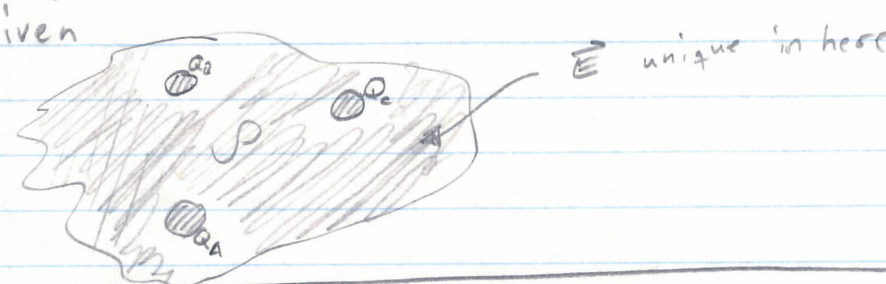
- We can obtain $\vec{E}(\vec{r})$ by solving the laplace's equation
  $$\nabla^2 V = 0$$
  but is our solution unique, What if $\nabla^2 V_1 = 0 = \nabla^2 V_2$ which is correct?

FIRST UNIQUENESS Theorem: The solution to Laplaces Eq. in some volume τ is unique if V is specified on the surface of τ.

Pf/ Let $\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$ where V_1, V_2 are both subject to the same boundary conditions on τ.
$V_1 - V_2 = 0$ then Let $V_3 = V_1 - V_2$ on boundary
then $\nabla^2 V_3 = 0$ and $V_3 = 0$ on boundary.
So we may conclude $V_3 = 0$ on all of τ.
as if it has any value at all it would have a largest number and thus a maximum

SECOND UNIQUENESS Theorem

In a volume τ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given



Pf/ Let $\vec{E}_1, \vec{E}_2$ both give same ρ, Q_A, Q_B

$$\nabla \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{E}_2 = \rho/\epsilon_0 \qquad \vec{E}_3 = \vec{E}_1 - \vec{E}_2 \qquad \nabla \cdot \vec{E}_3 = 0$$

$$\oint (\vec{E}_1 - \vec{E}_2) \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = Q_A - Q_A = 0 \Rightarrow \oint \vec{E}_3 \cdot d\vec{a} = 0$$

div. theorem

$$\nabla \cdot (V_3 \vec{E}_3) = V_3 (\nabla \cdot \vec{E}_3) + \vec{E}_3 \cdot (\nabla V) = 0 - E_3^2 \Rightarrow \int_\tau \nabla \cdot (V_3 \vec{E}_3) d\tau = -\int_\tau + E_3^2 = \sum_i \oint V_3 \vec{E}_3 \cdot d\vec{a}_i \ (*)$$

$$(*) = 0 \Rightarrow 0 = -\int E_3^2 \Rightarrow \vec{E}_3 = 0 \text{ everywhere } \Rightarrow \vec{E}_2 = \vec{E}_1$$

What is the energy of the system

$$W = \int_{\infty}^{d} (-\vec{F}) \cdot d\vec{l} = \int_{\infty}^{d} \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2z)^2} dz$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4z}\Big|_{\infty}^{d} = \boxed{-\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d}}$$

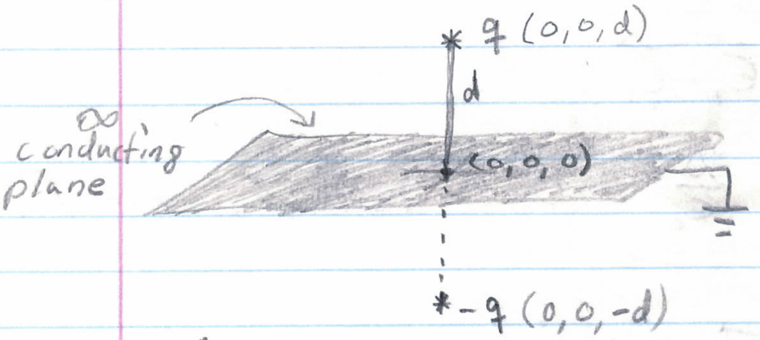
$$W = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$$

1/2 because $\frac{1}{2}$ the space is not real.

3.6, 3.9

$\frac{\partial V}{\partial n}$ = derivative of potential along the normal.

§ 3.2 METHOD OF Images



We know V = 0 on conducting plane and we care not what is below it.

Sol$^{\underline{n}}$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2)^{1/2}}\right] = V(x,y,z).$$

$\nabla^2 V = \frac{-\rho}{\epsilon_0}$ all we care about is region of interest and it's boundarys

we try to produce a physical situation

- What is surface charge? We know $V(x,y,z)$ near surface and $\nabla V = -\vec{E}$ and $\vec{E} = \frac{\sigma}{\epsilon_0}$ near a conductor

$E_\perp = E_z = -\frac{\partial V}{\partial z}$

$$\frac{\partial V}{\partial z} = \frac{-1}{4\pi\epsilon_0}\ \frac{q(z-d)}{(x^2+y^2+z-d)^{3/2}} - \frac{q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}$$

at surface $z=0$, $\frac{\partial V}{\partial z} = \frac{1}{2\pi\epsilon_0}\frac{qd}{(x^2+y^2+d^2)^{3/2}} = -E_z = -\frac{\sigma}{\epsilon_0}$

- What is the total charge induced

$$\int \sigma\, dx\,dy = \int \frac{-1}{2\pi}\frac{qd\ r\,dr\,d\theta}{(r^2+d^2)^{3/2}} = -\int \frac{qd\,r\,dr}{(r^2+d^2)^{3/2}} = \frac{qd}{(r^2+d^2)^{1/2}}\Bigg|_0^\infty$$

$$= -\frac{qd}{(d^2)^{1/2}} = -q.$$

- What is $\vec{F}$ on $q(*)$?

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2}\right]$$

$$\vec{F} = q\vec{E}(0,0,d) = \frac{q}{4\pi\epsilon_0}\left(-\nabla\frac{-q}{(x^2+y^2+(z+d)^2)^{1/2}}\right) = \frac{-q^2}{4\pi\epsilon_0}\left(\frac{x\hat{i}+y\hat{j}+(z+d)\hat{k}}{(x^2+y^2+(z+d)^2)^{3/2}}\right)\Bigg|_{0,0,d}$$

$$= \frac{-q^2\,2d\hat{k}}{4\pi\epsilon_0\,(2d)^3} = \frac{-q^2\hat{k}}{4\pi\epsilon_0(2d)^2}$$ = force of interaction between real and imagined charge.

PROBLEM 3.4 his Sol^{n}

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Assume $\vec{E}_1, \vec{E}_2$ Def^{n} $\vec{E}_3 \equiv \vec{E}_2 - \vec{E}_1$

$$\nabla \cdot \vec{E}_3 = \nabla(\vec{E}_2 - \vec{E}_1) = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0$$ As $\vec{E}_1, \vec{E}_2$ share ρ.

Let $\vec{E}_3 = -\nabla V_3$

$$\nabla \cdot (V_3 \vec{E}_3) = V_3 \nabla \cdot \vec{E}_3 + \vec{E}_3 \cdot \underbrace{\nabla V_3}_{-\vec{E}_3}$$

$$\int (\nabla \cdot V_3 \vec{E}_3) d\tau = \oint V_3 \vec{E}_3 \cdot d\vec{a} = -\oint V_3 \frac{\partial V_3}{\partial n} da = \int |\vec{E}_3|^2 d\tau$$

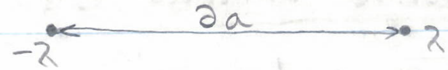
As $V_1 = V_2$ we would know $V_3 = 0$

the $-\oint V_3 \frac{\partial V_3}{\partial n} da = 0 = \int |\vec{E}_3| d\tau \Rightarrow \vec{E}_3 = 0$ as we are [illegible] $\int$ positive $\vec{E}$.

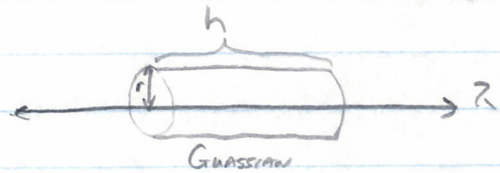
if $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ ($\frac{\partial V}{\partial n}$ is given) then

the $-\oint V_3 \underbrace{\frac{\partial V_3}{\partial n}}_{0} da = 0$ and so $\int |\vec{E}_3| d\tau = 0 \Rightarrow \vec{E}_3 = 0$

Infinitely Long Wires with Linear Charge Density λ.



(a) Find Potential $\longleftarrow$ 1st Find $\vec{E}$
(b) Show Equipotential Surface

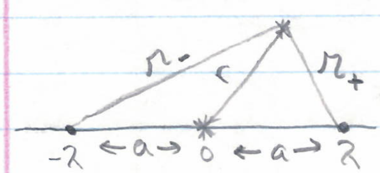


$$\Phi_E = EA = E(2\pi r \cdot h) = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda h}{\epsilon_0}$$

$$E = \frac{\lambda h}{\epsilon_0 2\pi r h} = \frac{\lambda}{2\pi\epsilon_0 r} = E$$

$$V = -\int \vec{E}\cdot d\vec{l} = -\int_a^r \frac{\lambda}{2\pi\epsilon_0 r}\underbrace{\hat{r}\cdot d\vec{l}}_{dr} = \text{potential at arbitrary } r.$$

$$= \frac{-\lambda}{2\pi\epsilon_0}\ln r\Big|_a^r = \frac{-\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{a}\right)$$



$$V = \frac{-\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_+}{a}\right) + \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_-}{a}\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_-}{r_+}\right)$$

For constant $V \Rightarrow \left(\frac{r_-}{r_+}\right)^2 = c \Rightarrow \frac{r^2+a^2+2ar\cos\theta}{r^2+a^2-2ar\cos\theta} = C$

$(c-1)(r^2+a^2) = (c+1)2ar\cos\theta$, Lets Define $d = \frac{c+1}{c-1}$

$r^2+a^2 = 2ad(r\cos\theta)$

$x^2+y^2+a^2 = 2adx$

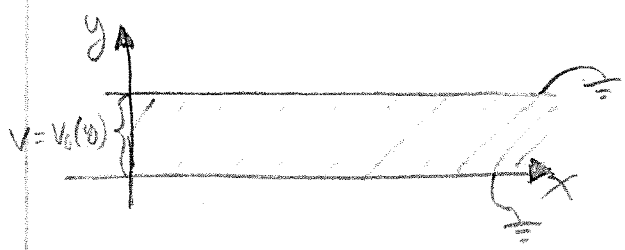
$x^2-2adx+y^2 = -a^2$

$x^2-2adx+a^2d^2+y^2 = a^2d^2-a^2$

$(x-ad)^2+y^2 = a^2(d^2-1) = \frac{4ca^2}{c-1}$

AVERAGE TEST 2 = 67 %

3.3 SEPERATION OF VARIABLES, SOLVING LAPLACES Eq.



2 ∞ conducting planes and another plane along y axis with $V_0(y)$ given as $V_0(y)$.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = X(x)\,Y(y) \Rightarrow Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{k^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-k^2} = 0,$$

So $k^2 - k^2 = C$ where we claim $\frac{X''}{X} = k^2$ and $\frac{Y''}{Y} = -k^2$

$$\frac{\partial X^2}{\partial x^2} = k^2 X \Rightarrow X = Ae^{kx} + Be^{-kx}$$

$$\frac{\partial Y^2}{\partial y^2} = -k^2 Y \Rightarrow Y = C\cos(ky) + D\sin(ky)$$

from $V(\infty, y) = \text{finite} \Rightarrow A = 0$. So our product sol^{n} is just

$$V(x,y) = e^{-kx}(C\sin(ky) + D\cos(ky))$$

$$V(x,0) = 0 \Rightarrow V(x,0) = e^{-kx} D = 0 \Rightarrow D = 0$$

$$V(x,a) = 0 \Rightarrow V(x,a) = e^{-kx}(C\sin(ka)) = 0 \Rightarrow \sin ka = 0$$

$$\Rightarrow ka = n\pi \quad \text{so} \quad k = \frac{n\pi}{a}$$

$$V(x,y) = \sum_n C_n e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

now choose C_n to satisfy verticle bound. conditions

$$V(0,y) = \sum_n C_n \sin\left(\frac{n\pi}{a}\right)y = V_0(y) \leftarrow \text{some given function}$$

$$\sin\left(\frac{m\pi}{a}\right)y\, V(0,y) = \sum_n C_n \sin\left(\frac{n\pi}{a}\right)y \sin\left(\frac{m\pi}{a}\right)y$$

$$\int_0^a \sin\frac{m\pi}{a}y\left(\sum_n C_n \sin\left(\frac{n\pi}{a}y\right)\right)dy = \int_0^a V_0(y)\sin\left(\frac{m\pi}{a}y\right)dy$$

3.12, 3.14

$$\int_0^a \sin\left(\frac{m\pi}{a}y\right) \sum_n c_n \sin\left(\frac{n\pi}{a}y\right) dy = \int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy$$

$$\sin\theta_1 \sin\theta_2 = \frac{1}{2}\left(\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)\right)$$

$$\int_0^a \left[\cos\left(\frac{m-n}{a}\right)\pi y - \cos\left(\frac{m+n}{a}\right)\pi y\right] dy = \frac{a}{(m-n)\pi} \sin\frac{(m-n)\pi y}{a} + \frac{a}{(m+n)\pi} \sin\left(\frac{(m+n)\pi y}{a}\right)\Bigg|_0^a$$

only when $m = n$ does this integral give non-zero relsult.

$$\frac{c_m}{2} a = \int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy.$$

$$c_m = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy.$$ exchange m for n and we have c_n.

$$\boxed{V(x,y) = \sum_n c_n e^{-\left(\frac{n\pi x}{a}\right)} \sin\left(\frac{n\pi}{a}y\right), \quad c_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy}$$

$\nabla^2 V = 0$ with Legendre Polynomial

- $\nabla^2 V(r,\theta,\phi) = 0$

  If $V(r,\theta,\phi) = V(r,\theta)$

  then $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}} P_l \cos\theta\right)$ if spherical symetry exists

  Space with no charge

  $P_l(\cos\theta)$ is the legendre P.

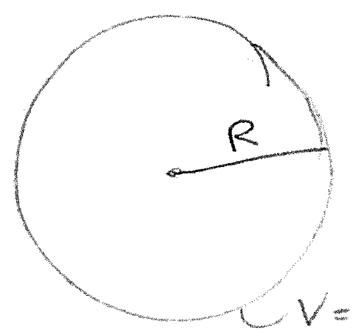
  $P_0(x) = 1$
  $P_1(x) = x$
  $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

$$\int_{-1}^{1} P_l(x)\, P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0 & \text{if } l = l \\ \frac{2}{2l+1} & \text{if } l = l \end{cases}$$

$$= \delta_{ll'}$$

- Example/ $V_{out} = ?$

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l/r^{l+1}\right)P_l(\cos\theta)$$



- **RODRIGUES FORMULA**

$$\boxed{P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l}$$

back to Ex) as $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l/r^{l+1}\right) P_l(\cos\theta)$

a $V(\infty,\theta)$ = finite then $A_l = 0\ \forall l$. as $r \to \infty$.

$V(R,\theta) = \sum_l \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$ applying the boundary condition

multiplying by $\int_0^\pi P_{l'}(\cos\theta)\sin\theta\, d\theta$.

$$\sum_{l=0}^{\infty}\int_0^{\pi} P_{l'}(\cos\theta)\frac{B_l}{R^{l+1}} P_l(\cos\theta)\sin\theta\, d\theta = \int_0^{\pi} P_{l'}(\cos\theta)\sin\theta\, V_0(\theta)\, d\theta$$

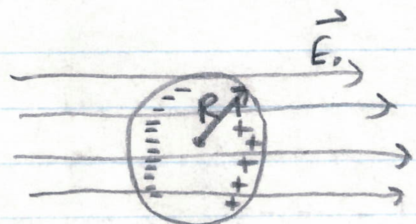
$\boxed{3.16,\ 3.18}$

$$\int_0^\pi P_{\ell'}(\cos\theta) V_0(\theta)\sin\theta\, d\theta = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} \frac{2}{\ell+1}\delta_{\ell\ell'} = \frac{B_{\ell'}}{R^{\ell'+1}} \frac{2}{2\ell'+1}$$

$$B_{\ell'} = \left(\frac{2\ell'+1}{2}\right) R^{\ell'+1} \int_0^\pi P_{\ell'}(\cos\theta)\, V_0(\theta)\sin\theta\, d\theta$$

change $\ell \to \ell'$ then we have A_ℓ, B_ℓ so we have $V(r,\theta)$.

Ex//



uniform electric field is applied to a nuetral metal sphere, find $\sigma(\theta) = ?$

we cant claim $V(\infty,\theta) = \emptyset$ but we may state

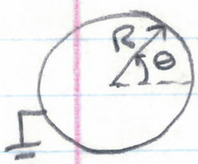
$$\nabla V(\infty,\theta) = -E_0(\infty,\theta) = -E_0$$

$$\nabla V(\vec{r} \to \text{large}) = -E_0 \hat{z}$$

$$\text{if } V = -E_0 z \Rightarrow -\nabla V = E_0 \hat{z}$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell / r^{\ell+1}\right) P_\ell(\cos\theta) \underset{r\to\infty}{=\!=} \sum_{\ell=0}^{\infty} A_\ell r^\ell = -E_0 r\cos\theta$$

$$\Rightarrow A_\ell = \left\{\begin{matrix} -E_0 & \text{for } \ell = 1 \\ 0 & \text{for } \ell \neq 1 \end{matrix}\right\}$$



$V(R,\theta) = 0$ for any θ, remember $A_\ell = -\delta_{1\ell} E_0$

$$V(R,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell R^\ell + B_\ell / R^{\ell+1}\right) P_\ell(\cos\theta) = 0$$

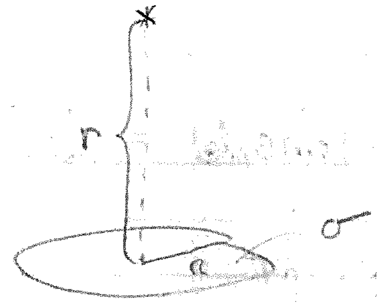
$$\left.\begin{matrix} \Rightarrow \dfrac{B_\ell}{R^{\ell+1}} = -A_\ell R^\ell \\ V(r,\theta) = -E_0\left(r - \dfrac{R^3}{r^2}\right)\cos\theta \end{matrix}\right\} \Rightarrow \quad B_1 = -A_1 R^3$$

$$\sigma = \epsilon_0 \vec{E}_n = \epsilon_0\left(\frac{-\partial}{\partial r} V\right) = \epsilon_0 E_0 \frac{\partial}{\partial r}\left(r - \frac{R^3}{r^2}\right)\cos\theta = \epsilon_0 E_0\left(1 + \left.\frac{2R^3}{r^3}\right|_{r=R} \cos\theta\right)$$

$$\sigma = 3\epsilon_0 E \cos\theta$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)(n-2)}{3!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$V = \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$



$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad (*)$$

$$\text{or} \quad \sum_{\ell=0}^{\infty} B_\ell / r^{\ell+1} P_\ell(\cos\theta) \quad (*)$$

if $r \to \infty$ then $r^\ell \to \infty$
if $r \to 0$ then $1/r^{\ell+1} \to \infty$
neither we want

$$V(r,0,\phi) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$$\theta = 0 \Rightarrow \sum_\ell A_\ell r^\ell$$

$$\Rightarrow \sum_\ell B_\ell r^{-\ell-1}$$

If $r > R$ then $\sqrt{r^2+R^2} = r\left(1+\frac{R^2}{r^2}\right)^{1/2}$ using bin. exp.

$$= r\left[1 + \frac{1}{2}\frac{R^2}{r^2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{R^4}{r^4} + \dots\right]$$

$$V(R,0,\phi) = \sum^{\infty} B_\ell r^{-(\ell+1)} = \frac{\sigma}{2\epsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$$\frac{2\epsilon_0}{\sigma}\sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} = \frac{1}{2}\frac{R^2}{r} - \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{2!}\frac{R^4}{r^3} + \dots$$

$$B_0 = \frac{\sigma}{2\epsilon_0}\frac{R^2}{2} \quad B_1 = 0 \quad B_2 = \frac{-\sigma}{2\epsilon_0}\frac{R^4}{8}, \quad B_3 = 0 \dots$$

If $r < R$ $\sqrt{r^2+R^2} = R\left[1 + \frac{1}{2}\frac{r^2}{R^2} - \frac{1}{8}\frac{r^4}{R^4} + \dots\right]$

$$\frac{2\epsilon_0}{\sigma}\sum_{\ell=0}^{\infty} A_\ell r^\ell = R - r + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \dots$$

$$A_0 = \frac{\sigma}{2\epsilon_0}R \quad A_1 = \frac{-\sigma}{2\epsilon_0} \quad A_2 = \frac{\sigma}{2\epsilon_0}\frac{1}{2R} + \dots$$

$$V(r,\theta) = \frac{\sigma}{2\epsilon_0}\left[R P_0 - r P_1 + \frac{1}{2}\frac{r^2}{R}P_2 - \frac{1}{8}\frac{r^4}{R^3}P_4 + \dots\right]$$

$\boxed{3.20,\ 3.24}$

Cylindrical Coordinates

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = \nabla^2 V = 0$$

Let $V = \mathcal{S}(s)\,\Phi(\phi)$

$$\frac{s}{\mathcal{S}}\frac{\partial}{\partial s}\left(s\frac{\partial \mathcal{S}}{\partial s}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m^2 \Phi \Rightarrow \Phi = A\cos m\phi + B\sin\phi$$

but physical hoohah $\Rightarrow \cos(m\pi + \phi) = \cos m\phi$

$\therefore$ m is an integer

$$s\frac{\partial}{\partial s}\left(s\frac{\partial \mathcal{S}}{\partial s}\right) = m^2 \mathcal{S} \Rightarrow \mathcal{S} = A_m s^m + B_m s^{-m},\ m \neq 0$$

$$m = 0 \Rightarrow s\frac{\partial \mathcal{S}}{\partial s} = c \Rightarrow \mathcal{S} = c\ln(s),\ m = 0$$

$V(\phi, s)\ \Phi(\phi)\ \mathcal{S}(s)$

$$V(\phi, s) = a_0 + b_0\ln(s) + \sum_{m=1}^{\infty}\left\{s^m\left(a_m\cos(m\phi) + b_m\sin(m\phi)\right) + s^{-m}\left(a_m'\cos m\phi + b_m'\sin m\phi\right)\right\}$$

$$s\left(\frac{\partial s}{\partial s}\frac{\partial \mathcal{S}}{\partial s} + s\frac{\partial \mathcal{S}}{\partial s}\right) = m^2\mathcal{S}$$

$$s\mathcal{S}' + s\mathcal{S}'' = \mathcal{S}$$

$$\mathcal{S}' + \mathcal{S}'' = m^2\mathcal{S}/s$$

$$p^2 + p = m^2/s$$

$$s\frac{\partial}{\partial s}\left(s\frac{\partial \mathcal{S}}{\partial s}\right) = s\frac{\partial}{\partial s}\left(s\left(A_m m s^{m-1} - B_m m s^{-m-1}\right)\right)$$

$$= s\frac{\partial}{\partial s}\left(A_m m s^m - B_m m s^{-m}\right)$$

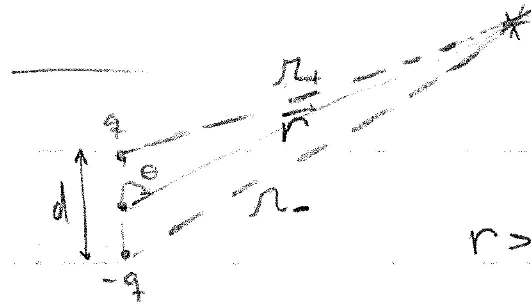
$$= s\left(A_m m^2 s^{m-1} + B_m m^2 s^{-m-1}\right)$$

$$= A_m m^2 s^m + B_m m^2 s^{-m}$$

$$= m^2\mathcal{S} = m^2\left(A_m s^m + B_m s^{-m}\right)$$

Multipole Expansion

point charge $V = \frac{q}{r}$



$r \gg d$

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

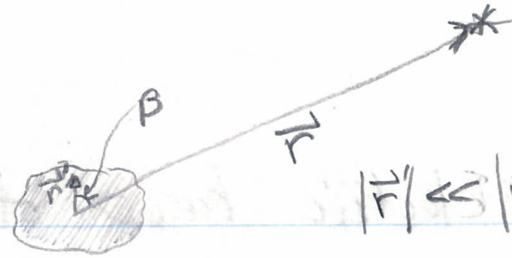
$$r_+ = \left(r^2 + \left(\frac{d}{2}\right)^2 - 2\frac{d}{2}r\cos\theta\right)^{1/2} \cong \left(r^2 - dr\cos\theta\right)^{1/2} = r\left(1 - \frac{d}{r}\cos\theta\right)^{1/2}$$

$$r_- = \left(r^2 + \left(\frac{d}{2}\right)^2 + 2\frac{d}{2}r\cos\theta\right)^{1/2} \cong \left(r^2 + dr\cos\theta\right)^{1/2} = r\left(1 + \frac{d}{r}\cos\theta\right)^{1/2}$$

$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left(\frac{1}{\left(1 - \frac{d}{r}\cos\theta\right)^{1/2}} - \frac{1}{\left(1 + \frac{d}{r}\cos\theta\right)}\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}\frac{d}{r}\cos\theta = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

$\boxed{\text{Def}^n \quad q\vec{d} = \vec{p} \quad \text{where the direction of } \vec{d} \text{ is from negative charge in dipole to positive charg}}$

§3.4 Multipole Expansion



$$V = \int \frac{\rho(\vec{r}')\, d\tau'}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} \qquad |\vec{r}'| \ll |\vec{r}|$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \frac{r'}{r^2} P_1(\cos\beta) + \frac{r'}{r^2} P_2(\cos\beta) + \cdots$$

$$= \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\beta)$$ how ill prove upto $l = 2$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\beta}} = \frac{1}{r}\left(1 + \frac{r'^2 - 2rr'\cos\beta}{r^2}\right)^{-1/2}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\frac{r'^2 - 2rr'\cos\beta}{r^2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{r'^2 - 2rr'\cos\beta}{r^2}\right)^2 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r}\cos\beta - \frac{1}{2}\frac{r'^2}{r^2} + \frac{\frac{3}{4}}{2}\,\frac{4r^2 r'^2 \cos^2\beta}{r^4} + \cdots\right]$$ just keep big terms ???

$$= \frac{1}{r}\left[1 + \frac{r'}{r} P_1 + \frac{r'^2}{r^2}\left(\frac{3}{2}\cos^2\beta - \frac{1}{2}\right) + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \frac{r'}{r} P_1 + \frac{r'^2}{r^2} P_2 + \cdots\right]$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int \rho(\vec{r}')\, r'^l\, P_l(\cos\beta)\, d\tau'}$$

$l = 0$ monopole : $\frac{Q}{4\pi\epsilon_0 r}$

$l = 1$ dipole : $\frac{1}{4\pi\epsilon_0 r^2}\int \rho(\vec{r}')\, r' \cos\beta\, d\tau = \frac{1}{4\pi\epsilon_0 r^2}\int \rho(\vec{r}')\, \vec{r}' \cdot \hat{r}\, d\tau' = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}$

$l = 1$

$$\boxed{\vec{p} = \int \rho(\vec{r}')\, \vec{r}'\, d\tau'}$$

$\boxed{3.31,\ 3.32}$

Electric Field of a Dipole

$$V(r,\theta) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = -\nabla V(r,\theta) = \underbrace{-\hat{r}\frac{\partial V}{\partial r}}_{E_r} - \underbrace{\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}}_{E_\theta} + \frac{1}{r\sin\theta}\overbrace{\frac{\partial V}{\partial \phi}}^{=0}\hat{\phi}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

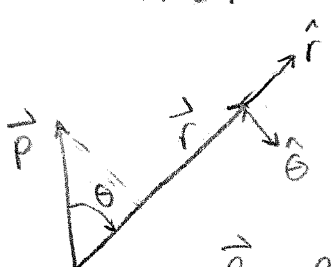
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$\vec{E} = E_r + E_\theta = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}, \quad \vec{E} = -\nabla V = \frac{p}{4\pi\epsilon_0 r^2}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$



$$\vec{p} = p\cos\theta\,\hat{r} - p\sin\theta\,\hat{\theta}$$

$$\Rightarrow \left[3(\vec{p}\cdot\hat{r}) - \vec{p}\right] = 3(\vec{p}\cdot\hat{r})\hat{r} - p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$

$$= 2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$