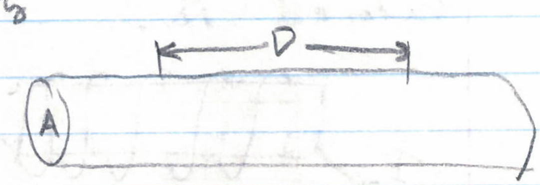
# Chapter Seven – Electrodynamics

HWK. 7.1, 7.3

## § 7.1 Electomotive Force

- Ohm's law $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$ where $\boxed{\sigma \equiv \text{conductivity}}$
  for all pratical purposes $\vec{v} \times \vec{B} \longrightarrow$ zero
  thus $\vec{J} = \sigma \vec{E}$

- $\sigma \equiv \frac{1}{\rho}$ where $\rho \equiv$ resistivity




ohm's law derivation

$$JA = \sigma A E$$

$$\underbrace{JA}_{\text{current}} = \frac{\sigma A}{D} \underbrace{DE}_{\text{voltage}}$$

$$I = \frac{\sigma A}{D} V$$

Now Def$^n$ $\frac{\rho D}{A} \equiv R$ $\quad \therefore \quad I = \frac{A}{\rho D} V = \frac{V}{R} \Rightarrow V = IR.$

$$dw = dq\, V$$

$$\frac{dw}{dt} = \frac{dq}{dt} V = IV \Longrightarrow P = IV = I^2 R$$

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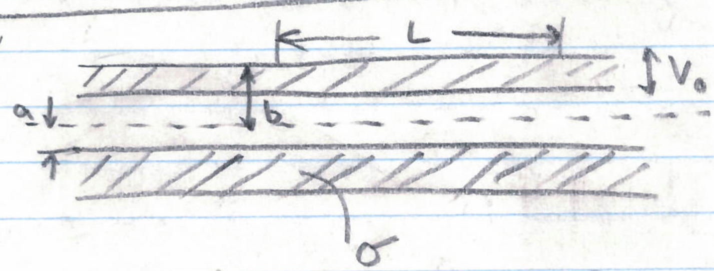
Ex//




two long cylindrical surfaces with $\Delta V = V_o$

what is current in L $\quad I = \int \vec{J} \cdot d\vec{a} = \int J_s \cdot dl \cdot s\, d\phi$

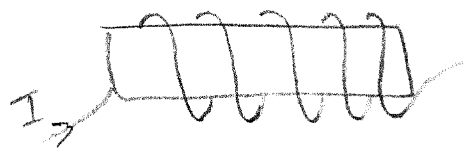
$I = \sigma \int E_s\, dl \cdot s\, d\phi$ note $E_s = \frac{\lambda}{2\pi\epsilon_o s} \Rightarrow I = \frac{\sigma\lambda}{2\pi\epsilon_o} \int dl\, d\phi = \frac{\sigma\lambda}{2\pi\epsilon_o} 2\pi L$

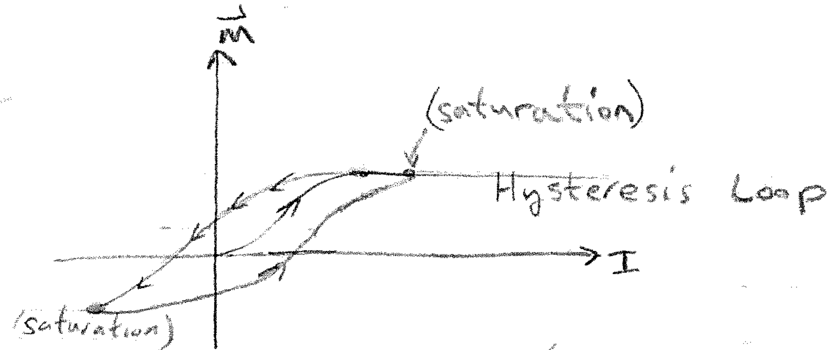
$I = \frac{\sigma\lambda}{\epsilon_o} L$. Also $V_o = -\int_b^a \vec{E} \cdot d\vec{s} = \int_a^b E_s\, ds = \frac{\lambda}{2\pi\epsilon_o} \ln\left(\frac{b}{a}\right)$

$\frac{\lambda}{\epsilon_o} = \frac{I}{\sigma L} \Rightarrow V_o = \frac{1}{2\pi} \frac{I}{\sigma L} \ln\left(\frac{b}{a}\right) \Rightarrow \boxed{R = \frac{1}{2\pi\sigma L} \ln\left(\frac{b}{a}\right)}$ and $I = \frac{2\pi\sigma L V_o}{\ln(b/a)}$

# Ferromagnetism (Fe, Ni)

each piece has random $\vec{M}$, when we apply $\vec{B}$ the molecule is favored to happen the boundary, unfavored domains get eaten up by favored domain as $\vec{B}$ is increased, as we decrease $\vec{B}$ field we don't go back to original situation although we do lose a little of favored $\vec{M}$.




now in a linear material we would expect the material to follow same curve back as I decreased. The history of a Ferromagnetic material matters that is One value of $\vec{I}$ maps to many $\vec{I}$

# BEYOND THE STUPID MAGNETISM GARBAGE $\boxed{8.1, 8.2}$

## Chapter 8 — Conservation

### § 8.1 Poyntings Theorem

If a field does work on a charge then does the field loss energy?

$$W_e = \frac{\epsilon_0}{2}\int E^2 d\tau \qquad W_m = \frac{1}{2\mu_0}\int B^2 d\tau$$

$$W_{em} = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$$

Place $q$ in $(\vec{E}, \vec{B})$ then the work done on $q$ is

$$\vec{F}\cdot d\vec{l} = q(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{v}\,dt$$

$$d\vec{W} = q(\vec{E}\cdot\vec{v})dt$$

$$\frac{d\vec{W}}{dt} = q(\vec{E}\cdot\vec{v}) \quad \text{rate of work done on } q$$

If a distribution of charges then,

$$\int \rho\,\vec{E}\cdot\vec{v}\,d\tau = \int \vec{E}\cdot\vec{J}\,d\tau \qquad \text{work per unit vol/time}$$

note, $\vec{E}\cdot\vec{J} = \vec{E}\cdot\left(\frac{\nabla\times\vec{B}}{\mu_0} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$

note, $\vec{E}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot\underbrace{(\nabla\times\vec{E})}_{\frac{\partial B}{\partial t}} - \nabla\cdot(\vec{E}\times\vec{B})$

thus, $\vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$

then

$$\underbrace{\vec{E}\cdot\vec{J}}_{\text{mechanical}} + \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\right.}_{\text{out of } \epsilon, b} + \underbrace{\left.\frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})\right)}_{\text{out of } \epsilon, b} = 0$$

$$\boxed{\text{Def}^{\underline{n}}\ \text{Poynting Vector}:\ \vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})}$$

Theorem

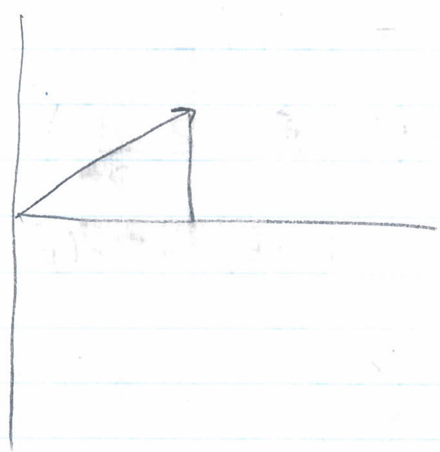
$$\frac{dW}{dt} + \frac{d}{dt}\int_\tau \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau + \frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{a}$$

$$\underbrace{\frac{\partial u_{mech}}{\partial t} + \frac{\partial u_{em}}{\partial t}}_{\text{rate energy increase}} = \underbrace{-\nabla\cdot\vec{S}}_{\text{rate energy flow into system}}$$

$$W_{\text{in } \vec{B} \text{ field}} = \frac{1}{2\mu_0} \int \vec{B}^2 \, d\tau \quad \text{like} \quad W_e = \frac{\epsilon_0}{2} \int \vec{E}^2 \, d\tau$$

| | |
|---|---|
| $\nabla \cdot \vec{E} = \rho / \epsilon_0$ | $\nabla \cdot \vec{D} = \rho_f$ |
| $\nabla \cdot \vec{B} = 0$ | $\nabla \cdot \vec{B} = 0$ |
| $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ | $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$ |
| $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ | $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ |

$F = q(\vec{E} + (\vec{v} \times \vec{B}))$ and $\nabla \cdot J = \frac{\partial \rho}{\partial t}$



# Test One Review

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \nabla \times \vec{H} = \vec{J}_f \quad \left(\text{if } \frac{\partial \vec{D}}{\partial t} = 0\right)$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H} = \mu \vec{H}$$

$$\underline{\vec{J}_B} = \nabla \times \vec{M} = \nabla \times \chi_m \vec{H} = \underline{\chi_m \vec{J}_f}$$ *test for linear media*

Ferromagnetism (NoN! linear Media)

$$J = \sigma \vec{E} \qquad V = IR$$

$$R = \frac{\rho D}{A} \qquad \rho = \frac{1}{\sigma} \qquad P = IV = I^2 R$$

Charging $Q = CV_0(1 - e^{-t/RC})$

DisCharging $Q = Q_0 e^{-t/RC}$

$$F = q(\vec{v} \times \vec{B}) \Rightarrow \mathcal{E} = -\frac{d\Phi}{dt}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} \Rightarrow \mathcal{E} = \oint \vec{E} \cdot d\vec{l} \Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$M = \frac{\mu_0}{4\pi} \oint\oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$ Nuemann Formula

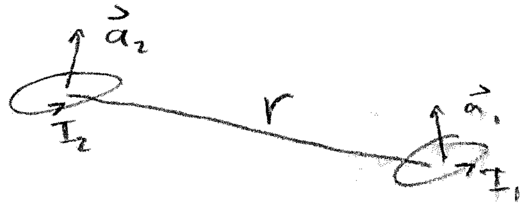
$$\Phi = MI \qquad \mathcal{E} = -M \frac{dI}{dt}$$

$$\Phi = LI \qquad \mathcal{E} = -L \frac{dI}{dt}$$

For Solenoid: $\frac{L}{l} = \mu_0 n^2 A$

$$I = \frac{\mathcal{E}_0}{R}\left(1 - e^{-Rt/L}\right)$$ for ?

Ex/ 7.30




tiny loops

$m_1 = I_1 \vec{a}_1$ $\qquad \vec{B}_{\text{due to 1}} = \frac{\mu_0}{4\pi r^3}\left((3\vec{m}_1\cdot\vec{r})\hat{r} - \vec{m}_1\right)$

$\Phi = \vec{B}\cdot\vec{a}_2 = \frac{\mu_0}{4\pi r^3}(3\vec{m}_1\cdot\vec{r})\hat{r}\cdot\vec{a}_2 - m_1\cdot\vec{a}_2$ $\qquad \mathcal{E} = -\frac{d\Phi}{dt}$

$\therefore M = \frac{\mu_0}{4\pi r^3}\left[(3\vec{a}_1\cdot\vec{r})\hat{r}\cdot\vec{a}_2 - \vec{a}_1\cdot\vec{a}_2\right]$ $\qquad \Phi = MI$

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$\frac{dW}{dt} = -\mathcal{E}_1 I_1 = -\left(-M\frac{dI_2}{dt}\right)I_1 = M_1 I_1 \frac{dI_2}{dt}$

$W = M I_1 I_2 = \mu_0 I_1 I_2\left[3(a_1\cdot\hat{r})(a_2\cdot\hat{r}) - \vec{a}_1\cdot\vec{a}_2\right]$

$= \frac{\mu_0}{4\pi r^3}\left[3(m_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r}) - \vec{m}_1\cdot\vec{m}_2\right]$

$0 \to I_2$ while $I_1$ = constant

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$\vec{M} = \chi_m \vec{H}$ $\qquad \left\{ \begin{array}{l} \chi_m \text{ positive paramagnets} \\ \chi_m \text{ negative diamagnets} \end{array} \right\}$

$\vec{M}$ = dipole moment per unit volume

$\vec{m} = \vec{M}$(Volume)

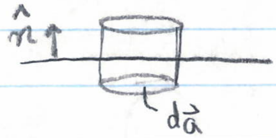
dipole in $\vec{B}$ field exp. torque $\vec{N} = \vec{m}\times\vec{B}$
if the $\vec{B}$ field is uniform then no net force
however if $\vec{B}$ non-uniform, $\vec{F} = \nabla(\vec{m}\cdot\vec{B})$ or $U = -\vec{m}\cdot\vec{B}$

$\vec{A} = \frac{\mu_0}{4\pi}\frac{m\times\hat{r}}{r^2}$ mag. vector potential for dipole

for dist. of dipoles $\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}_b\, d\tau'}{r} + \frac{\mu_0}{4\pi}\int\frac{\vec{K}_b\, da'}{r}$
where,
$\vec{J}_b = \nabla\times\vec{M}$
$\vec{K}_b = M\times\hat{n}$

# Boundary Conditions for Maxwells $\mathcal{E}_q$ in Matter

$$\nabla\cdot\vec{D} = \rho_f \Rightarrow \oint \vec{D}\cdot d\vec{a} = Q_{fenc}$$



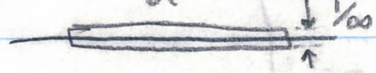
$$\Rightarrow D^{\perp}_{abo}\cdot da - D^{\perp}_{bel}\cdot da = dQ$$

$$D^{\perp}_{abo} - D^{\perp}_{bel} = \frac{dQ}{da_\perp} = \boxed{\sigma_f = D^{\perp}_{abo} - D^{\perp}_{bel}}$$

$$\nabla\cdot\vec{B} = 0 \Rightarrow \oint \vec{B}\cdot d\vec{a} = 0$$

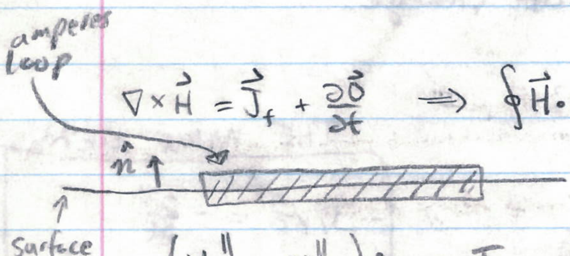
$$B^{\perp}_{abo}\cdot da - B^{\perp}_{bel}\cdot da = 0 \Rightarrow B^{\perp}_{abo} = B^{\perp}_{bel}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\oint \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$$



$$\Rightarrow d\vec{a} = 0 \Rightarrow \oint \frac{\partial B}{\partial t}\cdot d\vec{a} = 0$$

$$\Rightarrow E^{\parallel}_{abo}\cdot dl - E^{\parallel}_{bel}\cdot dl = 0 \Rightarrow E^{\parallel}_{abo} = E^{\parallel}_{bel}$$



$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \Rightarrow \oint \vec{H}\cdot d\vec{l} = \underbrace{\int \vec{J}_f\cdot d\vec{a}}_{I_{fenc}} + \underbrace{\frac{d}{dt}\int_s \vec{D}\cdot d\vec{a}}_{0 \text{ if } \vec{D} \text{ is finite}}$$

$$(H^{\parallel}_{abo} - H^{\parallel}_{bel})\,l = I_{fenc} \Rightarrow H^{\parallel}_{abo} - H^{\parallel}_{bel} = K_f \times \hat{n}$$

if I flows out of page then H is $\leftarrow$ which $H \perp I_{fenc}$ and $H \perp \hat{n}$

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Ex/(7.42) A perfect conducter $\equiv \{\sigma = \infty,\ \vec{E} = \vec{0}$ inside$\}$

(a) show $\frac{\partial\vec{B}}{\partial t} = 0$ inside

(b) the flux through the loop is constant

(c) super conductor $(\vec{B} = 0) \Rightarrow$ the current must be on surface

(a) $$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Longrightarrow \frac{\partial\vec{B}}{\partial t} = 0$$

(b) $$\int(\nabla\times\vec{E})\cdot d\vec{a} = \oint\vec{E}\cdot d\vec{l} = -\oint\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$$

$$= -\frac{d\Phi}{dt} = $$

$$\Rightarrow \Phi \neq \Phi(t$$

(c) $$\underset{0}{\nabla\times\vec{B}} = \underline{\underline{\mu_0\vec{J}}} + \underset{0}{\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$$

recall $\vec{D} = \epsilon_0 \vec{E} \Rightarrow \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ Let $\frac{\partial \vec{D}}{\partial t} \equiv \vec{J}_D$

$\vec{J}_D \equiv \frac{\partial \vec{D}}{\partial t}$ = displacement current.

$\therefore \nabla \times \vec{B} = \mu_0 (\vec{J} + \vec{J}_D)$

Maxwells Equations!

(i) $\nabla \cdot \vec{E} = \rho/\epsilon_0$ (iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) $\nabla \cdot \vec{B} = \vec{0}$ (iv) $\nabla \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$

Lorentz Force Law

$F = q(\vec{E} + v \times \vec{B})$

Continuity Eq.

$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

foundation of E/M.

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{E} = \rho_e/\epsilon_0 & \nabla \cdot \vec{B} = \mu_0 \rho_m \\ \nabla \times \vec{E} = -\mu_0 \vec{J}_m - \frac{\partial \vec{B}}{\partial t} & \nabla \times \vec{B} = \mu_0 \vec{J}_e + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{array}}$$

SYMETRICAL MAXWELLS Eq's with magnetic monopoles

note dirac used monopoles $\Rightarrow$ quantisation of charge.

## Maxwells Equations in Matter

THE WHOLE DEAL

$$\boxed{\begin{array}{l} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{D} = \rho_f \\ \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \\ \vec{D} = \epsilon \vec{E} \\ \vec{H} = \frac{1}{\mu} \vec{B} \end{array}}$$

$\rho_B = -\nabla \cdot \vec{P}$ $\vec{J}_B = \nabla \times \vec{M}$

$\nabla \cdot \vec{E} = \rho/\epsilon_0 = \frac{\rho_f}{\epsilon_0} + \frac{\rho_B}{\epsilon_0} \Rightarrow \nabla \cdot \vec{E} = \frac{\nabla \cdot \vec{P}}{\epsilon_0} = \frac{\rho_f}{\epsilon_0}$

$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f \Rightarrow \nabla \cdot \vec{D} = \rho_f$

$\nabla \cdot \vec{B} = 0$ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$(\vec{P} \cdot \hat{n} = \sigma_B)$  $\Rightarrow \frac{\partial \sigma_B}{\partial t} = \frac{\partial \vec{P} \cdot \hat{n}}{dt}$

$dI = \frac{\partial \sigma_B}{\partial t} \cdot da_\perp \Rightarrow \vec{J} = \frac{dI}{da_\perp} = \frac{\partial \sigma_B}{\partial t} \cdot da_\perp = \frac{\partial \vec{P}}{\partial t}$

$\therefore \vec{J} = \vec{J}_f + \vec{J}_B + \vec{J}_P = \vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$

$\Rightarrow \nabla \times \vec{B} = \mu_0 (\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t}) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

simplify $(\nabla \times \vec{B}) \Rightarrow \nabla \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J}_f + \mu_0 \frac{\partial}{\partial t}(\epsilon_0 \vec{E} + \vec{P})$

$\therefore \mu_0 (\nabla \times \vec{H}) = \mu_0 \vec{J}_f + \mu_0 \frac{\partial \vec{D}}{\partial t} \Rightarrow \boxed{\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$

$$W_{electric} = \frac{1}{2}\int V\rho \, d\tau = \frac{\epsilon_0}{2}\int \vec{E}^2 \, d\tau$$

$$W_{magnetic} = \frac{1}{2\mu_0}\int \vec{B}^2 \, d\tau$$

$W_{TOTAL} = W_{magnetic} + W_{electric}$ total field energy composed from $\vec{E}$ and $\vec{B}$

## § 7.3 Maxwells Equations

(i) $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ (Guass law)

(ii) $\nabla \cdot \vec{B} = \vec{0}$ (no source)

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

(iv) $\nabla \times \vec{B} = \mu_0 \vec{J}$ (amperes law) $\longrightarrow$ $\nabla \times \vec{B} = \mu_0\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$

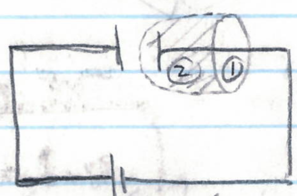
amperes law with maxwell's correction.

$\nabla \cdot \nabla \times \vec{E} = \vec{0} = \nabla \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right) = \vec{0}$.

$\nabla \cdot \nabla \times \vec{B} = \vec{0} = \nabla \cdot \mu_0 \vec{J} \stackrel{?}{=} \vec{0}$ is this true ?

$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$ always.

Ex)



$$\oint \vec{B} \cdot d\vec{l} = \oint \mu_0 \vec{J} \cdot d\vec{a}$$

consider another surface with same boundary but extendended surface then $\vec{J} = 0$ for ① $\oint \mu_0 \vec{J} \cdot d\vec{a} \neq 0$

② $\oint \mu_0 \vec{J} \cdot d\vec{a} = 0$.

$$\vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \frac{-\partial \rho}{\epsilon_0 \partial t} = \frac{\partial}{\epsilon_0 \partial t}(\nabla \cdot \vec{E}) \Rightarrow -\frac{\partial \rho}{\partial t}(\nabla \cdot \vec{E}) = \vec{J}$$

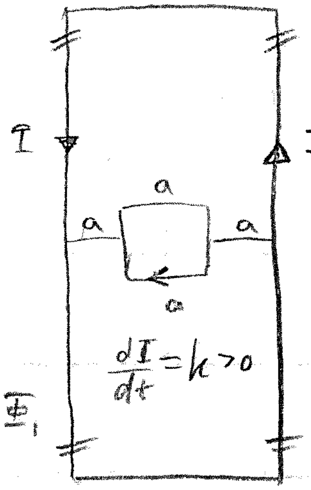
$\nabla \cdot \vec{J} \neq 0$ but $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

$$\nabla \cdot \vec{J} + \frac{\partial}{\partial t}(\epsilon_0(\nabla \cdot \vec{E})) = \nabla \cdot \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$$

$\therefore$ Let us write $\nabla \times \vec{B} = \mu_0\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$ now we have fixed the divergence, this is maxwells correction of Amperes Law.

7.28, 7.29

Ex/(7.21) find $\mathcal{E}$ in big loop if
$\frac{dI}{dt} = k > 0$




$\Phi$ I(big loop) = $\Phi$(little loop)

$\Phi_{\text{due to one side}} = \int_a^{2a} \frac{\mu_0 I}{2\pi s} ds \cdot a$

$= \frac{a\mu_0 I}{2\pi} \ln s \Big|_a^{2a} = \frac{a\mu_0 I}{2\pi} \ln 2 = \Phi_1$

$\Phi_{total} = 2\Phi_1 = \frac{a\mu_0 I}{\pi} \ln(2) = \Phi = MI \Rightarrow M = \frac{\mu_0 a}{\pi} \ln(2)$

then $\mathcal{E} = -M\dot{I} = \boxed{-\frac{\mu_0 a}{\pi} \ln(2) k = \mathcal{E}}$

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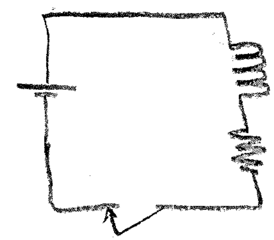
## Energy in Magnetic Field



$\frac{dW}{dt} = -\frac{dU}{dq}\frac{dq}{dt} = -\mathcal{E} I = LI\frac{dI}{dt}$

$\therefore W = \frac{1}{2}LI^2$

$\Phi = \int \vec{B} \cdot d\vec{a} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$

$\Phi = LI \Rightarrow W = \frac{1}{2} I \Phi \Rightarrow W = \frac{1}{2} I \int \vec{A} \cdot d\vec{l}$

$W = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l} = \frac{1}{2}\oint \vec{A} \cdot \vec{I}\, dl = \boxed{\frac{1}{2}\oint \vec{A} \cdot \vec{J} d\tau = W}$

$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \therefore W = \frac{1}{2\mu_0}\oint \vec{A} \cdot (\nabla \times \vec{B}) d\tau$

$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - A \cdot (\nabla \times \vec{B})$ vector identity

$= \vec{B}^2 - \vec{A} \cdot (\nabla \times \vec{B})$

$W = \frac{1}{2}\oint \vec{A} \cdot (\nabla \cdot \vec{B}) d\tau = \frac{1}{2\mu_0}\oint \vec{B}^2 - \nabla \cdot (A \times \vec{B})$

$= \frac{1}{2\mu_0}\oint \vec{B}^2 d\tau + \frac{1}{2\mu_0}\oint \nabla \cdot (A \times \vec{B}) d\tau$

$\nabla \times \vec{A} = \vec{B}$

$\vec{A} \times (\nabla \times \vec{A}) \stackrel{?}{=} \vec{0}$

$= \frac{1}{2\mu_0}\oint \vec{B}^2 d\tau + \frac{1}{2\mu_0}\oint \underbrace{(\vec{A} \times \vec{B})}_{\frac{1}{r^3}} \cdot \underbrace{d\vec{a}}_{r^2}$

$\boxed{W_{mag} = \frac{1}{2\mu_0}\oint \vec{B}^2 d\tau}$

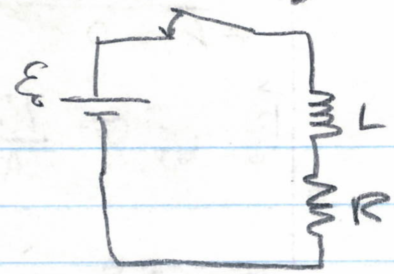
$\quad // \quad 0$

$\Rightarrow \frac{1}{r^3} r^2 = \frac{1}{r} \longrightarrow 0$ for large r.

since $\Phi_2 = M_{21} I_1$ , $\mathcal{E}_2 = -\frac{d\Phi_2}{dt}$.

Changing current not only induces emf on near by loop, but also itself

$$\Phi = LI \quad , \quad \mathcal{E} = -L\frac{dI}{dt}$$




If switch is closed at $t=0$ Find $I(t)$.

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR$$

$$L\frac{dI}{dt} = \mathcal{E}_0 - IR \Rightarrow \frac{L\,dI}{(\mathcal{E}_0 - IR)} = dt \Rightarrow \frac{-L}{R}\ln(\mathcal{E}_0 - IR) = t + k$$

$$\mathcal{E}_0 - IR = ke^{-R/L\,t}$$

$$\text{at } t=0 \Rightarrow k = \mathcal{E}_0 \Rightarrow I = \underbrace{\frac{\mathcal{E}_0}{R}}_{I_0}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\tau = L/R$$

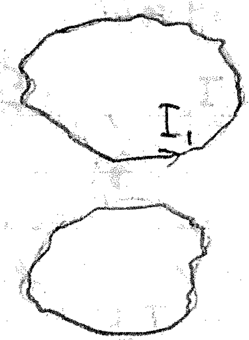
If current is already flowing and switch is suddenly open what is voltage accross $L$.

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$A\times(\nabla\times\vec{B}) + B\times(\nabla\times\vec{A}) + (A\cdot\nabla)B + (B\cdot\nabla)A = \nabla(A\cdot B)$$

# Mutual Inductance

$\boxed{7.20,\ 7.22}$



loop 1

loop 2

$$\vec{B}_{\text{due to }1} = \frac{\mu_0 I_1}{4\pi}\oint \frac{d\vec{l}_1 \times \hat{r}}{r^2}$$

$$\Phi_2 = \int \vec{B}_{\text{due to }1}\cdot d\vec{a}_2 = \int (\nabla \times \vec{A}_1)\cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2 \quad ; \quad A_1 = \frac{\mu_0}{4\pi}\oint \frac{I_1\, d\vec{l}_1}{r} \Rightarrow \Phi_2 = \frac{\mu_0}{4\pi}\oint\oint I_1 \frac{d\vec{l}_1\cdot d\vec{l}_2}{r}$$

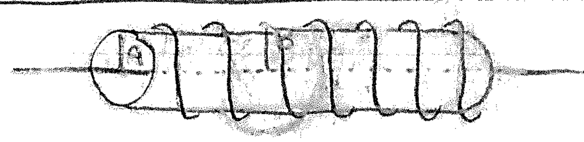
$$\Phi_2 = \left(\oint\oint \frac{d\vec{l}_1\cdot d\vec{l}_2}{r}\right) I_1 = M_{21} I_1 \qquad M_{21} = \oint\oint \frac{d\vec{l}_1\cdot d\vec{l}_2}{r}$$

obviously if $I_2$ flows through loop 2 $\quad \Phi_1 = \left(\oint\oint \frac{d\vec{l}_2\cdot d\vec{l}_1}{r} I_2\right) = M_{12} I_2 \quad$ as we have not used any character of loop 1 as preferential.

$$\boxed{M_{12} = M_{21}}$$

---

Ex/



$M = ?$

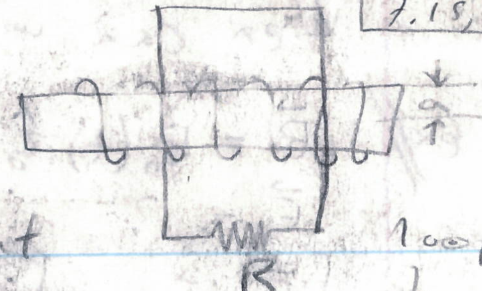
$\Phi = \mu_0 n I \cdot \pi a^2 = (\mu_0 n \pi a^2) I \Rightarrow M = \mu_0 n \pi a^2$

if instead of one loop, we have $n_2$ loops / length and length $l$,

$\Phi = (\mu_0 n I \pi a^2)\, n_2 l \Rightarrow M = \mu_0 n_2 n \pi a^2 l$

Ex// (a) If $\frac{dI_s}{dt} = k > 0$ in solenoid

7.17(a) $I_e = ?$ on R




7.15, 7.18

(b) If $I_s$ is constant solenoid is pull out and reinserted in the opposite direction. What is total Q pass through R.

loop ⊥ to solenoid

(a)

$$B = \mu_0 n I_s \qquad \Rightarrow \frac{d\Phi}{dt} = \pi a^2 \mu_0 n k$$

$$\Phi = B\pi a^2 = \mu_0 \pi a^2 n I_s$$

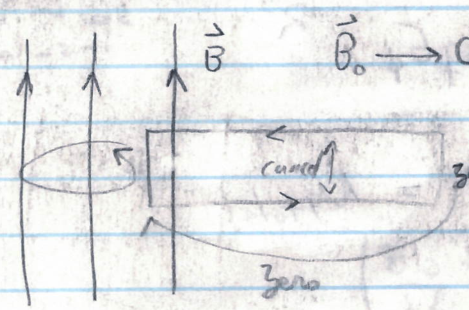
$$\mathcal{E} = -\frac{d\Phi}{dt} = -\pi a^2 \mu_0 n k = \varepsilon \quad \Rightarrow I_e = \frac{\pi a^2 \mu_0 n k}{R} \quad (ccw)$$

(b) $I = \frac{\mathcal{E}}{R}$

$$Q = \int I\,dt = \int \frac{\mathcal{E}}{R} dt = \int \frac{-1}{R}\frac{d\Phi}{dt} dt = \frac{-1}{R}\int d\Phi$$

$$= \frac{2\Phi}{R} = \boxed{\frac{2\pi a^2 \mu_0 n I_s}{R} = Q}$$

---




$\vec{B}_0 \longrightarrow 0 \Longrightarrow E_y$ without $E_x$ or $E_z$

amperes loop zero if any $\vec{E}$ field at all should be equal. because of symetry

$$\oint \vec{E} \cdot d\vec{l} =$$

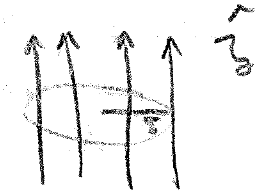
$$\nabla \cdot \vec{E} = \rho = 0 \Rightarrow \underbrace{\frac{1}{s}\frac{\partial}{\partial s}(sE_s)}_{0} + \overbrace{\frac{1}{s}\frac{\partial E_\phi}{\partial \phi}}^{=0} + \cancel{\frac{\partial E_z}{\partial z}} = 0$$

$$\Rightarrow E_s = 0.$$

$\vec{B}$ field with cylindrical symetry $\Rightarrow E_\phi \neq 0$ all others vanish

When can we apply Electrostatics

Ex// $\vec{B} = B_0(t)\hat{z}$ filled up a cylindrical space
$\vec{E} = ?$

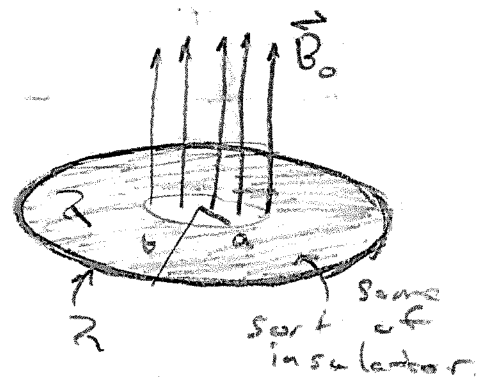


$$\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

symetry allows us to call $\vec{E}$ constant. if $\vec{E} \neq$ constant then use $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$E_\phi(2\pi s) = -\frac{\partial B}{\partial t}\cdot \pi s^2 \Rightarrow E_\phi = -\frac{1}{2}s\frac{\partial B_0}{\partial t}$$

Ex// A charged ring of fixed $\lambda$. $B_0$ is reduced to zero. What will happen




$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = -\pi a^2\frac{dB}{dt}$$

$$E_\phi = \frac{-\pi a^2}{2\pi b}\frac{dB}{dt} \qquad N_q = rF = bqE_\phi$$

$$N_q = \int b\frac{dq}{dl}E_\phi\, dl = b\lambda E_\phi\cdot 2\pi b = N_{q\,total}$$

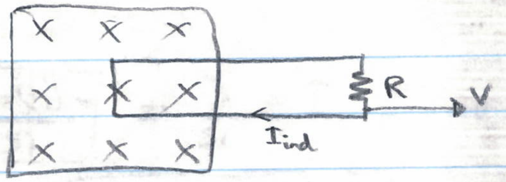
$$N_q = b\lambda\, 2\pi b\left(\frac{-\pi a^2}{2\pi b}\frac{dB}{dt}\right) = \lambda b\left(-\pi a^2\frac{dB}{dt}\right)$$

$$\vec{N} = \vec{r}\times\frac{d\vec{p}}{dt} \Rightarrow \int N\,dt = -\pi a^2\lambda b\int_{B_0}^{0} dB = \underline{\underline{\pi a^2\lambda b B_0}}$$

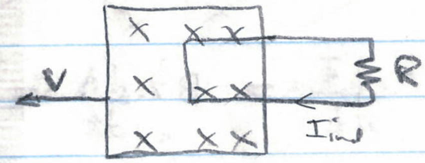
total end angular mom.

## 7.2 Electromagnetic Induction

Faraday's Law

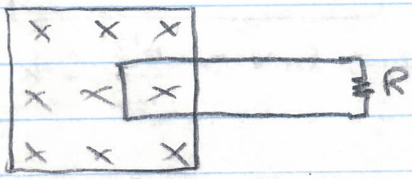



Expt. 1




Expt 2

lower $\vec{B}$




Exp 3

Experiments $\Rightarrow$ $\boxed{\mathcal{E} = \frac{-d\Phi}{dt}}$ FARADAYS LAW

note the negative sign $\Rightarrow$ the induced current is opposite is such that it reduces the $\Delta$ in flux.

$$\mathcal{E} = \int \vec{E} \cdot d\vec{l} = \frac{-d\Phi}{dt} = \frac{-d}{dt}\int \vec{B} \cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

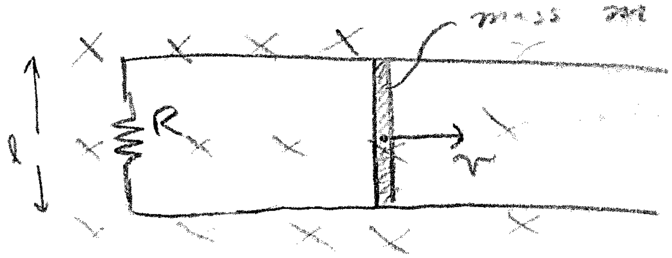
$$\mathcal{E} = \int \vec{E} \cdot d\vec{l} = \int (\nabla \times \vec{E}) \cdot d\vec{a} = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

$\therefore$ $\boxed{\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}}$ FARADAYS LAW ONE OF MAXWELLS EQUATIONS

- if $\vec{E}$ can be found by scalar potential then $\int E \cdot dl = 0$ but if not then $\int E \cdot dl \neq 0$ necessarily.
- Faradays Law $\Rightarrow$ that there are 2 ways to produce $\vec{E}$
  - (i) Charge Distribution
  - (ii) Change of $\vec{B}$ field

    compare $\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$ with $\nabla \times \vec{B} = \mu_0 \vec{J}$ (Amperes Law)

Ex//




(a) $I = ?$
(b) $F_{retarding} = ?$
(c) if $v = V_0$ at $t=0$, $v(t) = ?$
(d) What is $E_{dissipated}$ in $R$

(a) $\mathcal{E} = -\frac{d\Phi}{dt} = B \cdot \frac{dA}{dt} = -Bl v$ $\quad I = \frac{\mathcal{E}}{R} = \boxed{\frac{-Blv}{R} = I}$

(b) $I^2 R = F \cdot v \Rightarrow F = \frac{B^2 l^2 V^2}{R^2} \cdot \frac{R}{V} = \boxed{\frac{B^2 l^2 V}{R} = F_{retarding}}$

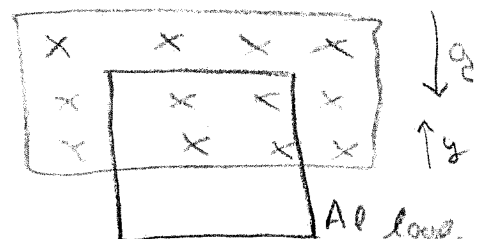
(c) $F = \frac{B^2 l^2}{R} V = -m \frac{dV}{dt} \Rightarrow \frac{dV}{V} = -\frac{B^2 l^2}{mR} dt \Rightarrow \ln V = -\frac{B^2 l^2}{mR} + k$

$\Rightarrow \boxed{V = V_0 e^{-\frac{B^2 l^2}{mR} t}}$

(d)

$$\int_0^\infty I^2 R = \int_0^\infty \frac{B^2 l^2 V^2}{R^2} R\,dt = \frac{B^2 l^2}{R} \int_0^\infty V_0^2 e^{-\frac{2B^2 l^2}{mR} t} dt = \frac{-B^2 l^2 V_0^2}{R} \frac{mR}{2B^2 l^2} \left(-1\right) = \frac{1}{2} m V_0^2$$

Ex//(7.11) An Aluminum Square loop of side $l$
Find terminal velocity $v_t$, how long does it take to reach $0.9\, V_t$




$\mathcal{E} = \frac{-d\Phi}{dt} = -Bl \frac{dy}{dt} = -Blv$

$\mathcal{E} = IR \Rightarrow I = \frac{Blv}{R}$ (Clock wise) $\quad$ note $R = \frac{4\rho l}{A}$

$F_y = ilB = \frac{B^2 l^2 V}{R} = \frac{B^2 l^2 V}{4\rho l / A}$

$mg - \frac{B^2 l^2 v}{R} = m \frac{dv}{dt} \Longrightarrow \frac{dt}{mR} = \frac{dv}{mgR - B^2 l^2 v}$

$\Rightarrow \frac{t}{mR} + k' = -\ln(mgR - B^2 l^2 v)$

$\Rightarrow mgR - B^2 l^2 v = k e^{\frac{-B^2 l^2 t}{mR}} \xrightarrow{v(0)=0} k = mgR$

$\Rightarrow v = \frac{mgR}{B^2 l^2}\left(1 - e^{-\frac{tB^2 l^2}{mR}}\right)$

$\therefore v_{terminal} = \frac{mgR}{B^2 l^2} = \frac{4mg\rho}{B^2 l A} = \frac{16 g \rho d}{B^2}$ ← density

if $|\vec{B}| = 1$ Tesla the $v_t = 1.2 \times 10^{-2}$ m/s

if $V = 0.9 V_t \Rightarrow t = 2.8 \times 10^{-3}$ sec.

7.5, 7.8

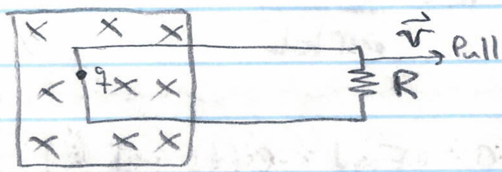
## Electromotive force

$f_s \equiv$ not from electrostatic

$$\oint \vec{f}\,(\text{force}/\text{charge}) \cdot d\vec{l} = \oint \left(\vec{f}_s + \vec{E}_{\text{from electrostatic charge}}\right) \cdot d\vec{l} \quad \text{as } \nabla \times \vec{E} = 0$$

$$= \oint \vec{f}_s \cdot d\vec{l} \equiv \mathcal{E}\ (\text{emf})$$

motional emf - caused by motion through $\vec{B}$ field




$$\vec{f}_s = \vec{v} \times \vec{B}$$

$$\oint \vec{f}_s \cdot d\vec{l} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = (\vec{v} \times \vec{B}) \cdot \vec{l} + 0 + 0 + 0$$

$$= (\vec{l} \times \vec{v}) \cdot \vec{B}$$

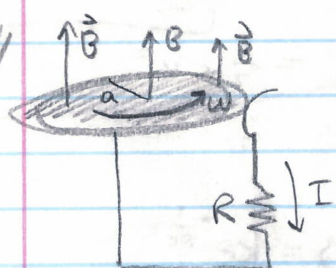
$$= \frac{d\vec{a}}{dt} \cdot \vec{B} = -\frac{d\Phi}{dt} = \mathcal{E}$$

$$\therefore\ \mathcal{E} = -\frac{d\Phi}{dt}$$

In general,

$$\left\{ \begin{aligned} &\mathcal{E} = \oint \vec{f}_s \cdot d\vec{l} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = \oint (d\vec{l} \times \vec{v}) \cdot \vec{B} = \oint \frac{d\vec{l} \times d\vec{r}}{dt} \cdot \vec{B} \\ &\mathcal{E} = \int \frac{d\vec{a}}{dt} \cdot \vec{B} = \boxed{-\frac{d\Phi}{dt} = \mathcal{E}} \end{aligned} \right\}$$

Ex//




Given $\vec{B}, \omega, a$, metal disk

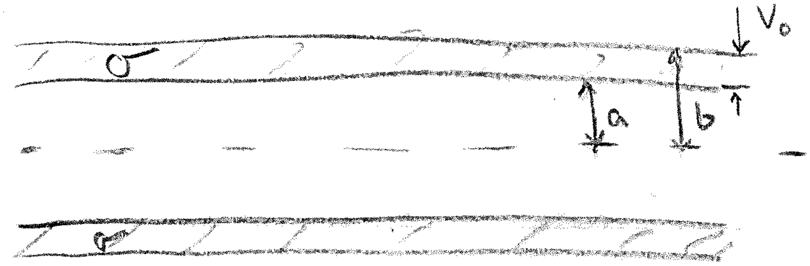
$$\frac{dA}{dt} = \frac{av}{2} = \frac{a^2\omega}{2}$$

$$\mathcal{E} = \frac{-d\Phi}{dt} = -B\frac{dA}{dt} = \frac{Ba^2\omega}{2}$$

$$I = \mathcal{E}/R = \frac{Ba^2\omega}{2R}$$

think about which way I goes

Ex//




$\sigma$ not constant that is $\sigma = \sigma(s) = \frac{k}{s}$

note $E \neq \frac{\lambda}{2\pi\epsilon_0 s}$

$I = J(s) \cdot 2\pi s \cdot L \Rightarrow J(s) = \frac{I}{2\pi s} L$ $\quad E = J/\sigma = \frac{I}{2\pi s \sigma L}$

thus, $E = \frac{I}{2\pi s \frac{k}{s} L} = \frac{I}{2\pi k L}$

$V = -\int_b^a \vec{E} \cdot d\vec{s} = \frac{-I}{2\pi k L}(a-b) \Rightarrow R = \frac{b-a}{2\pi k L}$

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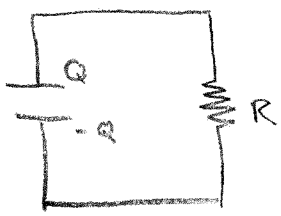
Ex// (7.2)




(a) $V = V_0$ for $t=0$ Find $Q(t)$, $I(t)$
(b) what is energy dissipated through $R$.

(a) $I = V/R \quad \frac{dQ}{dt} = -I = \frac{-V}{R} = \frac{-Q}{RC}$

$\Rightarrow \frac{dQ}{Q} = \frac{-dt}{RC} \Rightarrow \ln(Q) = -t/RC + \text{const.} \Rightarrow Q = Q_0 e^{-t/RC}$

(b) $\text{Energy} = \int I^2 R\, dt = \int_0^\infty \frac{Q_0^2}{R^2C^2} e^{-2t/RC} \cdot R\, dt = \frac{Q_0^2}{R^2C^2}\frac{RC}{2} R \left( e^{-2t/RC} \Big|_0^\infty \right)$

$= \frac{Q_0^2}{2RC} R = \boxed{\frac{Q_0^2}{2C} = E}$

$= \frac{1}{2} C V_0^2$

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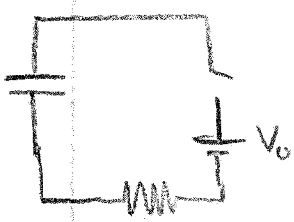


$IR = V_0 - \frac{Q}{C}$

$\frac{dQ}{dt} R = \frac{CV_0 - Q}{C} \Rightarrow \frac{-dQ}{CV_0 - Q} = \frac{-1}{RC} dt$

$\ln(CV_0 - Q) = \frac{-t}{RC} + \text{const.} \Rightarrow CV_0 - Q = ke^{-\frac{1}{RC}t}$

$t=0 \Rightarrow k = CV_0$

$\boxed{Q = CV_0(1 - e^{-t/RC})}$

(c) find $Q(t)$ and $I(t)$
(d) find $E$ delivered by battery

$E = \frac{1}{2}CV_0^2 + \int_0^\infty I^2 R\, dt \qquad I = \frac{CV_0}{RC} e^{-t/RC} \Rightarrow I^2 = \frac{V_0^2}{R^2} e^{-2t/RC}$

$= \frac{1}{2}CV_0^2 + \int \frac{V_0^2}{R} dt\, e^{-2t/RC} = \frac{1}{2}CV_0^2 + \frac{1}{2}CV_0^2 = \boxed{CV_0^2 = E}$