Group Velocity always less than c $\longrightarrow$ $\boxed{9.25}$

away from resonance

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

For transparent material $\omega_j(\text{min})$ is in the ultraviolet hence $n > 1$

If $\omega << \omega_j$, $\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega^2}\left(\frac{1}{1 - \omega^2/\omega_j^2}\right) \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$

$$n \cong 1 + \underbrace{\left(\frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2}\right)}_{A} + \omega^2 \underbrace{\frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega^4}}_{B}$$

$\Rightarrow \quad n = 1 + A + B\omega^2$ (Cauchy Formula)

$\boxed{\text{! TEST 2 UP TO THIS POINT !}}$

Test 2 Review

Chapter 8

$$\frac{dW_{mech}}{dt} + \frac{dU_{em}}{dt} = -\oint \vec{S} \cdot d\vec{a}$$

$$\frac{\partial u_{mech}}{\partial t} + \frac{\partial u_{em}}{\partial t} = -\nabla \cdot \vec{S}$$ Conservation, of energy

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) \qquad u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$T_{ij} = \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} - \epsilon_0 \mu_0 \frac{d}{dt}\int \vec{S}\, d\tau$$

momentum carried by the field $\wp_{em} = \mu_0 \epsilon_0 \vec{S}$

$$\frac{\partial}{\partial t}\left(\vec{\wp}_{em} + \vec{\wp}_{mech}\right) = \nabla \cdot \overleftrightarrow{T}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J} \implies$$ $\left(-\overleftrightarrow{T}\right.$ flow of momentum density$\left.\right)$

$$\vec{l}_{em} = \vec{r} \times \vec{\wp}_{em} = \epsilon_0 \vec{r} \times \left(\vec{E} \times \vec{B}\right)$$

go over examples given in class

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad f = g(z - vt) + h(z + vt)$$

$$f = A\cos\left[k(z - vt) + \delta\right]$$

$$\tilde{f} = \tilde{A}e^{i(kz - \omega t)}$$

$$k = \frac{2\pi}{\lambda} \quad \Big| \quad kv = \omega = \frac{2\pi}{T} = 2\pi\nu$$

$$\tilde{f}(z,t) = \int \tilde{A}(k)e^{i(kz - \omega t)}\,dk$$

at interface : $\begin{cases} \tilde{f}(0^-, t) = \tilde{f}(0^+, t) \\ \left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+} \end{cases}$

EM wave: $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

in vacuum $\vec{k}\cdot\tilde{\vec{E}}_0 = 0$ (know why!)

$$\vec{B} = \frac{\vec{k}\times\tilde{\vec{E}}_0}{\omega}e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon_r\epsilon_0\mu_0(1+\chi_m)}} \approx \frac{c}{\sqrt{\epsilon_r}}$$ as χ_m small.

$\therefore n = \sqrt{\epsilon_r}$

$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \Rightarrow \epsilon_0 E^2 = u$ in vacuo. $\epsilon_0 E^2 = \frac{1}{\mu_0}B^2$

? $\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) \Rightarrow \frac{1}{\mu}\frac{E^2}{v} \Rightarrow \langle S\rangle_{\text{time avg.}} = \frac{1}{2}\epsilon v E_0^2 = I = \text{Intensity}$

Normal Incidence

$$\tilde{E}_{0R} = \left(\frac{V_2 - V_1}{V_2 + V_1}\right)\tilde{E}_{0I} \qquad \tilde{E}_{0T} = \left(\frac{2V_2}{V_1 + V_2}\right)\tilde{E}_{0I}$$

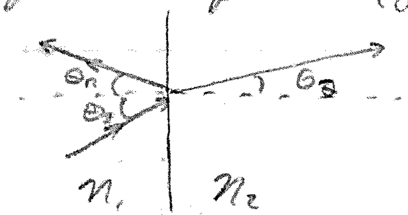
$$R = \frac{I_R}{I_{in}} \qquad T = \frac{I_T}{I_{IN}} \qquad R + T = 1$$

Oblique incidence

first law $\vec{k}_i, \vec{k}_R, \vec{k}_T$ lying in same plane (general property)

second law $\theta_I = \theta_R$

third law $n_1 \sin\theta_1 = n_2 \sin\theta_2$



$\theta_T = \theta_2$

$\theta_I = \theta_1$

If define:

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} \qquad \alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

- Polarization on plane of k_{i}, R, T

$$E_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)E_{0I} \qquad E_{0T} = \left(\frac{2}{\alpha + \beta}\right)E_{0I}$$

- Perpendicular

$$E_{0R} = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right) \qquad E_{0T} = \left(\frac{2}{1 + \alpha\beta}\right)E_{0I}$$

- In both cases $R + T = 1$.

In a Conductor with conductivity σ.

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

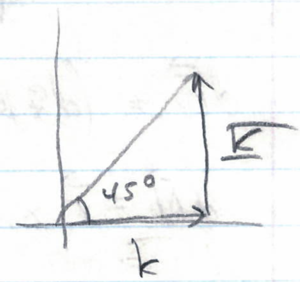
$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}$$

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \Rightarrow \tilde{k} = k + i\kappa$$

no memorize $\rightarrow$ $k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(1+\sqrt{1+\sigma^2/(\epsilon\omega)^2}\right)^{1/2}$ $\quad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\sigma^2/\epsilon^2\omega^2} - 1\right)^{1/2}$

$T = 0$ for perfect cond. while $R = 1$
in fact for good conductor $\kappa = k$
but good conductor has perpendicular $\vec{E}, \vec{B}$ 45°

The skin depth $= \frac{1}{\kappa}$



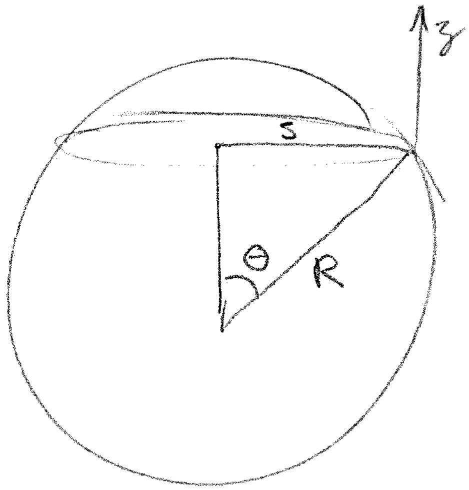
$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} = n^2$$

$$k \cong \frac{\omega}{c}\left(1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2}\right), \quad n = \frac{ck}{\omega} \qquad k = \kappa$$

$$\alpha = 2\kappa = \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2 + \gamma_j^2\omega^2}$$ <u>anomalous dispersion</u>

most of time α small except when $\omega_j^2 \approx \omega^2$
then small $\gamma_j \Rightarrow \frac{1}{\gamma_j}$ very big this is
called anomonlous dispersion
drastic change of index of refraction.

If ω_j is in the ultraviolet, $n \cong 1 + A + B\omega^2$ Cauchy formula
A, B meterial propertys



$$\vec{B} = \frac{2\mu_0 \vec{M}}{3}$$

$$\frac{d\vec{M}}{dt} = \vec{C}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\int \nabla\times\vec{E}\cdot d\vec{a} = -\int \vec{B}\cdot d\vec{a} \Rightarrow \oint \vec{E}\cdot d\vec{l} = E_\varphi (2\pi s) = \int -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}$$

$$E_\varphi (2\pi s) = -\pi s^2 \frac{dB}{dt} \Rightarrow E_\varphi (2\pi s) = -\pi s^2 \frac{2\mu_0}{3}\frac{dM}{dt}$$

$$E_\varphi = \frac{-\mu_0 R \sin\theta}{3} \frac{dM}{dt}$$

$$d\vec{F} = (dq)\vec{E}_\varphi = -(\sigma\, da)\vec{E}_\varphi \qquad F = q(\vec{E} + \cancel{(\vec{v}\times\vec{B})})^{0}$$ no relative motion.

$$d\vec{N} = \vec{r}\times d\vec{F} = \vec{r}\times(-\sigma\, da\, E_\varphi \hat{\varphi}) = (R\sigma\, da\, E_\varphi)\hat{\theta}$$

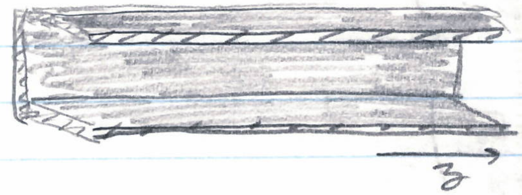
Know net torque in z direction, $\hat{\theta}\cdot\hat{z} = -\sin\theta$

$$(d\vec{N})_z = R\sigma \underbrace{E_\varphi}_{-\frac{\mu_0 R\sin\theta}{3}\frac{dM}{dt}} \underbrace{da}_{2\pi R\sin\theta\, R\, d\theta} \sin\theta$$

$$N = \int (d\vec{N})_z = -\frac{8\pi}{9} R^4 \mu_0 \sigma \frac{dM}{dt} = -\frac{2}{9} Q R^2 \mu_0 \frac{dM}{dt}$$

$$\vec{L} = \int \vec{N}\, dt = \int_M^0 -\frac{2}{9} Q R^2 \mu_0\, dM\, \hat{z} = \frac{2}{9} Q R^2 \mu_0 \vec{M}$$

$\boxed{9.5}$ GUIDED WAVES



Assume
perfect conductor
$\vec{E}=0$, $\vec{B}=0$. inside conductor

at boundary, (i) $E^{\parallel}=0$, (ii) $B^{\perp}=0$
thus we cannot send a plane wave as $\vec{E}_0$ is constant in $\vec{E}=\vec{E}_0 e^{i(kz-\omega t)}$ $\Rightarrow$ $\vec{E}_0$ goes to zero instantaneously at boundary $\therefore$ $\vec{E}_0 = \vec{E}_0(x,y)$, look for sol^{n}'s of form

① $\tilde{\vec{E}} = \tilde{\vec{E}}_0(x,y)\, e^{i(kz-\omega t)}$
② $\tilde{\vec{B}} = \tilde{\vec{B}}_0(x,y)\, e^{i(kz-\omega t)}$

wave nolonger necessarily transverse, in a wave guide, the wave will have longintudenal component, we can prove that the wave cannot be entirely transverse. Subst ①, ② into max eq.

$$\nabla\cdot\vec{E}=0 \qquad \nabla\times\vec{E} = i\omega\vec{B}$$
$$\nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{B} = -\frac{i\omega}{c^2}\vec{E}$$

Components,

$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega$	$\frac{\partial B_x}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$
$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$	$\frac{\partial B_z}{\partial y} - ikB_y = \frac{-i\omega}{c^2}E_x$
$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$	$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$

Sol^{n}
$$iB_y = \frac{ik}{\omega}E_x - \frac{1}{\omega}\frac{\partial E_z}{\partial x}$$
$$iB_y = \frac{1}{k}\frac{\partial B_z}{\partial y} + i\frac{\omega}{c^2}E_x$$

$$\left(\frac{ik}{\omega} - \frac{i\omega}{c^2}\right)E_x = \frac{1}{\omega}\frac{\partial E_z}{\partial x} + \frac{1}{k}\frac{\partial B_z}{\partial y}$$

$$\text{(i)}\quad E_x = \frac{i}{\frac{\omega^2}{c^2}-k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

Similarly

(ii) $$E_y = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x}\right)$$

(iii) $$B_x = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y}\right)$$

(iv) $$B_y = \frac{i}{\frac{\omega^2}{c^2} - k^2}\left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x}\right)$$

□ Wave Equation,

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad , \quad \hat{\vec{E}} = \hat{\vec{E}}_0(x,y)\, e^{i(kz - \omega t)}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right) E_z = -\frac{\omega^2}{c^2} E_z$$

just take z-components for wave eq.

and,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2\right) B_z = \frac{-\omega^2}{c^2} B_z$$

for B_z of B wave equation.

- △ If we take $E_z = 0 \Rightarrow$ TE wave
- △ If we take $B_z = 0 \Rightarrow$ TB wave

TEM ($E_z = B_z = 0$) exists in free space but not in single tube wave guide, it may exist between 2 conductors.

$E^{\parallel} = 0 \qquad B^{\perp} = 0$

$\vec{E}, \vec{B} \qquad E_z, B_z$

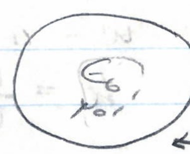
$$\left(\frac{\partial^2}{\partial x^2} \; \frac{\partial^2}{\partial y^2} - k^2\right) \begin{matrix} E_z \\ B_z \end{matrix} = -\frac{\omega^2}{c^2} \begin{matrix} E_z \\ B_z \end{matrix}$$

Its not possible to have a single conductor to carry a TEM wave. Pf/

If $E_z = 0 \qquad \nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad$ if $B_z = 0 \qquad \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0$

$\Rightarrow \vec{E} = -\nabla \phi \qquad$ no ϕ_{max} or ϕ_{min} in region thus the region is a equipotential



Thus $\exists$ no $\vec{E}$ field inside as $\phi = \text{const.}$
In order to have TEM wave we need potential accross boundary.

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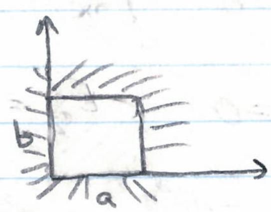
TE waves in a rectangular guide



$E_z = 0$
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0$$

$B^{\perp} = 0$ but,

$$\left.\begin{aligned} B_x &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} k \frac{\partial B_z}{\partial x} \\ B_y &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} k \frac{\partial B_z}{\partial y} \end{aligned}\right\} \Rightarrow$$

derivative must be $\sin \frac{m\pi x}{a}, \; \sin \frac{n\pi y}{b}$

thus $B_z = B_0 \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$

$$-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0 \Rightarrow \boxed{k = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}}$$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}$$

k must be real if the wave is not attenuated. Lowest possible frequency in guide if $(a > b)$

$$\omega = \omega_{10} = \frac{c\pi}{a} \quad \text{lowest cutoff frequency}$$

For each m, n $\quad \omega \leq \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

Def^n cutoff frequency

$$\boxed{\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

ω_{10} is best way to transmit wave

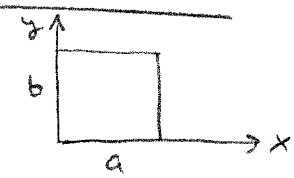
If ω is propagating with ω_{mn}, then

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

phase velocity $\quad v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{c}\right)^2}} > c$

$$V_{group} = \frac{1}{dk/d\omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c.$$

TM wave in rect. wave guide



$B_z = 0 \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right]E_z = 0$

$E'' = 0 \qquad E_z = E_0 \sin\frac{n\pi x}{a} \sin\frac{n\pi y}{b}$

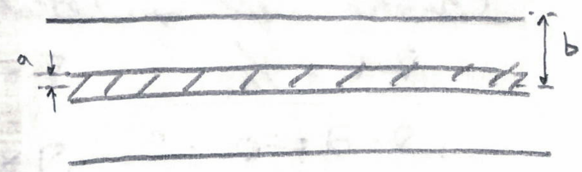
$$-\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0 \qquad k = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}$$

lowest possible freq. if $m = 1, n = 1$

$$\omega_{11} = \sqrt{\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}} > \omega_{10}$$

TEM wave : the coaxial transmission line

$\boxed{9.27,\ 9.28}$



$E_z = 0\ ,\ B_z = 0$

From Hanke.

$$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\cancelto{0}{\frac{\partial E_z}{\partial x}} + \omega\cancelto{0}{\frac{\partial B_z}{\partial y}}\right)$$

for TEM wave we must have $\frac{\omega}{c^2} = k^2$

No dispersion just one wave.

If we charge up the inside wire,

$$\vec{E}_0(s,\phi) = \frac{A}{s}\hat{s} \implies \vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$$

from $\nabla \times \vec{E} = i\omega\vec{B}$

thus $\frac{\partial E_s}{\partial z} - \cancelto{0}{\frac{\partial E_z}{\partial s}} = i\omega B_\varphi$ taking the φ component

$$\therefore\ i\frac{A}{s}k = i\omega B_\varphi \implies B_\varphi = \frac{A}{s}\frac{k}{\omega}e^{i(kz-\omega t)}$$

$$\vec{B}_0 = \frac{Ak}{s\omega}\hat{\varphi}\ ,\ B^{\perp} = 0 \text{ satisfied.}$$

$$\vec{E}_0 = \frac{A}{s}\hat{s}\ ,\ E^{\parallel} = 0.$$

In purely real notation

$$\vec{E}(s,\varphi,z) = \frac{A\cos(kz-\omega t)}{s}\hat{s}$$

$$\vec{B}(s,\varphi,z) = \frac{A\cos(kz-\omega t)}{cs}\hat{\varphi}$$

Chapter 10, Potential and Fields

$$\nabla \cdot E = \rho/\epsilon_0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Given $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$, what are the $\vec{E}$ and $\vec{B}$

- In static case,

$\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$ but (iii) contradict

$\nabla \cdot \vec{B} \Rightarrow \vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \qquad \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\Rightarrow \boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}}$$