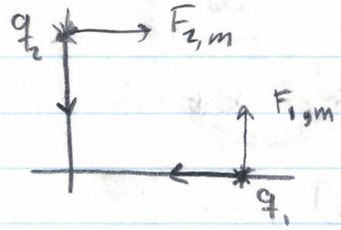
# §8.2 Momentum and Maxwell's stress tensor

Newtons 3rd law ⇒ conservation of linear momentum




the direction isn't even the same so how can we not violate Mr. Newton's 3rd law? ($F_{e_1} = -F_{e_2}$

The momenta are not conserved because there is also a momentum in the E/B field, We must consider that additional interaction to see the 3rd law isn't violated.

Maxwells Stress Tensor,

$$\vec{F} = \rho d\tau(\vec{E} + \vec{v}\times\vec{B}) = \int(\rho\vec{E} + \vec{J}\times\vec{B})d\tau$$

$$\Rightarrow \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} \quad \text{force density}$$

use maxwells equation to eliminate $\rho$ and $\vec{J}$!

we can then consider $\vec{f}$ from just $\vec{E}$ and $\vec{B}$ themselves.

$$f = \epsilon_0(\nabla\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

use $\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E} + \frac{\partial\vec{B}}{\partial t}$ and $\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E}$

we find $\vec{f} = \epsilon_0\left[(\nabla\cdot\vec{E})\vec{E} - \vec{E}\times(\nabla\times\vec{E}) - \frac{1}{\mu_0}[\vec{B}\times(\nabla\times\vec{B})] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right]$

$$\vec{f} = \epsilon_0\left[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E}\right] - \frac{1}{2}\nabla\epsilon_0\vec{E}^2 + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B} - \frac{1}{2\mu_0}\nabla\vec{B}^2\right.$$
$$- \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

define,

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

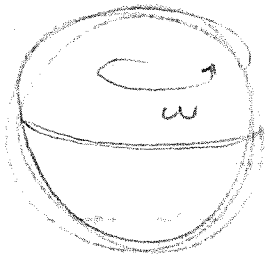
$(\vec{A}\cdot\overleftrightarrow{T})_j = \sum_i a_i T_{ij}$ then $\vec{f} = \nabla\cdot\overleftrightarrow{T} - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}$

$$\vec{F} = \int\vec{f}\cdot d\tau = \int\nabla\cdot\overleftrightarrow{T}\,dt - \epsilon_0\mu_0\frac{d}{dt}\int\vec{S}\,d\tau$$

$$\vec{F} = \oint\overleftrightarrow{T}\cdot d\vec{a} - \epsilon_0\mu_0\frac{d}{dt}\int\vec{S}\cdot d\tau$$

$\boxed{8.4, 8.5}$

Ex/ 8.3 Uniformly charged $(\sigma, \omega, R)$ spinning shell
Find force on the upper hemisphere with max. Tensor




$$\vec{B}_{inside} = \frac{2}{3}\mu_0 \sigma R \omega \hat{z} \quad (\text{Ex } 5.11 \text{ text})$$

$$\vec{B}_{out} = \frac{\mu_0}{4\pi r^3}\left(3(m\cdot\hat{r})\hat{r} - \vec{m}\right) \qquad \vec{m} = \frac{4}{3}\pi\sigma\omega R^4 \hat{z}$$

$$\vec{E}_{in} = 0 \qquad \vec{E}_{out} = \frac{\sigma(4\pi R^2)}{4\pi\epsilon_0 r^2}\hat{r} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r}$$

on the x-y plane

$$s<R, \quad E=0, \quad B = \frac{2}{3}\mu_0\sigma R\omega\hat{z} = B_z\hat{z}, \quad \vec{E}\times\vec{B} = 0$$

$$B_z = \frac{m\mu_0}{4\pi s^3}, \quad B_x = 0, \quad B_y = 0 \qquad \vec{E}\times\vec{B} \perp \hat{z}$$

$$F_z = -\int T_{zz}\, da_z \quad (\text{force on the upper half of hemi})$$

$$= -\int_0^R T_{zz}^{in}\cdot 2\pi s\, ds - \int_R^\infty T_{zz}^{out}\cdot 2\pi s\, ds$$

$$T_{zz}^{in} = \frac{1}{2\mu_0}B_z^2 \quad - \int T_{zz}^{in} da_z = \frac{-1}{2\mu_0}B_z^2\pi R^2 = -\frac{2\pi}{9}\mu_0\sigma^2\omega^2 R^4$$

$$T_{out} = -\frac{\epsilon_0}{2}\left(\frac{\sigma R^2}{\epsilon_0 s^2}\right)^2(x^2+y^2) + \frac{1}{2\mu_0}B_z^2 = \frac{\epsilon_0\sigma^2 R^4}{2\epsilon_0 s^4} + \frac{1}{2\mu_0}\frac{m^2\mu_0^2}{16\pi^2 s^6}$$

$$= \frac{-\sigma^2 R^2}{2\epsilon_0 s^4} + \frac{\mu_0 m^2}{32\pi^2 s^6}$$

$$-\int_R^\infty T_{zz}^{out}\cdot 2\pi s\, ds = \int_R^\infty \frac{\sigma^2 R^4}{2\epsilon_0 s^4} 2\pi s\, ds - \int_R^\infty \frac{\mu_0 m^2}{32\pi^2 s^6} 2\pi s\, ds = \left.\frac{-\pi\sigma^2 R^4}{2 s^2 \epsilon_0}\right|_R^\infty + \left.\frac{\mu_0 m^2}{16\pi 4 s^4}\right|$$

$$= \frac{-\pi\sigma^2 R^2}{2\epsilon_0} - \frac{\mu_0 m^2}{64\pi R^4} = \frac{\pi\sigma^2 R^2}{2\epsilon_0} - \frac{\pi\sigma^2\omega^2 R^8\mu_0}{4\times 9 R^4}$$

$$F_z = \frac{\pi\sigma^2 R^2}{2\epsilon_0} - \frac{1}{4}\pi\sigma^2\omega^2 R^4\mu_0$$

8.6, 8.8 a

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$(\vec{A}\cdot\overleftrightarrow{T})_j = \sum_i a_i T_{ij}$$

$$(\overleftrightarrow{T}\cdot\vec{A})_j = \sum_k T_{jk} a_k$$

$$\vec{f} = \nabla\cdot\overleftrightarrow{T} - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}$$

$$\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} - \epsilon_0\mu_0\frac{d}{dt}\int S\,d\tau$$

## Conservation of Momentum

$\vec{F} = \dfrac{d\vec{P}_{mech}}{dt}$ // the momentum carried by the field is $\vec{P}_{em} = \mu_0\epsilon_0\int\vec{S}\,d\tau$

$$\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} - \epsilon_0\mu_0\frac{d}{dt}\int\vec{S}\,d\tau \qquad \text{Eq. 8.21}$$

$$\vec{f} = \nabla\cdot\overleftrightarrow{T} - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}$$

$$\vec{P}_{em} = \mu_0\epsilon_0\vec{S}$$

thus $\dfrac{\partial P_{mech}}{\partial t} + \dfrac{\partial P_{em}}{\partial t} = \nabla\cdot\overleftrightarrow{T}$ $\quad\overleftrightarrow{T}$ is the flow of momentum density!

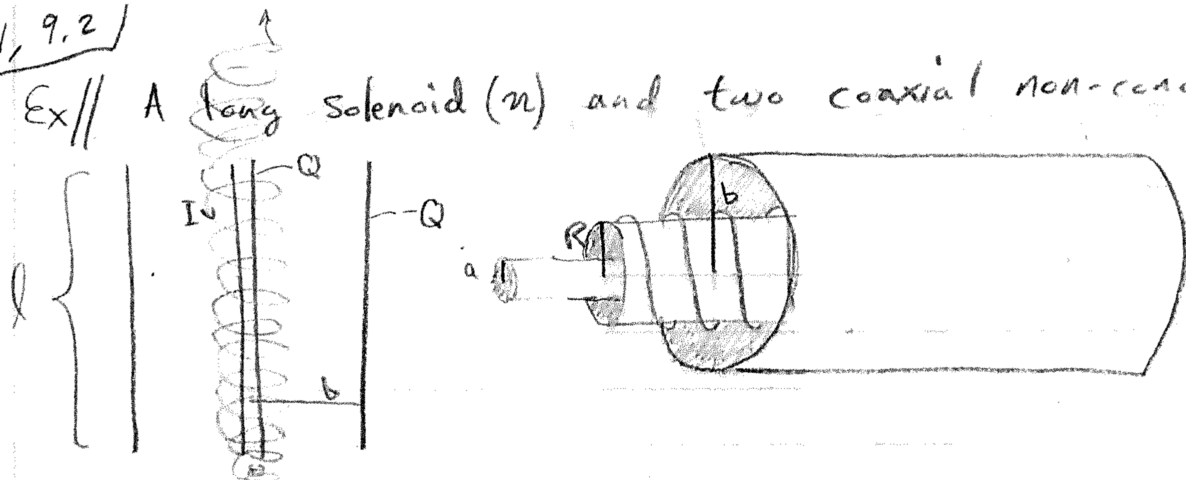
we understand this from the analogy to $\nabla\cdot J = -\dfrac{\partial\rho}{\partial t}$ !

Linear Mom. Density $\vec{P}_{em} = \mu_0\epsilon_0\vec{S} = \epsilon_0(\vec{E}\times\vec{B})$

Angular Mom. Density $\vec{l}_{em} = \vec{r}\times\vec{P}_{em} = \epsilon_0\left[\vec{r}\times(\vec{E}\times\vec{B})\right]$

9.1, 9.2

Ex// A long solenoid ($n$) and two coaxial non-conducting shells




If we reduce $I$, what will happen? Why?

$$\vec{E}_s = \frac{Q}{2\pi\epsilon_0 l s}\hat{s} \qquad \vec{B} = \mu_0 n I \hat{z}$$

$$\vec{l}_{em} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) = \vec{r} \times \left(\frac{Q\mu_0 n I}{2\pi l s}\right)(-\hat{\phi}) = \left(\frac{-Q\mu_0 n I}{2\pi l}\right)\hat{z}$$

$$\text{Total } \vec{l} = \int \vec{l}_{em}\, d\tau = \frac{-Q\mu_0 n I}{2\pi l} \cdot l(R^2 - a^2)\pi$$

$$\vec{l}_{total} = -\frac{Q\mu_0 n I}{2}(R^2 - a^2)$$


where $E, B \neq 0$

When $I$ changes, the $\vec{E}$ field induced on outer shell is

$$\oint \vec{E}_b \cdot d\vec{l} = -\pi R^2 \mu_0 n \frac{dI}{dt} \implies \vec{E}_b = \frac{-\pi R^2}{2\pi b} \mu_0 n \frac{dI}{dt} \hat{\phi}$$

thus torque

$$N_b = b(-Q)E_b = \frac{R^2 Q}{2} \mu_0 n \frac{dI}{dt} \hat{z}$$

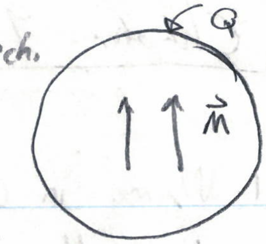
$$\oint \vec{E}_a \cdot d\vec{l} = -\pi a^2 \mu_0 n \frac{dI}{dt} \implies N_a = a\,Q\,E_a = -\frac{a}{2}\mu_0 Q n \frac{dI}{dt}\hat{z}$$

$$\vec{L}_b = \int_I^0 \vec{N}_b\, dt = -\frac{R^2 Q}{2}\mu_0 n I \hat{z}$$

$$\vec{L}_a = \int_I^0 \vec{N}_a\, dt = \frac{a^2}{2}\mu_0 Q n I \hat{z}$$

$$\therefore L_A + L_B = \vec{l}_{total} \qquad \vec{l}_{em} \longrightarrow \vec{l}_{mech.}$$

$\boxed{\text{Ex } 8.8b,c}$




If we heat up sphere and lose $\vec{M}$ then $\vec{L}_{em} \to \vec{L}_{mech.}$
find $\vec{E}_{induc.}$, $\vec{N}$, and $L_{total}$ for the system

For a uniform $\vec{M}$ sphere

$\Rightarrow$ surface current $\vec{K} = \vec{M} \times \hat{n} = M \sin\theta \, \hat{z} \times \hat{r}$

$$\underline{\vec{K} = M \sin\theta \, \hat{\varphi}}$$

in Ex 5.11, if you have a spinning surface charge then current $\sigma R \sin\theta \, \omega \, \varphi \Rightarrow \vec{B} = \frac{2}{3}\mu_0 \sigma R \omega \hat{z}$

by comparison $\Rightarrow \vec{B} = \frac{2}{3}\mu_0 \vec{M}$

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also another comparison

$\uparrow\uparrow\uparrow \; \vec{P} = P\hat{z} \Rightarrow \vec{E}_{in} = -\frac{\vec{P}}{3\epsilon_0}$ by laplace bound. cond.

then $\vec{D}_{in} = \epsilon_0 \vec{E}_{in} + P = -\frac{2}{3}P$

$$\frac{\vec{B}}{\mu_0} = \vec{H} + \vec{M} = -\frac{1}{3}\vec{M} + \vec{M} = \frac{2}{3}\vec{M} \Rightarrow \vec{B} = \frac{2}{3}\mu_0 \vec{M}$$

?

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$$\nabla \times \vec{E} = -\dot{\vec{B}}$$

$$\oint \vec{E} \cdot d\vec{l} = -\pi s^2 \frac{\partial B}{\partial t} = -\frac{2}{3}\mu_0 \pi s^2 \frac{dM}{dt}$$

$$E_\varphi = -\frac{\mu_0 s}{3}\frac{dM}{dt} = -\frac{\mu_0 R \sin\theta}{3}\frac{dM}{dt}$$

$dF = (dq)E_\varphi = \sigma \, dA \, E_\varphi$, $\quad d\vec{N} = \vec{r} \times d\vec{F} = R\sigma E_\varphi \, dA(-\hat{\theta})$

$dA = s\,d\varphi \cdot R\,d\theta$

$d\vec{N} = R^3 \sigma E_\varphi \sin\theta \, d\varphi \, d\theta (-\hat{\theta})$ by sym. we just want $dN_z \hat{z}$.

$(d\vec{N})_z = R^3 \sigma E_\varphi \sin^2\theta \, d\varphi \, d\theta \, \hat{z}$ as $(-\hat{\theta} \cdot \hat{z}) = \sin\theta$

$$N_z = \int dN_z = \int \left(-\frac{\mu_0 R \sin\theta}{3}\frac{dM}{dt}\right)\left(R^3 \sigma \sin^2\theta \, d\varphi \, d\theta\right)$$

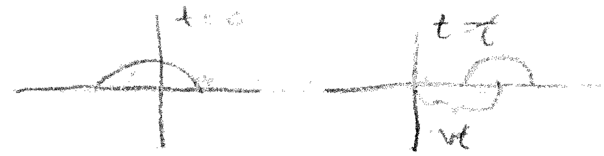
$$= -\frac{\mu_0 R^4}{3}\int \sin\theta (1 - \cos^2\theta)\frac{dM}{dt}\sigma \, d\theta \, d\varphi$$

$$= -\frac{2\pi R^4 \mu_0 \sigma}{3}\frac{4}{3}\frac{dM}{dt} = -\frac{2}{9}QR^2\mu_0\frac{dM}{dt}$$

$$\vec{L} = \int \vec{N} dt = -\int_M^0 \frac{2}{9}QR^2\mu_0 \, dM \hat{z} = \boxed{\frac{2}{9}QR^2\mu_0\vec{M} = \vec{L}}$$

# Chapter 9 — Electromagnetic Waves

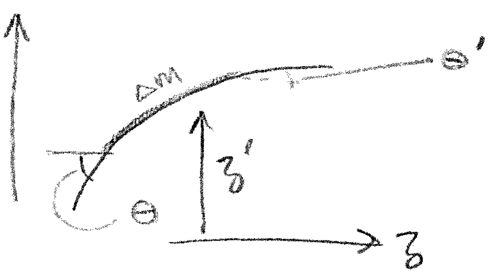
## §9.1 Waves in One Dimension



at $t=0$ $\quad y = f(z, 0)$

at $t=t$ $\quad y = f(z, t) = f(z - vt, 0)$

a wave is a shape that is maintained as it travels



$$\Delta F = T\sin\theta' - T\sin\theta$$
$$\cong T(\tan\theta') - T(\tan\theta) \quad \text{if } \theta \text{ small}$$
$$= T(\tan\theta' - \tan\theta)$$
$$= T\left(\left.\frac{\partial f}{\partial z}\right|_{z'} - \left.\frac{\partial f}{\partial z}\right|_{z}\right)$$
$$= T\frac{\partial^2 f}{\partial z^2}\Delta z$$
$$= \Delta m \frac{\partial^2 f}{\partial t^2}$$

$\boxed{\text{Def}^{n} \,//\, v^2 = \frac{T}{\mu}}$

$$\therefore \boxed{\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} \quad \text{wave eq.}}$$

Then of course $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v}\frac{\partial^2 f}{\partial t^2}$ we know $\frac{\mu}{T} = \frac{1}{v^2}$ from g(3-

Consider if $f(z,t) = g(z - vt)$

$\frac{\partial f}{\partial z} = g' \quad \frac{\partial^2 f}{\partial z^2} = g'' \quad \frac{\partial f}{\partial t} = -vg' \quad \frac{\partial^2 f}{\partial t^2} = v^2 g''$

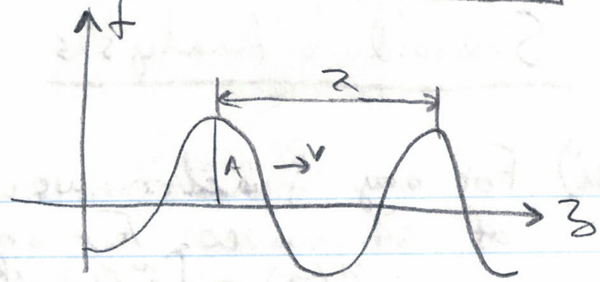
$\therefore \frac{\partial^2 f}{\partial z^2} = g'' = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \therefore \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ if $\underline{f(z,t) = g(z-}$

3.2K background
$\Downarrow$
Eave (Harmo
$\Downarrow$ grad co
$E = mc^2$ $\Downarrow$
$\Downarrow$ C ≠ con
$m = \frac{E}{c^2}$ $m \neq c$

9.3, 9.4

# Sinusoidal Wave

(i) terminology




$$f(z,t) = A\cos\left[k(z - vt) + \delta\right]$$

$$k\lambda = 2\pi \Rightarrow k = \frac{2\pi}{\lambda}$$

at a fixed $z$, the oscillation goes through 1 cycle in 1 period T !

$$kvT = 2\pi \Rightarrow kv = \frac{2\pi}{T} = \text{angular freq} = \omega$$

$$\frac{2\pi}{\lambda} v = \frac{2\pi}{T} \Rightarrow v = \frac{\lambda}{T} = f\lambda = \nu\lambda$$

(ii) Complex Notation

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \cos\theta = \mathrm{Re}(e^{i\theta})$$

$$f(z,t) = \mathrm{Re}(Ae^{i(kz - \omega t + \delta)})$$
$$= \mathrm{Re}\left[(Ae^{i\delta})e^{i(kz-\omega t)}\right]$$

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$$ make $\delta$ into imag. constant multiplicative factor.

Ex// $A_1\cos(kz - \omega t + \delta_1) + A_2\cos(kz - \omega t + \delta_2)$

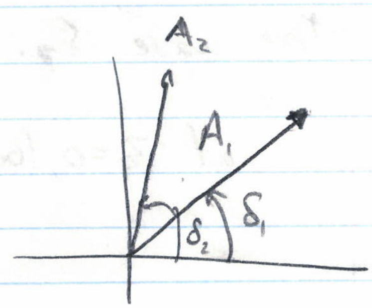
$f_1 = \mathrm{Re}\,\tilde{f}_1 \qquad f_2 = \mathrm{Re}\,\tilde{f}_2$

$f_1 + f_2 = \mathrm{Re}(\tilde{f}_1 + \tilde{f}_2) = \mathrm{Re}(\tilde{f}_3)$

$\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)}$

$f_3 = \mathrm{Re}(\tilde{f}_3) = \mathrm{Re}(\tilde{A}_3 e^{i(kz-\omega t)}) = \underline{A_3}\cos(kz - \omega t + \underline{\delta_3})$

$A_3 = ? \qquad \delta_3 = ?$ prob 9.3




$$\frac{T}{\mu} = v^2 = c^2 = \frac{1}{\mu_0\epsilon_0} \qquad c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$\frac{T}{\mu} = \frac{1}{\mu_0\epsilon_0} = \frac{c^2}{\epsilon}$$

# Sinusoidal Analysis

(iii) For any function we may write as superposition of sin waves. For any ~~function~~ $f(z)$

$$\tilde{f}(z) = \int \tilde{A}(k) e^{ikz} dk \quad \text{Fourier Transform}$$

$$= \int \tilde{A}(k) e^{i(kz - vt)} dk = \int \tilde{A}(k) e^{i(kz - \omega t)} dk$$

If $\omega = kv$ with constant $v$ many cases in reality $k$ depends on $v$ often the <u>wave packet</u> will spread out. But here our wave packet travels w/o $\Delta$ shape

If $v$ differs for different $k$ then the shape may $\Delta$

$v(k) \Rightarrow$ differing freq. move at diff velocities $\Rightarrow$ shape $\Rightarrow$ sh.

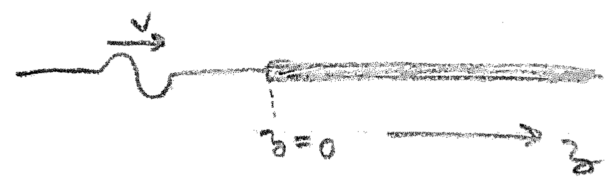
## Boundary Conditions: (Reflection and Transmission)

Incident wave $\quad \tilde{f}_I = \tilde{A}_I e^{i(k_1 z - \omega t)} \quad z < 0$

Reflected wave $\quad \tilde{f}_R = \tilde{A}_R e^{i(k_1 z - \omega t)} \quad z < 0$

Transmitted wave $\quad \tilde{f}_T = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad z > 0$

Hence for $z < 0 \quad \tilde{f} = \tilde{f}_I + \tilde{f}_R$
for $z > 0 \quad \tilde{f} = \tilde{f}_T$




The wave Eq. req. that $f$ be cont. and $f'$ as well!

at $z=0$, (boundary) $\quad \tilde{f}(0^-, t) = \tilde{f}(0^+, t)$

$$\left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

We are looking for magnitudes of reflected and transmitted wave

First bound. cond $\Rightarrow$ $\boxed{\tilde{A}_R + \tilde{A}_I = \tilde{A}_T}$

Second vel. bound cdt. $\Rightarrow$ $(-ik_1)\tilde{A}_R + (ik_1)\tilde{A}_I = (ik_2)\tilde{A}_T$

$$\Rightarrow \boxed{(\tilde{A}_I - \tilde{A}_R) = \frac{k_2}{k_1}\tilde{A}_T}$$

Combining;

$$2\tilde{A}_I = \left(1+\frac{k_2}{k_1}\right)\tilde{A}_T \quad , \quad \boxed{\tilde{A}_T = \frac{2k_1}{k_1+k_2}\tilde{A}_I}$$

$$\tilde{A}_R = \left(\frac{2k_1}{k_1+k_2} - 1\right)\tilde{A}_I = \boxed{\left(\frac{k_1-k_2}{k_1+k_2}\right)\tilde{A}_I}$$

recall, $kv = \omega \Rightarrow k = \omega/v$.

$$\tilde{A}_R = \tilde{A}_I\left(\frac{\omega/v_1 - \omega/v_2}{\omega/v_1 + \omega/v_2}\right) = \boxed{\left(\frac{V_2-V_1}{V_2+V_1}\right)\tilde{A}_I = \tilde{A}_R}$$

$$\tilde{A}_T = \frac{2\,\omega/v_1}{\omega/v_1 + \omega/v_2}\tilde{A}_I = \boxed{\frac{2V_2}{V_1+V_2}\tilde{A}_I = \tilde{A}_T}$$

$$\tilde{A}_R = A_R e^{i\delta_R} = \left(\frac{V_2-V_1}{V_2+V_1}\right)A_I e^{i\delta_I}$$

$$\tilde{A}_T = A_T e^{i\delta_T} = \left(\frac{2V_2}{V_1+V_2}\right)A_I e^{i\delta_I}$$

$$A_R = A_I\left(\frac{V_2-V_1}{V_2+V_1}\right)\left(\cos(\delta_I-\delta_R) + i\sin(\delta_I-\delta_R)\right) \qquad A_R \in \mathbb{R}^+$$

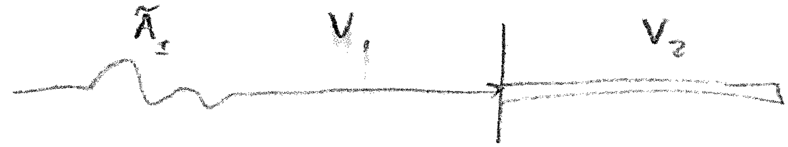
$$A_T = A_I\frac{2V_2}{V_1+V_2}\left(\cos(\delta_I-\delta_T) + i\sin(\delta_I-\delta_R)\right) \qquad A_T \in \mathbb{R}^+$$

as $A_R, A_T \in [0,\infty) \Rightarrow i$ terms $\longrightarrow 0$ !

now since $A_R, A_T > 0 \Rightarrow \delta_I - \delta_T = 0$ for $\cos(\delta_I-\delta_T) = 1$

$\therefore$ $i\sin(\delta_I - \delta_T) = i\sin 0 \Rightarrow i$ drops out

$\therefore$ the transmitted wave has same phase as incident wave !



$$A_R = A_I\left(\frac{V_2 - V_1}{V_2 + V_1}\right)\left\{\cos(\delta_I - \delta_R) + i\sin(\delta_I - \delta_R)\right\}$$

now if $V_2 > V_1$ then $\delta_I - \delta_R = 0$
but if $V_1 > V_2$ then $\delta_I - \delta_R = \pi$ $\Big\}$ as $A_R > 0$

Whats the point?, Read $\S$.

## Polarization

If we have a transverse wave then we are vibrating $\perp$ to direction of wave propagation, the direction of vibration gives the polarization.

$$\vec{\tilde{f}} = \tilde{A}e^{i(kz-\omega t)}\hat{x} \quad \text{, vibration about } y=0 \text{ plane}$$

in general

$$\vec{\tilde{f}} = \tilde{A}e^{i(kz-\omega t)}\hat{n} \quad \text{such that } \hat{n}\cdot\hat{z} = 0.$$

$$\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\vec{\tilde{f}}(z,t) = \tilde{A}\cos\theta\, e^{i(kz-\omega t)} + \tilde{A}\sin\theta\, e^{i(kz-\omega t)}\hat{y}$$

## §9.2 : Electromagnetic Wave [illegible]

9.9, 9.10

$\boxed{E}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla(\cancel{\nabla \cdot \vec{E}}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

0 as $\rho = 0$

$$\Rightarrow \boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \qquad \text{recall } \nabla^2 \vec{f} = \frac{1}{v^2}\frac{\partial \vec{f}}{\partial t}$$

$$\epsilon_0 \mu_0 = (4\pi\epsilon_0)\left(\frac{\mu_0}{4\pi}\right) = (9\times10^{9})^{-1}\,10^{-7} = \frac{1}{9\times10^{16}} \Longrightarrow v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \sqrt{9\times10^{16}} = 3\times10^{8}\ !$$

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c\ .$$

$\boxed{B}$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\nabla \cdot (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \Rightarrow \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- Monochromatic Plane Wave in Vacuum

$$\tilde{\vec{E}}(\vec{r}, t) = \tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\nabla \cdot \tilde{\vec{E}} = \tilde{\vec{E}}_0 \cdot \nabla e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \nabla = \hat{i}\frac{\partial}{\partial x}(\cdot) + \hat{j}\frac{\partial}{\partial y}(\cdot) + \hat{k}\frac{\partial}{\partial z}(\cdot)$$

$$= i\tilde{\vec{E}}_0 \cdot \vec{k}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad i\hat{x}k_x + i\hat{y}k_y + ik_z = i\vec{k}$$

$= 0$ in vaccum

$\therefore\ \tilde{\vec{E}}_0 \cdot \vec{k} = 0 \Rightarrow \vec{E}$ is a transverse wave

$k$ is direction of wave propagation

$\tilde{\vec{E}}_0$ defines plane of vibration (Polarization direction)

$$\nabla \times \vec{E} = -\vec{E}_0 \times i\vec{k}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = -\frac{\partial \vec{B}}{\partial t}$$

If we $\beta$ $\vec{B} = \tilde{\vec{B}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \Longrightarrow \frac{\partial \vec{B}}{\partial t} = -i\omega \tilde{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

where, $\tilde{B}_0 = \dfrac{\vec{k} \times \tilde{\vec{E}}_0}{\omega}$

$$|\vec{B}_0| = \frac{k}{\omega}|\vec{E}_0| = \frac{2\pi}{\lambda}\frac{E_0}{2\pi\nu} = \frac{1}{c}B_0 = E_0$$

$\underline{E = hf}$

$B_0 = \frac{1}{c} E_0 \Rightarrow E_0 \gg B_0$ we forget about B field often as $E \gg B$ for E/M waves interacting, Basically E is major effect.

## §9.3 E/M WAVES IN MATTER

- In homogeneous medium, $\epsilon, \mu$ are constants, Assume $\rho_f \equiv J_f = 0$

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

In terms of $\vec{E}, \vec{B}$

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \epsilon\mu \frac{\partial \vec{E}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

$\nabla \times (\nabla \times \vec{E}) \Rightarrow$ wave equation $\nabla^2 E = \epsilon\mu \frac{\partial^2 E}{\partial t^2} \Rightarrow v = \frac{1}{\sqrt{\epsilon\mu}}$

$\epsilon = \epsilon_r \mu_0$ $\quad \mu = \mu_0(1+\chi_m)$ for practical purp. $\chi_m \ll 1 \Rightarrow \mu = \mu_0$

then $v = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_0}} = \frac{c}{\sqrt{\epsilon_r}}$ with $n \cong \sqrt{\epsilon_r}$

in vacuum $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \Rightarrow u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu} B^2\right)$

$B = \frac{E}{c} = \sqrt{\epsilon_0 \mu_0}\, E \Rightarrow B^2 = \epsilon_0 \mu_0 E^2 \Rightarrow u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{\epsilon_0 \mu_0}{\mu_0} E^2\right)$

See in vacuum E and B carry same energy.

$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \Rightarrow \vec{S} = \frac{1}{\mu}(\vec{E} \times \vec{B})$ for monochromatic waves $|B_0| = \left|\frac{E_0}{v}\right|$

$S = \frac{1}{\mu}\frac{E^2}{v} = \frac{\epsilon\mu v}{\mu} E^2 = \epsilon v E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$ take time average of S to find power. $\frac{1}{T}\int_0^T \cos^2(\vec{k}\cdot\vec{r} - \omega t)\,dt = \frac{1}{2}$

$\langle S \rangle_{\text{time avg}} = I = \frac{1}{2} v \epsilon E_0^2$ = Intensity = Energy flow. = (Energy density)(velocity)

# Reflection and transmission of normal incidence

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad E_1^{\parallel} = E_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$$

Incident Wave:

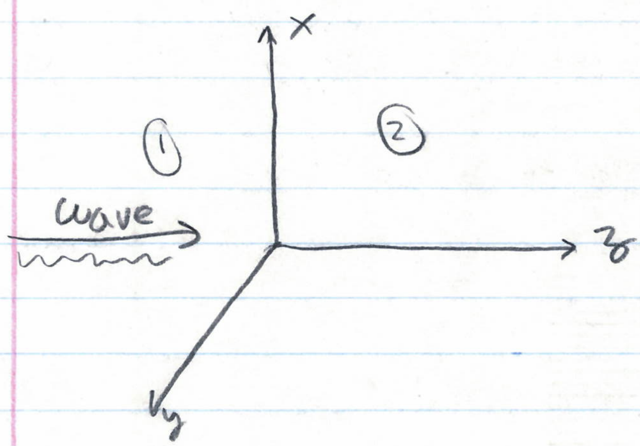
$$\tilde{E}_I = \tilde{E}_0 e^{i(k_1 z - \omega t)} \hat{x} \qquad \tilde{B}_I = \frac{1}{v_1} \tilde{E}_0 e^{i(k_1 z - \omega t)} \hat{y}$$

reflected Wave:

$$\tilde{E}_R = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x}$$
$$\tilde{B}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} (-\hat{y})$$

Transmitted:

$$\tilde{E}_T = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x}$$
$$\tilde{B}_T = \frac{1}{v_2} e^{i(k_2 z - \omega t)} \hat{y}$$




Incident Wave

$$\tilde{E}_I = \tilde{E}_0 e^{i(k_1 z - \omega t)} \hat{x}$$
$$\tilde{B}_I = \frac{1}{v_1} \tilde{E}_0 e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected Wave

$$\tilde{E}_R = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x}$$
$$\tilde{B}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} (-\hat{y})$$

Transmitted Wave

$$\tilde{E}_T = \tilde{E}_{0T} e^{i(k_2 z - \omega t)}$$
$$\tilde{B}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(k_2 z - \omega t)}$$

for $t = 0$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$
$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0T}\right)$$

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

If $\mu = \mu_0 \Rightarrow$

$$\tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I} = \left(\frac{2v_2}{v_1+v_2}\right)\tilde{E}_{0I}$$

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0I}$$

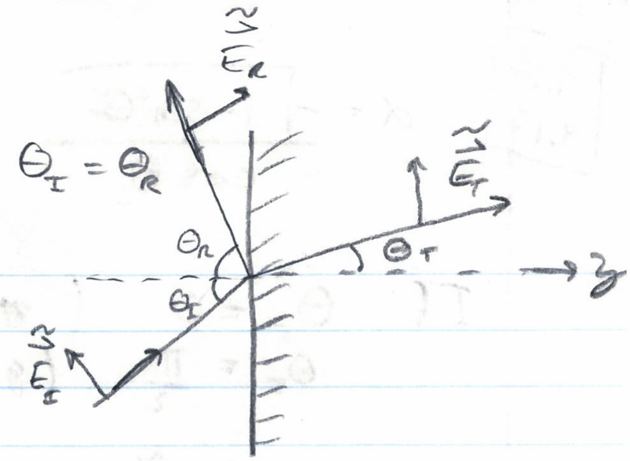
What fraction of energy is reflected or transmitted

$$\epsilon_1(\tilde{E}_{0I} + \tilde{E}_{0R})_z = \epsilon_2(\tilde{E}_{0T})_z$$

$$(\tilde{B}_{0I} + \tilde{B}_{0R})_z = (\tilde{B}_{0T})_z$$

$$(\tilde{E}_{0I} + \tilde{E}_{0R})_{x,y} = (\tilde{E}_{0T})_{x,y}$$

$$\frac{1}{\mu_1}(\tilde{B}_{0I} + \tilde{B}_{0R})_{x,y} = \frac{1}{\mu_2}(\tilde{B}_{0T})_{x,y}$$




For polarization in the plane of incidence
• just use 2 of the above BC's

$$\epsilon_{01}(-\tilde{E}_{0I}\sin\Theta_I + \tilde{E}_{0R}\sin\Theta_R) = -\epsilon_2\tilde{E}_{0T}\sin\Theta_T$$

$$\tilde{E}_{0I}\cos\Theta_I + \tilde{E}_{0R}\cos\Theta_R = \tilde{E}_{0T}\cos\Theta_T$$

$$\theta_I = \theta_R \Rightarrow \quad \tilde{E}_{0R} - \tilde{E}_{0I} = -\frac{\epsilon_2}{\epsilon_1}\frac{\sin\Theta_T}{\sin\Theta_I}\tilde{E}_{0T}$$

$$\tilde{E}_{0R} - \tilde{E}_{0I} = -\sqrt{\frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}\,\tilde{E}_{0T} = \sqrt{\frac{\epsilon_2\mu_2}{\epsilon_1\mu_1}}\cdot\frac{\mu_1}{\mu_2} = \underbrace{-\frac{v_1}{v_2}\frac{\mu_1}{\mu_2}}_{\beta} = \tilde{E}_{0T}$$

$$\begin{cases} \tilde{E}_{0I} - \tilde{E}_{0R} = \beta\tilde{E}_{0T} \\ \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}\underbrace{\frac{\cos\Theta_T}{\cos\Theta_I}}_{\alpha} = \alpha\tilde{E}_{0T} \end{cases}$$

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}$$

$\boxed{\begin{matrix}9.16\\9.17\end{matrix}}$ $\alpha = \dfrac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \dfrac{\sqrt{1-\left(\frac{n_1 \sin\theta_I}{n_2}\right)^2}}{\cos\theta_I}$ $\boxed{\beta = \dfrac{n_2}{n_1}}$

If $\theta_I = 0$ (normal incidence, same as before)
$\theta_I = \frac{\pi}{2}$ (grazing incidence

$\Rightarrow \tilde{E}_{0T} = 0 \quad \tilde{E}_{0I} = \tilde{E}_{0R}$ total reflection

If $\alpha = \beta$ no reflection, total transmission called Brewster Angle. ($\tilde{E}_{0T} = \tilde{E}_{0I}/\alpha$

$$\frac{\sqrt{1-\left(\frac{n_1\sin\theta_B}{n_2}\right)^2}}{\sqrt{1-\sin^2\theta_B}} = \beta \Rightarrow \sin^2\theta_B = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2-\beta^2} \quad : \text{If } \mu_1 \cong 1$$

$$\sin^2\theta_B = \frac{\beta^2}{1+\beta^2} \quad \therefore \quad \beta = \tan\theta_B$$

$$\cos^2\theta_B = \frac{1}{1+\beta^2}$$

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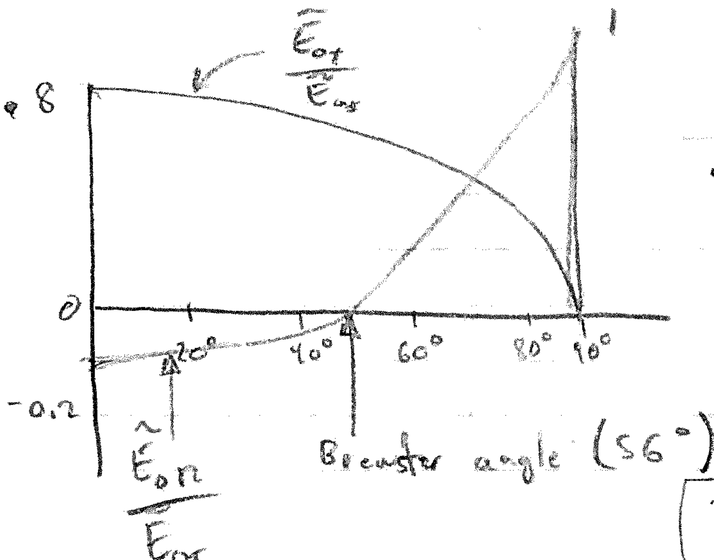



Power/unit area striking surface

$$S \cdot \hat{z} = \frac{E_{0I}^2}{\mu_1 V_1}\cos^2(\vec{k}_1 \cdot \vec{r} - \omega t)\cos\theta$$

$$\langle S \cdot \hat{z}\rangle_{AVG.} = \frac{1}{2}\frac{E_{0I}^2}{\mu_1 V_1}\cos\theta_I$$

$$= \frac{1}{2}\frac{\epsilon_1}{\mu_1\epsilon_1 V_1}\cos\theta_I$$

$$\boxed{I_{inc} = \frac{1}{2}\epsilon_1 V_1 E_{0I}^2 \cos\theta_I}$$

$$\boxed{I_R = \frac{1}{2}\epsilon_1 V_1 E_{0R}^2\cos\theta_R} \qquad \boxed{I_T = \frac{1}{2}\epsilon_1 V_2 E_{0T}^2\cos\theta_T}$$

Reflection coefficient //

$$R = \frac{I_R}{I_{inc}} = \frac{E_{0R}^2\cos\theta_I}{E_{0I}^2\cos\theta_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 \qquad T = \frac{I_T}{I_{inc}} = \frac{\frac{1}{2}\epsilon_2 V_2 E_{0T}^2\cos\theta_T}{\frac{1}{2}\epsilon_1 V_1 E_{0I}^2\cos\theta_I} = \left(\frac{2}{\alpha+\beta}\right)^2\alpha\beta$$

$\beta \quad \frac{2}{\alpha+\beta} \quad \alpha$

AND $\boxed{R+T=1}$

## § 9.4 Absorption and Dispersion

In a Conductor $\vec{J}_f = \sigma \vec{E}$

$$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$$

$$\sigma \nabla \cdot \vec{E} = \frac{\sigma}{\epsilon} \nabla \cdot \vec{D} = \frac{\sigma}{\epsilon} \rho_f = -\frac{\partial \rho_f}{\partial t}$$

$$\rho_f = \rho_f(0) e^{-\sigma/\epsilon\, t}$$

For conductors $\sigma$ is large $\Rightarrow \rho_f = 0$ quickly hence

(i) $\nabla \cdot \vec{E} = 0$

(ii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ take curl of ii and subst into iv.

(iii) $\nabla \cdot \vec{B} = 0$ take curl of ? and subs. into ?

(iv) $\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \sigma \vec{E}$

2 eq's $\Downarrow$

$$\left[\begin{array}{ll} (v) & \nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \\ (vi) & \nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t} \end{array}\right]$$

Solve! Find gen. sol$^{\underline{n}}$.

$k \neq \kappa$

If we assume $\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i k (z - \omega t)}$

using (v) $\Rightarrow \tilde{k}^2 = \mu \epsilon \omega^2 + i \mu \sigma \omega$

Define, $\tilde{k} \equiv k + i\kappa \Rightarrow \tilde{k}^2 = k^2 - \kappa^2 + 2k\kappa i$

$\therefore \quad 2k\kappa = \mu\sigma\omega$

$\therefore \quad k^2 - \kappa^2 = \mu\epsilon\omega^2$

$$\Rightarrow k^2 - \left(\frac{\mu\sigma\omega}{2k}\right)^2 = \mu\epsilon\omega^2$$

Algebra,

$$k^4 - \mu\epsilon\omega^2 k^2 - \frac{\mu^2\sigma^2\omega^2}{4} = 0 \quad \text{quad. in } k^2 \quad k^2 = \frac{\mu\epsilon\omega^2 + \sqrt{(\mu\epsilon\omega^2)^2 - \mu^2\sigma^2\omega^2}}{2}$$

thus $k^2 = \frac{\mu\epsilon\omega^2}{2}\left(1 + \sqrt{1 + \sigma^2/\epsilon^2\omega^2}\right) \Rightarrow k = \sqrt{\frac{\epsilon\mu}{2}}\left(1 + \sqrt{1 + \sigma^2/\epsilon^2\omega^2}\right)^{1/2}$

KAPPA $= \kappa = \omega \sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1 + \sigma^2/\epsilon^2\omega^2} - 1\right)^{1/2}$

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(1+\sqrt{1+\sigma^2/\epsilon^2\omega^2}\right)^{1/2}$$

note $\kappa \Rightarrow e^{-\kappa z}$ !
exp. decay

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\sigma^2/\epsilon^2\omega^2}-1\right)^{1/2}$$

skin depth : $\boxed{d \equiv \frac{1}{\kappa}}$ $\quad k = \frac{2\pi}{\lambda} \Rightarrow \boxed{\lambda = \frac{2\pi}{k}}$

Now check $\vec{B}$ eq'n //

$$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \left(\frac{\partial \tilde{E}_x}{\partial z} - \underbrace{\frac{\partial \tilde{E}_z}{\partial x}}_{=0}\right)\hat{y}$$

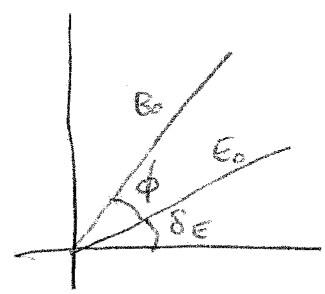
$$\tilde{\vec{B}}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0\,\hat{y} \quad\Rightarrow\quad \boxed{\tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0}$$

$\tilde{B}_0$ and $\tilde{E}_0$ no longer in phase necessarily by $\tilde{k} \in \mathbb{C}$

$$\tilde{k} = k + i\kappa = K e^{i\phi}$$

$$K = \sqrt{k^2+\kappa^2} = \sqrt{\frac{\omega^2\epsilon\mu}{2}\left(1+\sigma^2/\epsilon^2\omega^2\right)}$$

$$\phi = \tan^{-1}\left(\frac{\kappa}{k}\right) = \tan^{-1}\left(\frac{\sqrt{1+\sigma^2/\epsilon^2\omega^2}-1}{\sqrt{1+\sigma^2/\epsilon^2\omega^2}+1}\right)^{1/2}$$




$$\tilde{B}_0 = \frac{K e^{i\phi}}{\omega}\tilde{E}_0$$

$B_0$ and $E_0$ are out of phase by $\phi$ moreover,

$$\delta_B = \delta_E + \phi$$

the real fields are,

$$E(z,t) = E_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E)\,\hat{x}$$
$$B(z,t) = B_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E + \phi)\,\hat{y}$$

at the same phase angle

$$kz - \omega t_E + \delta_E = kz - \omega t_B + \delta_E + \phi$$

$$t_B - t_E = \phi/\omega \Rightarrow \text{phase of } B \text{ lags behind}$$

B reaches that phase at later time

9.18

(b) How thick a silver coating is needed on microwave ($f \doteq 10^{10}$ Hz) experimental instruments

$$\frac{1}{d} \cong k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\sigma^2/\epsilon^2\omega^2}-1\right)^{1/2}$$

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \cancel{\vec{P}}^{\to 0}$$ for good cond use $\epsilon_0$

$$\therefore \frac{1}{d} = \omega\sqrt{\frac{\epsilon_0\mu}{2}}\left(\sqrt{1+\sigma^2/\epsilon_0^2\omega^2}-1\right)^{1/2}$$

$$\frac{\sigma}{\epsilon\omega} = \frac{10^9}{8.8} \Rightarrow d = \sqrt{\frac{2}{\sigma\omega\mu}} = \boxed{0.63\times10^{-5}\,m = d}$$

$$\boxed{63\ \mu m = d}$$

(c) find the wavelength and $v$ in copper for $\omega = 2\pi\times10^6$ Hz

$$k = \frac{2\pi}{\lambda} = \omega\sqrt{\frac{\epsilon\mu}{2}}\left(\sqrt{1+\sigma^2/\epsilon^2\omega^2}+1\right)^{1/2} = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\omega\sigma\mu}{2}}$$

$$\Rightarrow \lambda = 2\pi\sqrt{\frac{2}{\omega\sigma\mu}} = \boxed{400\ \mu m = \lambda}$$

$$v = \frac{\omega}{k} = \frac{2\pi\times10^6}{2\pi/\lambda} = \boxed{0.4\times10^3\,m/s = v}$$

$\boxed{9.23,\ 9.24}$

# Reflection at a Conducting Surface

(i) $\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f$

(ii) $E_1^\parallel - E_2^\parallel$

(iii) $B_1^\perp - B_2^\perp = 0$

(iiii) $\frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel = \vec{K}_f \times \hat{n}$

① Non conducting

$$\vec{\tilde{E}}_I = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x}$$

$$\vec{\tilde{B}}_I = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y}$$

$$\vec{\tilde{E}}_R = \tilde{E}_{0R} e^{i(k_1 z - \omega t)} \hat{x}$$

$$\vec{\tilde{B}}_R = \frac{-\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y}$$

② Conducting

$$\vec{\tilde{E}}_T = \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{x}$$

$$\vec{\tilde{B}}_T = \frac{\tilde{k}_2}{\omega} \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t} \hat{y}$$

at $z = 0$

$$E_1^\parallel = E_2^\parallel \Rightarrow \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

If $K_f = 0$ then $\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0I} - \tilde{E}_{0R}\right) = \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{0T}$

$$\Rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta} \tilde{E}_{0T} \qquad \text{where} \quad \tilde{\beta} \equiv \frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega}$$

$$\therefore \quad \tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I} \qquad \text{and} \qquad \tilde{E}_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$$

$\gamma_2 = K$

$$\tilde{k}_2 = k_2 + i\gamma_2 = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\left(1 + \sqrt{1 + \sigma^2/\epsilon^2\omega^2}\right)^{1/2} + i\left(\sqrt{1+\sigma^2/\epsilon^2\omega^2} - 1\right)^{1/2}\right]$$

$\left.\begin{array}{l}\text{if } \sigma \to \infty \\ \text{then } k_2 \to \infty\end{array}\right\} \Rightarrow$ for perfect conductor reflection is complete $E_{0T} = 0$ $\quad \vec{\tilde{E}}_{0R} = -\vec{\tilde{E}}_{0I}$

Frequency dependent permitivity ($\epsilon$)

$$n \cong \sqrt{\epsilon_r} \quad , \quad \epsilon = \epsilon_r \epsilon_0 \quad \text{if} \quad \epsilon_r = \epsilon_r(\lambda) \text{ ... dispersive}$$

$$f = A\cos(k(z - vt) = A\cos[kz - \omega t]$$

If $v$ is not a const, then the shape of the packet changes so then what is the velocity of the packet?

$$\tilde{y}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)} dk$$

the packet velocity or group velocity ultimately the particle velocity is, we derive next year,

$$v = \frac{\omega}{k} \quad \text{but} \quad V_{group} = \left.\frac{d\omega}{dk}\right|_{k_0}$$

$\frac{\omega}{k} > c$ not contradict relativity, why?

$V_{group}$ is the "real" phenomenon.

---

Since $\epsilon_r$ is related to the polarization which is dipole moment/volume We need to study the molecular structure to get $\epsilon_r$

$$m\frac{d^2x}{dt^2} = F_{tot} = F_{binding} + F_{damping} + F_{driving}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{qE_0}{m}\cos\omega t$$

or

$$\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = \frac{qE_0}{m}e^{-i\omega t}$$ , solve for $\tilde{x}$ get Re part.

$$\tilde{x} = \tilde{x}_0 e^{-i\omega t} \Rightarrow -\omega^2\tilde{x} - i\omega\gamma\tilde{x} + \omega_0^2\tilde{x} = \frac{qE_0}{m}e^{-i\omega t}$$

thus $x_0 = (qE_0/m) \setminus (\omega_0^2 - \omega^2 - i\gamma\omega)$

$$x_0 q \longrightarrow p$$

833-7916
ALEX

The dipole moment of molecule

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$

more generally,

# of $e^-$ in $j^{th}$ level

$$\tilde{p}(t) = \sum_i q\tilde{x}_i = \frac{q^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right) E_0 e^{-i\omega t}$$

$$P(\text{polarization}) = \frac{Nq^2}{m}\left( \qquad \right) E_0 e^{-i\omega t}$$

$$N = \frac{\#\text{ of molecules}}{\text{volume}}$$

$$\tilde{P} = \epsilon_0 \tilde{\chi}_e \tilde{E} \Rightarrow \epsilon_0 \tilde{\chi}_e = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

$$\tilde{\epsilon}_r = 1 + \tilde{\chi}_e = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j\left(\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

$$m\frac{d^2x}{dt^2} = F_{total} = \underset{\sim}{F_{BINDING}} + F_{DAMPING} + F_{DRIVING}$$

$$-m\omega_0^2 x - m\gamma\frac{dx}{dt} + qE_0\cos\omega t$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{qE_0}{m}\cos\omega t$$

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{qE_0}{m}e^{-i\omega t}$$

assume, $\tilde{x} = \tilde{x}_0 e^{-i\omega t}$

$$-\omega^2\tilde{x} - i\omega\gamma\tilde{x} + \omega_0^2\tilde{x} = \frac{qE_0}{m}e^{i\omega t}$$

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The dipole moment,

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}$$

For a molecule

$$\tilde{p}(t) = \sum_i q\tilde{x}_j(t) = q^2/m\sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\gamma_i\omega}\epsilon_0 e^{-i\omega t}$$

↑ eigen freq. of $i$ orbital

$f_i$ = # of $e^-$ in $i$ orbital

$$\underline{P}(\text{polarization}) = \frac{Nq^2}{m}\left(\quad\right)E_0 e^{-i\omega t},\quad N = \frac{\text{No molecules}}{\text{volume}}$$

$$\tilde{P} = \epsilon_0\tilde{\chi}_e\tilde{E} \Rightarrow \epsilon_0\tilde{\chi}_e = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)$$

$$\tilde{\epsilon}_r = 1 + \tilde{\chi}_e = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

$$\nabla^2\left(\vec{\tilde{E}}\right) = \tilde{\epsilon}\mu_0 \frac{\partial^2 \vec{\tilde{E}}}{\partial t^2} \Rightarrow \vec{\tilde{E}} = \vec{\tilde{E}} e^{i(\tilde{k}\cdot z - \omega t)}$$

$$\Rightarrow \vec{\tilde{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

$$\tilde{k}^2 = \tilde{\epsilon}\mu_0\omega^2 \qquad \tilde{k} = \sqrt{\tilde{\epsilon}_0\mu_0}\,\omega = k + i\kappa$$

$$|E^2| \propto |\tilde{E}_0|^2 e^{-2\kappa z} \qquad \alpha = 2\kappa \text{ is called the absorption coeff.}$$

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\,\omega = \sqrt{\tilde{\epsilon}/\epsilon_0\,\epsilon_0\,\mu_0}\quad \omega = \sqrt{\tilde{\epsilon}_r}\,\frac{\omega}{c}$$

if $$\frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} << 1$$

$$\sqrt{1+\delta} \cong 1 + \frac{1}{2}\delta$$

$\gamma_j$ ~ width res
$\omega_j$ ~ freq of res.

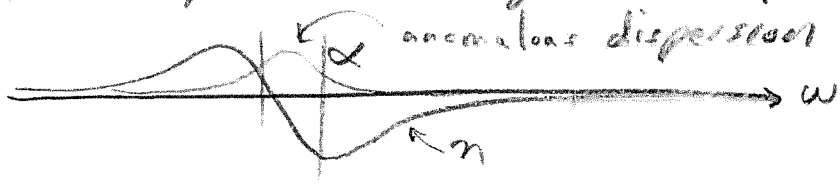
$$\tilde{k} \cong \frac{\omega}{c}\left[1 + \frac{1}{2}\frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$\frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} = \frac{f_j(\omega_j^2 - \omega^2 + i\gamma_j\omega)}{(\omega_j^2 - \omega^2)^2 + (\gamma_j}$$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2-\omega^2)^2 + (\gamma_j\omega)^2}$$

$$\alpha \cong 2\kappa = \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2-\omega^2)^2 + \gamma_j\omega^2}$$

near $\omega_j$ (resonance frequency) $n$ changes quickly
Hence $n$ is not always $> 1$



Group Velocity always less than c → $\boxed{9.25}$

away from resonance

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

For transparent material $\omega_j(\text{min})$ is in the ultraviolet
hence $n > 1$
If $\omega << \omega_j$, $\frac{1}{\omega_j^2-\omega^2} = \frac{1}{\omega_j^2}\left(\frac{1}{1-\omega^2/\omega_j^2}\right) \cong \frac{1}{\omega_j^2}\left(1+\frac{\omega^2}{\omega_j^2}\right)$

$$n \cong 1 + \underbrace{\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right)}_{A} + \omega^2 \underbrace{\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega^4}}_{B}$$

$\Rightarrow \quad n = 1 + A + B\omega^2$ (Cauchy Formula)

**! TEST 2 UP TO THIS POINT !**

# Test 2 Review

## Chapter 8

$$\frac{dW_{mech}}{dt} + \frac{dU_{em}}{dt} = -\oint \vec{S} \cdot d\vec{a}$$

$$\frac{\partial u_{mech}}{\partial t} + \frac{\partial u_{em}}{\partial t} = -\nabla \cdot \vec{S}$$ Conservation, of energy

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) \qquad u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$T_{ij} = \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} - \epsilon_0 \mu_0 \frac{d}{dt}\int \vec{S}\, d\tau$$

momentum carried by the field $\mathcal{P}_{em} = \mu_0 \epsilon_0 \vec{S}$

$$\frac{\partial}{\partial t}\left(\vec{\mathcal{P}}_{em} + \vec{\mathcal{P}}_{mech}\right) = \nabla \cdot \overleftrightarrow{T}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}. \implies \left(-\overleftrightarrow{T} \text{ flow of momentum density}\right)$$

$$\vec{l}_{em} = \vec{r} \times \vec{\mathcal{P}}_{em} = \epsilon_0 \vec{r} \times \left(\vec{E} \times \vec{B}\right)$$

go over examples given in class