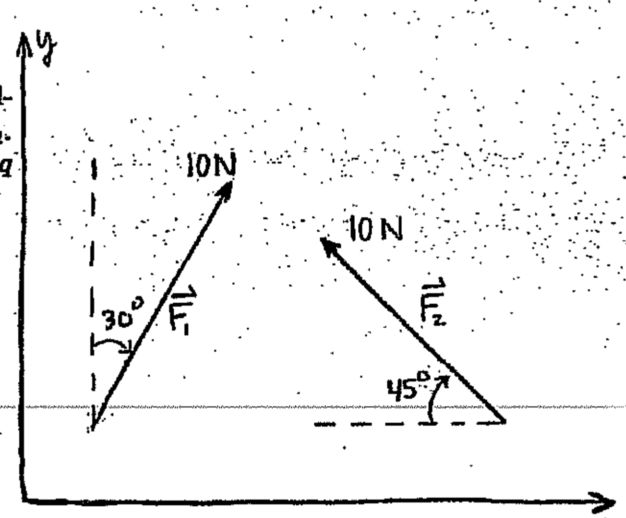
$$\epsilon_0 = 8.854 \times 10^{-12} \frac{C^2}{Nm^2}$$

Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Thanks. Note the term "find" in this test does not merely mean to guess the answer. Without exception the term "find" means to derive from proper physical and mathematical arguments the desired item.

Problem 1 [10pts] You are given forces $\vec{F}_1$ and $\vec{F}_2$ as pictured. Calculate the magnitude and standard angle for $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$. In addition, if $\vec{F}_{net}$ is the net-electric force acting on a charge q then what is the electric field acting on the charge?



(LET $q = 2\mu C$)

Let $F_0 = 10N$.

Let $\vec{F}_1 = \langle F_{1x}, F_{1y} \rangle$ then

$$F_{1x} = F_0 \sin 30° = 0.5 F_0$$
$$F_{1y} = F_0 \cos 30° = 0.866 F_0$$

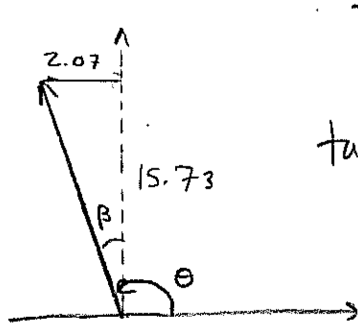
from given diagram.

Likewise, $\vec{F}_2 = \langle F_{2x}, F_{2y} \rangle$ and we can calculate from diagram that

$$F_{2x} = -F_0 \cos 45° = -0.707 F_0$$
$$F_{2y} = F_0 \sin 45° = 0.707 F_0$$

Hence,

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = \langle 5N, 8.66N \rangle + \langle -7.07N, 7.07N \rangle$$
$$= \langle -2.07N, 15.73N \rangle$$



$$\tan^{-1}\left(\frac{2.07}{15.73}\right) = 7.5° = \beta$$

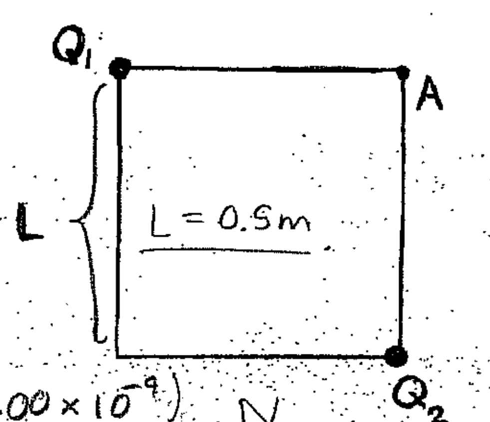
$$\Rightarrow \boxed{\Theta \cong 97.5°}$$

$$F_{net} = \left(\sqrt{(2.07)^2 + (15.73)^2}\right) N \cong \boxed{15.87 N = F_{net}}$$

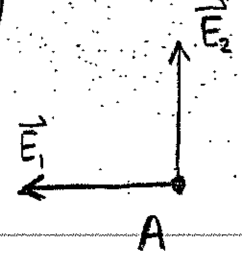
The electric field $\vec{E} = \vec{F}/q = \boxed{7.93 \times 10^6 \frac{N}{C} \text{ at } \Theta = 97.5°}$

Problem 2 [10pts] Two point charges, Q_1 and Q_2, are placed, as shown, at two corners of a square, 50cm on a side. Answer the following:

- (a.) If $Q_1 = -2.00\ nC$ and $Q_2 = +3.00\ nC$, find the electric field (magnitude and direction) at corner A.
- (b.) If a $+3.00\ nC$ charge is placed at A, find the force which acts on it.



(a.)

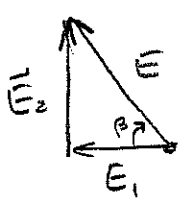


$$E_1 = \frac{kQ_1}{L^2} = \frac{(8.99\times10^{9})(2.00\times10^{-9})}{(0.5)^2}\ \frac{N}{C}$$

$$\Rightarrow \underline{E_1 \cong 71.92\ \frac{N}{C}}$$

$$E_2 = \frac{kQ_2}{L^2} = \frac{(8.99\times10^{9})(3.00\times10^{-9})}{(0.5)^2}\ \frac{N}{C}$$

$$\Rightarrow \underline{E_2 \cong 107.88\ \frac{N}{C}}$$



$$\vec{E} = \vec{E}_1 + \vec{E}_2 = -E_1\,\hat{i} + E_2\,\hat{j}$$

$$E = \sqrt{E_1^2 + E_2^2} = \boxed{129.66\ \frac{N}{C}}$$ ← magnitude

standard angle

$$\Theta = 180^\circ - \tan^{-1}\left(\frac{E_2}{E_1}\right) = 180^\circ - 56.31^\circ = \boxed{123.7^\circ}$$

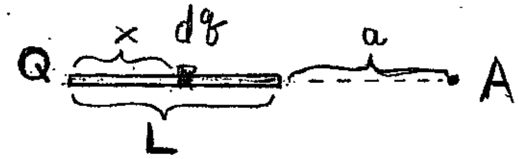
(b.)

$$\vec{F} = q_3\vec{E} = (3.00\times10^{-9}\,C)\left\langle -71.92\,\frac{N}{C},\ 107.88\,\frac{N}{C}\right\rangle$$

$$= \langle -2.158\times10^{-7}\,N,\ 3.236\times10^{-7}\,N\rangle$$

$$\Rightarrow \boxed{\vec{F} = 3.89\times10^{-7}\,N \text{ at } 123.7^\circ}$$

Problem 3 [10pts] A uniformly charges rod of length L carries a total charge $+Q$. Find the electric field (magnitude and direction) at point A, a distance x from the end of the rod.



corrected during test (should be a not x)

$\lambda = \frac{Q}{L}$

$$dE_x = \frac{k\,dq}{r^2} = \frac{k\lambda\,dx}{(a+L-x)^2}, \quad \underbrace{r = a+L-x}$$

($r = a$ for $x = L$, $r = a+L$ for $x = 0$)

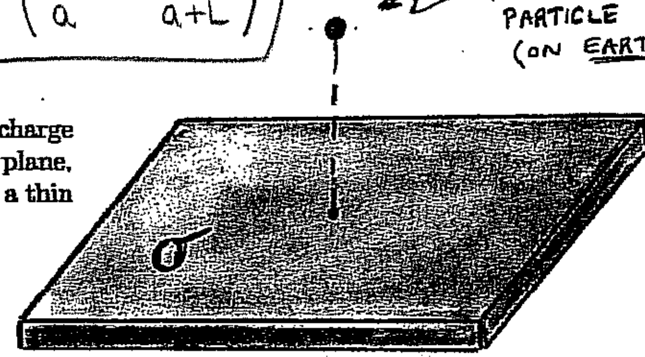
all $\vec{E}$ in x-direction by geometry.

$$E_x = \int_0^L \frac{kQ}{L}\frac{dx}{[a+L-x]^2} \quad : \text{let } u = a+L-x, \; du = -dx, \; u(0) = a+L,\; u(L) = a.$$

$$= \frac{-kQ}{L}\int_{a+L}^{a}\frac{du}{u^2}$$

$$= \frac{kQ}{L}\frac{1}{u}\Big|_{a+L}^{a} = \boxed{\frac{kQ}{L}\left(\frac{1}{a} - \frac{1}{a+L}\right)}$$ in the $\hat{i}$ direction

Problem 4 [10pts] A particle with a mass of $m = 0.005\ kg$ and charge $q = 4\ \mu C$ is observed to hover motionless over a horizontal plane. Find the area charge density σ of the plane assuming it is a thin insulator with a uniformly distributed charge.



(VERY LARGE AREA, PARTICLE NEAR MIDDLE)

$$E = \frac{\sigma}{2\epsilon_0}$$

Both $\vec{F}_e$ and $\vec{F}_g$ are purely vertical.

$$\frac{q\sigma}{2\epsilon_0} = mg \longrightarrow \sigma = \frac{2mg\epsilon_0}{q}$$

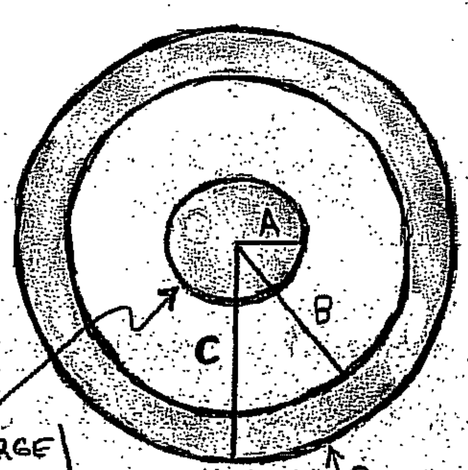
$$= \frac{2(0.005\ kg)(9.8\ m/s^2)(8.854\times10^{-12}\ C^2/Nm^2)}{4\times10^{-6}\ C}$$

$$= \boxed{2.169\times10^{-7}\ \frac{C}{m^2} = \frac{dQ}{dA} = \sigma}$$

(I gave bonus points if you showed why $E = \frac{\sigma}{2\epsilon_0}$ here.)

Problem 5 [10pts] A solid conducting sphere of radius A carries a total charge Q. It is surrounded by a concentric conducting spherical shell of inner radius B and outer radius C. The conducting shell carries a total charge $2Q$. Use Gauss's law to:

- (a.) Find the magnitude of the electric field at a distance r from the center where $A < r < B$.
- (b.) Find the magnitude of the electric field at a distance r from the center where $r > C$.
- (c.) Find the total amount of charge which is on the inner surface($r = B$) of the shell.
- (d.) Find the total amount of charge on the outer surface($r = C$) of the shell.



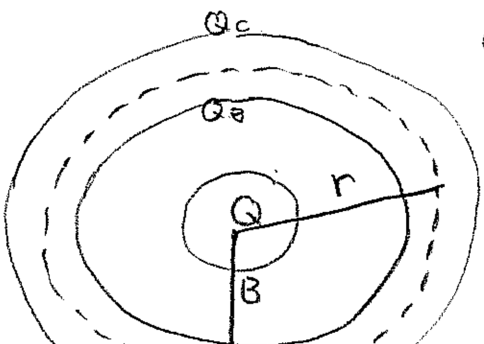
a.) for $A < r < B$ we enclose charge Q from the inner sphere hence,

$$\Phi = \frac{Q_{enc}}{\epsilon_0} \Rightarrow \underbrace{E(4\pi r^2)}_{\text{By spherical symmetry.}} = \frac{Q}{\epsilon_0} \quad \therefore \boxed{E = \frac{Q}{4\pi\epsilon_0}} \quad A < r < B.$$

b.) for $r > C$ we enclose $Q + 2Q = 3Q$ since the inner & outer conductors are inside the sphere of radius r for $r > C$. Hence,

$$\Phi = \frac{Q_{enc}}{\epsilon_0} \Rightarrow \underbrace{E(4\pi r^2)}_{\text{By spherical symmetry.}} = \frac{3Q}{\epsilon_0} \quad \therefore \boxed{E = \frac{3Q}{4\pi\epsilon_0}} \quad r > C$$

c.)



Consider S_r with $B < r < C$ as pictured. By symmety $\Phi = (4\pi r^2)E$ But $E = 0$ inside conductor thus

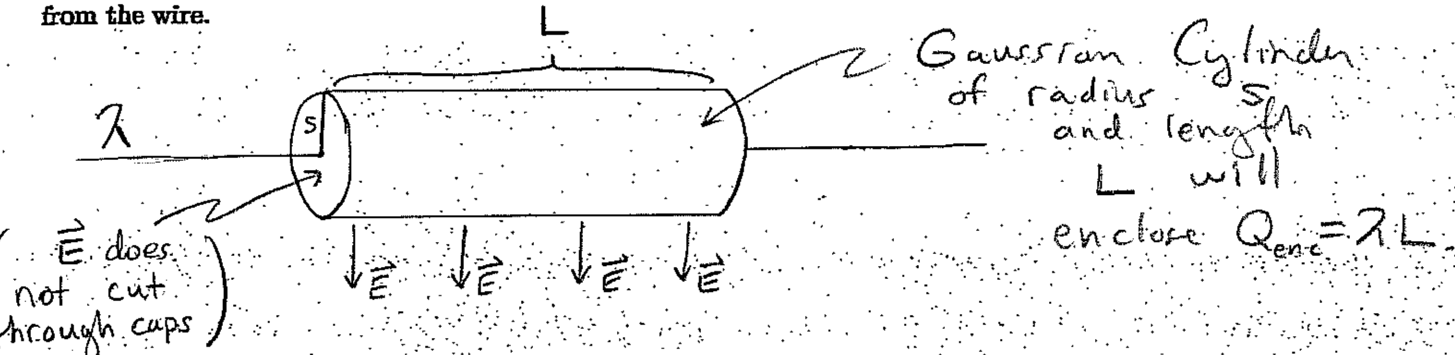
$$\Phi = 0 = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}(Q + Q_B)$$

Thus $\boxed{Q_B = -Q}$ ← charge at $r = B$.

d.) $Q_{\substack{TOTAL \\ OUT\ SHELL}} = Q_B + Q_C = -Q + Q_C = 2Q$

$$\therefore \boxed{Q_C = 3Q}$$

Problem 6 [10pts] A very long wire has a uniform linear charge density of λ. Show that the electric field due to the wire has a magnitude which is proportional to $1/s$ where s is the distance from the wire.

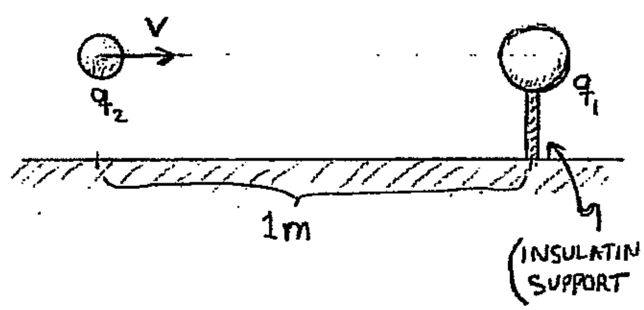


$$\Phi = E(2\pi s)L = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

$$\therefore \quad \boxed{E = \frac{\lambda}{2\pi\epsilon_0 s}}$$

Problem 7 [10pts] A small sphere carrying a uniformly distributed charge $q_1 = 8.00\ \mu C$, is held in stationary postion by insulating supports. A second small sphere, carrying a uniformly distributed charge $q_2 = 2.00\ \mu C$, and having a mass of 4.00 grams, is projected toward q_1. When the two spheres are 1.00 m apart, q_2 is moving with speed $v = 30.0\ m/s$.

- (a.) What is the speed of q_2 when it is 0.500 m from q_1 ?
- (b.) How close does q_2 get to q_1?



(a.) conservation of energy:

$$\frac{kq_1q_2}{r_1} + \frac{1}{2}mv^2 = \frac{kq_1q_2}{r_2} + \frac{1}{2}mV_2^2$$

Given $r_1 = 1.00m$, $r_2 = 0.500m$, $m = 0.004kg$
$q_1 = 8.00 \times 10^{-6}\,C$ and $q_2 = 2.00 \times 10^{-6}\,C$. Solve for V_2

$$V_2 = \sqrt{\frac{2kq_1q_2}{m}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] + V^2}$$

$$= \left\{\left[\frac{2 \times 8.99 \times 10^9 \times 8.00 \times 10^{-6} \times 2.00 \times 10^{-6}}{0.004}\right]\left(1 - \frac{1}{0.5}\right) + 30^2\right\}^{\frac{1}{2}}$$

$$= \sqrt{828.08}\ \frac{m}{s} \Rightarrow \boxed{V_2 = 28.78\ \frac{m}{s}}$$

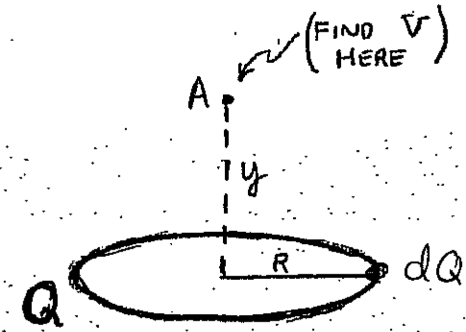
(b.) when $\underbrace{\frac{kq_1q_2}{r_1} + \frac{1}{2}mv^2}_{1.94J} = \frac{kq_1q_2}{r_f}$ ($V_f = 0$ just as it turns around.)

$$1.94J = \frac{kq_1q_2}{r_f} \longrightarrow r_f = \frac{(8.99\times10^9)(8\times10^{-6})(2\times10^{-6})}{1.94}\ m$$

$$\therefore \boxed{r_f = 0.0741\ m}$$

Problem 8 [10pts] A uniformly charged circular ring of radius R carries a total charge Q. Find the electrostatic potential V at a point on the axis of the ring a distance y from the center. Verify that your answer reduces to $V = \frac{Q}{4\pi\epsilon_0 R}$ for $y = 0$.

(ASSUME $V(\infty) = 0$)



dV due to dQ at A is given by

$$dV = \frac{k\,dQ}{\sqrt{R^2+y^2}}$$

(each dQ is the same distance from A in this geometry)

$$V = \int_{\text{Ring}} \frac{k\,dQ}{\sqrt{R^2+y^2}}$$

$$= \frac{k}{\sqrt{R^2+y^2}} \int_{\text{Ring}} dQ$$

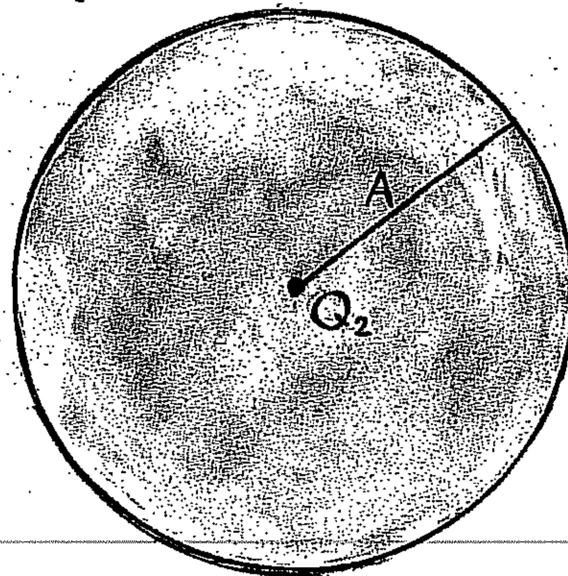
$$\therefore \quad \boxed{V(y) = \frac{kQ}{\sqrt{R^2+y^2}}}$$

Note that $V(0) = \frac{kQ}{\sqrt{R^2+0^2}} = \frac{kQ}{R} = \frac{Q}{4\pi\epsilon_0 R}$

(as it ought)

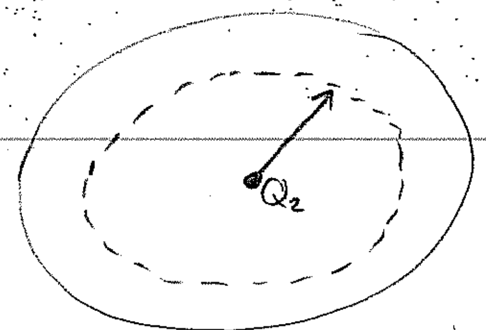
Problem 9 [20pts] Suppose a charge Q_1 is uniformly distributed over a sphere of radius A. Furthermore, suppose an additional Q_2 is placed at the center of the sphere. Find the magnitude of the electric field for **(a.)** $0 < r \leq A$ and **(b.)** $r \geq A$ where r is the radial distance from the center of the sphere. **(c.)** calculate the potential V as a function of r assuming that $V(\infty) = 0$. **(d.)** sketch the electric field strength and potential as a function of radius on a single plot.

(Q_2 IS POINT CHARGE AT $r=0$



(Q_1 IS SPREAD OUT OVER THE WHOLE SPHERE.)

(a.) Draw a Gaussian sphere at r for $0 < r \leq A$



$$Q_{enc} = Q_2 + \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi A^3}\right) Q_1$$

$$\Phi = \underbrace{E(4\pi r^2)}_{\text{by spherical symmetry.}} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow \boxed{E = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_2}{r^2} + \frac{r\,Q_1}{A^3}\right)}$$

(b.) If $r \geq A$ then the sphere at r will enclose $Q_1 + Q_2$ hence $\Phi = (4\pi r^2)E = \frac{Q_1 + Q_2}{\epsilon_0}$ $\therefore$ $\boxed{E = \frac{Q_1 + Q_2}{4\pi\epsilon_0 r^2}}$

(c.) Since $E = E(r)$ it follows $E = -\frac{dV}{dr} \Rightarrow V(r) = -\int E(r)\,dr$

$$V(r) = \begin{cases} k\left(\frac{Q_2}{r} - \frac{Q_1}{2A^3} r^2\right) + C_1 & \text{if } 0 \leq r \leq A. \\ k\left(\frac{Q_1 + Q_2}{r}\right) + C_2 & \text{if } r \geq A. \end{cases}$$

Since $V(\infty) = 0 \Rightarrow C_2 = 0$. Then continuity at $r = A \Rightarrow k\left(\frac{Q_2}{A} - \frac{Q_1}{2A}\right) + C_1 = k\left(\frac{Q_1 + Q_2}{A}\right)$

Thus $C_1/k = Q_1\left(\frac{1}{A} + \frac{1}{2A}\right) + Q_2\left(\frac{1}{A} - \frac{1}{A}\right) = \frac{3Q_1}{2A}$

$$\therefore \boxed{V(r) = \begin{cases} k\left(\frac{Q_2}{r} - \frac{Q_1}{2A^3} r^2 + \frac{3Q_1}{2A}\right) & : 0 \leq r \leq A \\ k\left(\frac{Q_1 + Q_2}{r}\right) & : r \geq A \end{cases}}$$

Plot ↓

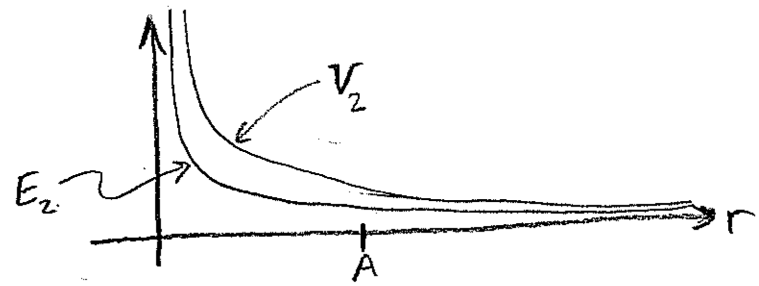
Problem 9 [20pts] Suppose a charge Q_1 is uniformly distributed over a sphere of radius A. Furthermore, suppose an additional Q_2 is placed at the center of the sphere. Find the magnitude of the electric field for **(a.)** $0 < r \le A$ and **(b.)** $r \ge A$ where r is the radial distance from the center of the sphere. **(c.)** calculate the potential V as a function of r assuming that $V(\infty) = 0$. **(d.)** sketch the electric field strength and potential as a function of radius on a single plot.

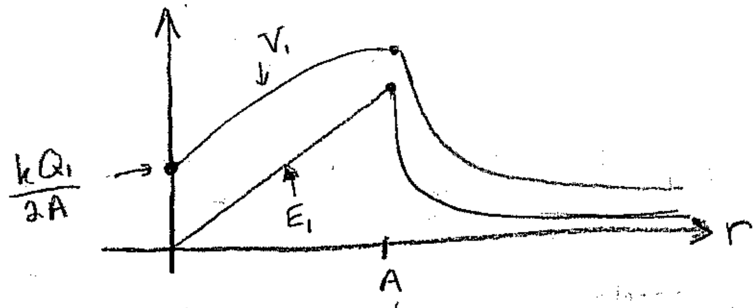
$$\text{(d.)}\quad V(r) = k \begin{cases} \dfrac{Q_2}{r} - \dfrac{Q_1}{2}\left(\dfrac{r^2}{A^3} - \dfrac{3}{A}\right) & : \ 0 \le r \le A \\[2ex] \dfrac{Q_1 + Q_2}{r} & : \ r \ge A \end{cases}$$

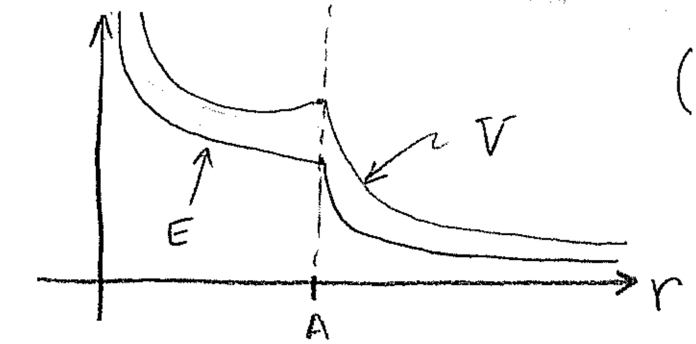
$$V_2 = \frac{kQ_2}{r} \qquad V_1 = \begin{cases} -\dfrac{kQ_1}{2}\left(\dfrac{r^2}{A^3} - \dfrac{3}{A}\right) & : \ 0 \le r \le A \\[2ex] Q_1/r & : \ r \ge A \end{cases}$$

$$E_2 = \frac{kQ_2}{r^2} \qquad E_1 = \begin{cases} \dfrac{kQ_1 r}{A^3} & : \ 0 \le r \le A \\[2ex] Q_1/r^2 & : \ r \ge A \end{cases}$$

We can view $V = V_1 + V_2$ and $\vec{E} = \vec{E}_1 + \vec{E}_2$
and in fact $E = E_1 + E_2$ since $\vec{E}_1$ & $\vec{E}_2$ are both radial.







(roughly)