<u>Definition</u> Let G be a group whose operation is written multipicatively and let $m: G\times G \to G$ and $i: G \to G$ be the functions defined by $m(x,y) = xy$, $i(x) = x^{-1}$ for $x, y \in G$. In this case G is called a <u>Lie group</u> provided G is a manifold and the maps m and i are both smooth mappings. It is understood that $G\times G$ is to be given the product manifold structure induced from the manifold structure of G.

Examples. It is easy to show that $(\mathbb{R}^n, +)$ is a Lie group in which the operation is written additively. Another example is the set of complex numbers with modulus 1. Thus $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$ is given the manifold structure induced on it as a submanifold of $\mathbb{R}^2$ and one must show the operations m and i are smooth relative to this structure. Still another example is the set $\mathcal{GL}(n)$ of all nonsingular matrices. Since the set of all $n\times n$ matrices is a complete normed linear space under the usual definition

$$(A+B)^i_j = A^i_j + B^i_j$$

$$(cA)^i_j = cA^i_j$$

$$\|A\| = \sqrt{\sum_{i,j=1}^{n} (A^i_j)^2}$$

and since $\mathcal{GL}(n)$ is an open subset of $gl(n)$, we see that $\mathcal{GL}(n)$ has a rather trivial manifold structure it inherits as an open subset of a finite dimensional complete normed linear space. The mapping $\hat{m}: gl(n) \times gl(n) \to gl(n)$ defined by $\hat{m}(A,B) = AB$ is bilinear and ~~thus~~ is smooth. Consequently $m = \hat{m} \mid (\mathcal{GL}(n) \times \mathcal{GL}(n)) : \mathcal{GL}(n) \times \mathcal{GL}(n) \to \mathcal{GL}(n)$ is smooth. If $A \in \mathcal{GL}(n)$ then A^{-1} may be written as $\left(\frac{1}{\det A}\right) A^{cf}$ where A^{cf} is the "cofactor" matrix of A. The entries of A^{cf} are polynomials of the entries A^i_j of A as is also $\det A$. Thus the entries of A^{-1} are rational functions of the entries of A and since $\det A \neq 0$ for all $A \in \mathcal{GL}(n)$, it follows that the mapping $i: \mathcal{GL}(n) \to \mathcal{GL}(n)$ defined by $i(A) = A^{-1}$ is smooth. Thus $\mathcal{GL}(n)$ is a Lie group.

Remark. Observe that the set of all $n \times n$ complex matrices, $gl(n, \mathbb{C})$, is a real vector space. Its real dimension is the same as that of $gl(n) \oplus gl(n)$ and so is $2n^2$. Moreover $gl(n, \mathbb{C})$ is an inner product space relative to the inner product

$$\langle A, B \rangle = \sum_{i,j=1}^{n} A^i_j \bar{B}^i_j$$

where $\bar{z}$ denotes the conjugate of the complex number z.

This real inner product space is complete in its induced norm and so $gl(n, \mathbb{C})$ is a finite dimensional real normed linear space.

The set of all nonsingular $n\times n$ complex matrices $Gl(n,\mathbb{C})$ is an open subset of $gl(n,\mathbb{C})$ and is given the induced manifold structure. Thus $Gl(n,\mathbb{C})$ is a <u>real</u> manifold. It is not difficult to prove that $Gl(n,\mathbb{C})$ is a real Lie group. Many Lie groups of interest are a subgroup of this group.

Remark Notice that if G is any Lie group and $a\in G$ then $\{a\}\times G$ is a submanifold of $G\times G$. Moreover the mappings from G to $\{a\}\times G$ and from $\{a\}\times G$ ~~to~~ G defined by $x\to(a,x)$ and $(a,x)\to ax$ are smooth mappings. Thus their composite is smooth. This <u>composite</u> is called <u>left multiplication</u> (by a) and is denoted l_a. This $l_a: G\to G$ is smooth and is given by $l_a(x)=ax$ for all $x\in G$. Similarly the mapping $r_a: G\to G$ defined by $r_a(x)=xa$ is called <u>right multiplication</u> (by a) and is smooth.

Theorem Let G be a Lie group, e the identity of G, and (U,x) an admissible chart of G with $e\in U$. Then $l_a(U)$ is open for each $a\in G$ and

$$\mathcal{A}=\{(l_a(U),\, x\circ l_a^{-1}) \mid a\in G\}$$

is an atlas of admissible charts of G.

Proof First observe that for each $a \in G$, l_a has an inverse and $(l_a)^{-1} = l_{a^{-1}}$. Thus $(l_a)^{-1}$ is smooth and l_a is a diffeomorphism of G. Thus $l_a(U)$ is open in G and, for each $a \in G$, $x \circ l_a^{-1} : l_a(U) \to x(U)$ is a smooth mapping (since x is an admissible chart of G, clearly x is smooth!). Thus $\mathcal{A}$ is a family of admissible charts and so is a subatlas of the differentiable structure of G.

Most of the Lie groups which occur in Physics are groups of matrices and many of those are isometry groups of some metric. We now prepare to show that a large class of isometry groups of finite dimensional vector spaces are indeed Lie groups.

Let V denote a finite dimensional vector space and g a metric on V. Since g is symmetric we know that there exists a basis of V relative to which the matrix G of g satisfies $G_{ij} = \pm \delta_{ij}$. For each $x \in V$ let $\vec{x}$ denote the column vector of components of x relative to this preferred basis. Then $g(x,y) = \vec{x}^t G \vec{y}$ for all $x, y \in V$. If α is an isometry of (V, g) then its matrix A has the property that

$$(A\vec{x})^t G (A\vec{y}) = g(\alpha(x), \alpha(y)) = g(x,y) = \vec{x}^t G \vec{y}$$

so that

$$\vec{x}^t A^t G A \vec{y} = \vec{x} G \vec{y}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. Thus

$$\vec{x}^t (A^t G A - G) \vec{y} = 0$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and thus $A^t G A = G$. It is easy to show that when A is the matrix of a linear transformation α relative to our preferred basis such that $A^t G A = G$ then α is an isometry, so the condition $A^t G A = G$ characterizes isometries of g.

Theorem If $J \in gl(n)$ has the properties that $J^2 = I$ and $J^t = J$ then

$$G_J = \{ A \in gl(n) \mid A^t J A = J \}$$

is a Lie group. Moreover, for any such J,

$$G_J^{\mathbb{C}} = \{ A \in gl(n, \mathbb{C}) \mid A^\dagger J A = J \}$$

is also a real Lie group. Here $A^\dagger$ denotes $\bar{A}^t$.

Remark. To prove the latter Theorem we must show G_J is a manifold. One way this can be done is to exhibit a single chart x of G_J at the identity $e \in G_J$ and then to left translate this chart over all of G_J as in Theorem ; i.e. define

$$\mathcal{A} = \{ (l_a(U), x \circ l_a^{-1}) \mid a \in G_J \}$$

and show that $\mathcal{A}$ is an atlas. There is a natural way to do this using the matrix exponential and we will discuss this idea in more detail later, but at the moment we choose to define a manifold structure on G_J by showing that it is the level surface of an appropriate function. This method establishes that G_J is actually a submanifold of $Gl(n)$. We will then show that the group operations are smooth relative to ~~this~~ submanifold structure. To show this we first need a technical lemma.

Lemma If S is a submanifold of a manifold M and $f : M \to N$ is a smooth function from M into a manifold N then $f|S : S \to N$ is smooth.

Proof To show that $f|S$ is smooth choose $p \in S$ and show that $f|S$ is smooth on some neighborhood of p in S. Since S is a submanifold of M there is an admissible chart (U, x) of M such that $p \in U$, $x(p) = 0$ and $x(U \cap S) = x(U) \cap \mathbb{R}^k \times \{0\}$ where $x(U) \subseteq \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^l$ is open for $k < m$. Recall that the mapping x_U defined on $U \cap S$ by $x_U = x|(U \cap S)$ is an admissible chart of S by the definition of the differentiable structure of S. Since $f : M \to N$ is smooth we know that each coordinate representative of f is smooth and so matrices in $gl(n)$. So we apply the theorem to $f: gl(n) \to S(n)$. We must show that $D_A f$ has rank $\dim S(n)$ for each $A \in f^{-1}(0)$. Let us compute $D_A f$ by finding the "linear approximation" to f at A. Note that

$$\begin{aligned} f(A+H)-f(A) &= (A+H)^t J(A+H) - A^t J A \\ &= A^t J A + H^t J A + A^t J H + H^t J H - A^t J A \\ &= H^t J A + A^t J H + H^t J H. \end{aligned}$$

Clearly the mapping $H \to H^t J A + A^t J H$, $H \in gl(n)$, is linear and since $\|BC\| \le \|B\|\|C\|$ for $B, C \in gl(n)$ we also see that $\|H^t J H\| \le \|H\|^2 \|J\|$. Thus if we choose

$$D_A f(H) = H^t J A + A^t J H$$

then

$$\frac{\|f(A+H) - f(A) - D_A f(H)\|}{\|H\|} \le \|H\| \|J\|$$

and since $\|H\|\|J\| \to 0$ as $\|H\| \to 0$ we see that the Frechet derivative of f at A is given by

$$D_A f(H) = (A^t J H) + (A^t J H)^t.$$

Recall that if $L: V \Rightarrow W$ is a linear map from one vector space V to another vector space W then the rank of L is $\dim V - \dim(\operatorname{Ker} L)$. Observe that if $H \in \operatorname{Ker} D_A f$, then $D_A f(H) = 0$ and this implies

$$A^t J H = -(A^t J H)$$

which implies that $B = A^t J H$ is skew-symmetric. Thus $H = (A^t J)^{-1} B$ for some skew-symmetric matrix B (note that $A^t J$ <u>is</u> invertible since

$A \in f^{-1}(0)$ implies that $A^t J A = J$ which in turn implies that $\det A \neq 0$). Conversely we claim that if $H = (A^t J)^{-1} B$ then H is in $\operatorname{Ker} D_A f$. Indeed in this case $A^t J H = B$ and

$D_A f(H) = A^t J H + (A^t J H)^t = B + B^t = 0$.

So

$$\operatorname{Ker} D_A f = (A^t J)^{-1}(\mathcal{a})$$

where $\mathcal{a}$ is the subspace of all skew-symmetric matrices in $gl(n)$. Since the mapping $X \to (A^t J)^{-1} X$ is an isomorphism from $gl(n)$ onto $gl(n)$ which carries $\mathcal{a}$ onto $\operatorname{Ker} D_A f$ we see that the mapping from $f^{-1}(0)$ into the integers defined by $A \longrightarrow \dim \operatorname{Ker} D_A f$ is constant and is equal to $\dim \mathcal{a}$. Since every matrix A may be written as $\frac{1}{2}(A+A^t) + \frac{1}{2}(A-A^t)$ we see that $gl(n) = \mathcal{S}(n) + \mathcal{a}$. Moreover since the only matrix which is both symmetric and anti-symmetric is 0 we see that

$$gl(n) = \mathcal{S}(n) \oplus \mathcal{a}$$

Thus

$$\begin{aligned} \operatorname{rk} D_A f &= \dim gl(n) - \dim \operatorname{Ker} D_A f \\ &= \dim gl(n) - \dim \mathcal{a} \\ &= \dim \mathcal{S}(n). \end{aligned}$$

It follows that f maps $gl(n)$ into $\mathcal{S}(n)$ and that the rank of $D_A f$ is the dimension of $\mathcal{S}(n)$ for each $A \in f^{-1}(0)$. Thus $f^{-1}(0)$ is a submanifold of $gl(n)$. Since $Gl(n)$ is an open subset of $gl(n)$ which contains $f^{-1}(0)$, $f^{-1}(0)$ is also a submanifold of $Gl(n)$.

The mappings $m: Gl(n) \times Gl(n) \to Gl(n)$ and $i: Gl(n) \to Gl(n)$ defined by $m(A,B) = AB$ and $i(A) = A^{-1}$, respectively are smooth mappings. Thus their restrictions to the respective submanifolds $f^{-1}(0) \times f^{-1}(0)$ and $f^{-1}(0)$ are smooth. It follows that $G_J = f^{-1}(0)$ is a Lie group.

Clearly the argument may be modified to show that $G_J^{\mathbb{C}}$ is a Lie group. One must replace A^t by $A^\dagger = \bar{A}^t$ in the argument and so $S(n)$ is just the set of hermitean matrices, $A^\dagger = A$ while a is the set of anti-hermitean matrices $A^\dagger = -A$. All dimensions are <u>real</u> dimensions, thus in <u>this</u> case $\dim S(n) = 2\left(\frac{1}{2}(n^2 - n)\right) + n = n^2$. We leave the details to the reader. ⊔

<u>Corollary</u> The dimension of G_J is $\frac{1}{2}(n^2 - n)$ while that of $G_J^{\mathbb{C}}$ is n^2.

<u>Proof</u> Recall that if $f: \mathbb{R}^n \to \mathbb{R}^m$ has constant rank on $f^{-1}(0)$ and $n > m$ then the dimension of $f^{-1}(0)$ is $n - m$. For $f: gl(n) \to S(n)$ defined by
$f(A) = A^t J A - J$

$$\dim G_J = \dim f^{-1}(0) = n^2 - \left(\frac{n^2 + n}{2}\right) = \frac{1}{2}(n^2 - n).$$

In the complex case f maps $gl(n, \mathbb{C})$ to the Hermitean matrices so

$$\dim G_J^{\mathbb{C}} = \dim f^{-1}(0) = 2n^2 - n^2 = n^2.$$

Another approach to obtaining charts for G_J uses the matrix exponential. This function is defined by $A \in gl(n)$ by the series

$$\exp(A) = e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \qquad A^0 = I.$$

To see that the series converges it is useful to consider the partial sums of the series. Let

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} A^k.$$

We show that $\{S_n\}$ is a Cauchy sequence in $gl(n)$ and use the fact that $gl(n)$ is a complete normed linear space to obtain the result. Notice that if $n > m$ then

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} \frac{1}{k!} A^k \right\| \le \sum_{k=m+1}^{n} \frac{1}{k!} \|A\|^k.$$

We used the fact that for $B, C \in gl(n)$, $\|BC\| \le \|B\|\|C\|$. Since $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ converges for each real number x we see that the partial sums $s_n = \sum_{k=0}^{n} \frac{1}{k!} \|A\|^k$ of $e^{\|A\|}$ must have a limit. Thus $\{s_n\}$ is a Cauchy sequence of real numbers. But

$$\|S_n - S_m\| \le |s_n - s_m|$$

from the estimate obtained above. Thus it follows that $\{S_n\}$ is a Cauchy sequence and that the exponential e^A is well-defined.

We now show that the mapping $X \to \exp(X)$ is Frechet differentiable and that

$$(D_A \exp)(H) = H + \tfrac{1}{2} AH + \tfrac{1}{2} HA + \tfrac{1}{6}[A^2H + AHA + HA^2] + \cdots$$

To show this we must examine the linear

approximation to exp near $X = A$. Observe that

$$\exp(A+H) - \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[(A+H)^k - A^k\right]$$
$$= \sum_{k=1}^{\infty} \frac{1}{k!}\left[(A+H)^k - A^k\right]$$
$$= H + \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=1}^{k} P(A^{k-i}, H^i)$$

where

$$\begin{cases} P(A^{2-1}, H^1) = P(A, H) = AH + HA \\ P(A^{2-2}, H^2) = P(A^0, H^2) = H^2 \end{cases}$$

$$\begin{cases} P(A^{3-1}, H^1) = P(A^2, H) = A^2H + AHA + HA^2 \\ P(A^{3-2}, H^2) = P(A, H^2) = AH^2 + HAH + H^2A \\ P(A^{3-3}, H^3) = P(A^0, H^3) = H^3 \end{cases}$$

and in general $P(A^a, H^b)$ denotes the sum of all distinct "products" where each "product" has A as a factor with multiplicity a and has H as a factor with multiplicity b.

Now for $\|H\| \le \|A\|$,

$$\left\| \exp(A+H) - \exp(A) - \left[H + \sum_{k=2}^{\infty} \frac{1}{k!} P(A^{k-1}, H)\right] \right\|$$
$$= \left\| \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2}^{k} P(A^{k-i}, H^i) \right\|$$
$$\le \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2}^{k} \|A\|^{k-i} \|H\|^i$$

$$\le \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2}^{k} \|A\|^{k-i+i-2} \|H\|^2$$

$$= \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=2}^{k} \|A\|^{k-2} \|H\|^2$$

$$= \sum_{k=2}^{\infty} \frac{1}{k!} (k-2+1) \|A\|^{k-2} \|H\|^2$$

$$= \sum_{k=0}^{\infty} \frac{k+1}{(k+2)!} \|A\|^{k} \|H\|^2$$

$$\le \sum_{k=0}^{\infty} \frac{1}{k!} \|A\|^{k} \|H\|^2$$

$$= e^{\|A\|} \|H\|^2 .$$

Thus

$$\frac{\left\| \exp(A+H) - \exp A - \sum_{k=1}^{\infty} \frac{1}{k!} P(A^{k-1}, H) \right\|}{\|H\|} \le e^{\|A\|} \|H\|$$

and so the Frechet derivative of exp exists at $X = A$ and

$$D_A(\exp)(H) = \sum_{k=1}^{\infty} \frac{1}{k!} P(A^{k-1}, H) .$$

~~Theorem There exists an open subset $\mathcal{O} \subseteq gl(n)$ about 0 such that $\exp(\mathcal{O}) \subseteq GL(n)$ is open and $\exp|\mathcal{O} : \mathcal{O} \to \exp(\mathcal{O})$ is a diffeomorphism. In particular $(\exp(\mathcal{O}), (\exp|\mathcal{O})^{-1})$ is a chart~~

Theorem There exists an open subset $\mathcal{O}$ about 0 in $gl(n)$ such that $\exp(\mathcal{O}) \subseteq Gl(n)$ is open and $\exp|\mathcal{O} : \mathcal{O} \to \exp(\mathcal{O})$ is a diffeomorphism. Thus $(\exp(\mathcal{O}), (\exp|\mathcal{O})^{-1})$ is a chart of $Gl(n)$ onto $\mathcal{O} \subseteq gl(n)$.

Proof For $A = 0$, $Df_A(H) = H$ for all H. Thus $Df_A : gl(n) \to gl(n)$ is an isomorphism. By the inverse function theorem f is a local diffeomorphism near 0.

Corollary ~~If~~ $\mathfrak{g}_J = \{ B \in gl(n) \mid B^t J + J B = 0 \}$ then $\exists\ \hat{\mathcal{O}}$ open about $0 \in gl(n) \ni$

$$\exp(\mathfrak{g}_J \cap \hat{\mathcal{O}}) = \exp(\hat{\mathcal{O}}) \cap G_J$$

and such that if

$$s\mathfrak{g}_J = \{ B \in \mathfrak{g}_J \mid Tr(B) = 0 \}$$

then

$$\exp(s\mathfrak{g}_J \cap \mathcal{O}) = \exp(\mathcal{O}) \cap S(G_J)$$

where

$$S G_J = \{ A \in G_J \mid \det(A) = 1 \}$$

Proof (See following pages for complete proof)

$$\cancel{B^t J + J B = 0 \Rightarrow B^t J = -JB}$$

$$\cancel{\Rightarrow B = -J B^t J = -J^{-1} B^t J}$$

$$\cancel{\Rightarrow \exp(-B) = J^{-1} \exp(B)^t J}$$

$$\cancel{\Rightarrow (\exp(B))^{-1} J = J \exp(B)^t}$$

$$\cancel{\Rightarrow J = \exp(B) J \exp(B)^t}$$

$$\Rightarrow \exp(B) \in G_J$$

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$$\exp(\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0) = G_J^{\mathbb{C}} \cap \exp(\mathcal{O}_0)$$

$\left(\frac{\text{Proof}}{J=I}\right)$ $A \in G_J^{\mathbb{C}} \cap \exp(\mathcal{O}_0) \iff A = \exp(B)$ and $\exp(B)^{\dagger}\exp(B) = I$, $B \in \mathcal{O}_0$

$\iff A = \exp(B)$, $\exp(B^{\dagger})\exp(B) = I$, $B \in \mathcal{O}_0$

$\iff A = \exp(B)$, $\exp(B^{\dagger}) = \exp(-B)$, $B \in \mathcal{O}_0$

(uses $\exp|\mathcal{O}_0$ one-one) $\iff A = \exp(B)$, $B^{\dagger} = -B$, $B \in \mathcal{O}_0$

Thus $\exp(\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0) = G_J^{\mathbb{C}} \cap \exp(\mathcal{O}_0)$ as claimed

<u>Remark</u> We used the fact that $B, -B$ commute in order that $\exp(-B) = \exp(B)^{-1}$. In general if A and B commute then $\exp(A+B) = \exp(A)\exp(B)$ and this implies that $\exp(B)^{-1} = \exp(-B)$.

Also observe that we used the fact that

$$\exp(B^{\dagger}) = \sum_{n=0}^{\infty} \frac{1}{n!}(B^{\dagger})^n = \sum_{n=0}^{\infty} \frac{1}{n!}(B^n)^{\dagger} = \exp(B)^{\dagger}.$$

$$\exp(\mathfrak{s}\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0) = \mathfrak{s}(G_J^{\mathbb{C}}) \cap \exp(\mathcal{O}_0)$$

$\left(\frac{\text{Proof}}{J=I}\right)$ Let $A \in \mathfrak{s}(G_J^{\mathbb{C}}) \cap \exp(\mathcal{O}_0)$. Then $A = \exp(B)$ where $B \in \mathcal{O}_0$ and $\exp(B)^{\dagger}\exp(B) = I$, $\det(\exp B) = 1$.

As in the above proof we see that $B^{\dagger} = -B$.

Since $1 = \det(\exp B) = \exp(\operatorname{Tr} B)$ we have in the real case $e^{\operatorname{Tr} B} = e^0$ and $\operatorname{Tr} B = 0$. <u>Note that in the complex case $\operatorname{Tr} B$ is pure imaginary and so this implies only that $\operatorname{Tr} B = 2\pi m i$ for $m \in \mathbb{Z}$</u>.

But our choice of $\mathcal{O}_0$ implies that $|\operatorname{Tr} B| < \frac{\pi}{2}$ and so $2\pi|m| < \frac{\pi}{2}$ and $m = 0$. So $\operatorname{Tr} B = 0$ in this case as well.

It follows that in both the complex and real cases, $A = \exp(B) \in \exp(s\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0)$ and $\exp(s\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0) \supseteq s(G_J^{\mathbb{C}}) \cap \exp(\mathcal{O}_0)$.

To show the converse let $A \in \exp(s\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0)$. Then $A = \exp(B)$ for $B \in s\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0$. Thus $B^\dagger = -B$ and $\operatorname{Tr} B = 0$. It follows as in the proof of Claim 3 that $\exp(B)^\dagger \exp(B) = I$. Moreover, since $\det(\exp(B)) = e^{\operatorname{Tr} B} = e^0 = 1$ we see that $A = \exp(B)$ is in $s(G_J^{\mathbb{C}}) \cap \exp(\mathcal{O}_0)$.

Thus $\exp(s\mathfrak{g}_J^{\mathbb{C}} \cap \mathcal{O}_0) \subseteq s(G_J^{\mathbb{C}}) \cap \exp(\mathcal{O}_0)$ and the claim follows.

Let $\mathcal{O}$ be open about $0 \in gl(n,\mathbb{C})$ such that $\exp(\mathcal{O})$ is open in $Gl(n,\mathbb{C})$ and $\exp|\mathcal{O}$ is a diffeomorphism. Choose $\mathcal{O}_0$ open about 0 such that $\mathcal{O}_0 \subseteq \mathcal{O}$ and $B \in \mathcal{O}_0 \Rightarrow \overline{B}, B^t, B^\dagger, -B$ all belong to $\mathcal{O}$ and $|\mathrm{Tr} B| < \frac{\pi}{2}$.

<u>Claim 1</u> $T_I U(2) = \{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B \}$

<u>Proof</u> First we show $T_I U(2) \subseteq \{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B \}$. Let $C \in T_I U(2)$. Then there is a curve $\gamma : (-a,a) \to U(2)$ such that $\gamma(0) = I$, $\gamma'(0) = C$. Since $\gamma(t) \in U(2)$ for all t, $\gamma(t)^\dagger \gamma(t) \equiv I$. Thus $\gamma'(t)^\dagger \gamma(t) + \gamma(t)^\dagger \gamma'(t) = 0$. Since $\gamma(0) = I$, $\gamma'(0)^\dagger + \gamma'(0) = 0$ and $C^\dagger + C = 0$. So $C \in \{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B \}$ as asserted above.

Next we show that $\{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B \} \subseteq T_I U(2)$. To see this let $B \in gl(2,\mathbb{C})$ such that $B^\dagger = -B$. Let $\gamma(t) = \exp(tB)$. We show that $\gamma(t) \in U(2)$ for all t and that $\gamma'(0) = B$. We have

$$\gamma(t)^\dagger \gamma(t) = [\exp(tB)]^\dagger [\exp(tB)] = \exp(tB^\dagger)\exp(tB)$$
$$= \exp(-tB)\exp(tB)$$
$$= \exp(0) = I,$$

so $\gamma(t) \in U(2)$, $\forall t$. Now $\gamma'(t) = D_{tB}(\exp)\left(\frac{d}{dt}(tB)\right)$ and $\gamma'(0) = D_0(\exp)(B) = id(B) = B$. So $B = \gamma'(0)$, $\gamma(0) = I$ and $\therefore$ $B \in T_I U(2)$. It follows that $T_I U(2) = \{ B \mid B \in gl(2,\mathbb{C}),\ B^\dagger = -B \}$

Claim 2 $T_I SU(2) = \{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B,\ \mathrm{Tr}\, B = 0 \}$

Proof First assume that $C \in T_I SU(2)$, then there exists a curve $\gamma : (-a, a) \to SU(2)$ such that $\gamma(0) = I$ and $C = \gamma'(0)$. Since $\gamma(t) \in SU(2)$ for all t, $\gamma(t)^\dagger \gamma(t) = I$ and $\det(\gamma(t)) = 1$ for all t.

As in the proof of the first claim we see that $\gamma'(0)^\dagger + \gamma(0) = 0$ and so $C^\dagger + C = 0$. To prove that $\mathrm{Tr}\, C = 0$ proceed as follows. Since $\gamma(0) = I$ there exists $\varepsilon > 0$, $\varepsilon < a$ such that $\gamma(t) \in \exp(\mathcal{O}_0)$ for $|t| < \varepsilon$. Let $B(t) = \exp^{-1}(\gamma(t))$, so that $\gamma(t) = \exp(B(t))$. We have that

$$1 = \det(\gamma(t)) = \det(\exp(B(t))) = e^{\mathrm{Tr}\, B(t)}$$

for $|t| < \varepsilon$. Differentiate to get

$$0 = e^{\mathrm{Tr}\, B(t)} \frac{d}{dt}[\mathrm{Tr}(B(t))] = e^{\mathrm{Tr}\, B(t)} \mathrm{Tr}(B'(t))$$

(Note that Tr is a continuous linear map so its Frechet derivative is given by $D_B(\mathrm{Tr}) = \mathrm{Tr}$). Thus at $t=0$ we have $0 = e^{\mathrm{Tr}(B(0))} \mathrm{Tr}(B'(0))$. But

$$B(0) = \exp^{-1}(\gamma(0)) = \exp^{-1}(I) = 0 \text{ and}$$

$$C = \gamma'(0) = D_0(\exp)(B'(0)) = B'(0).$$

So $0 = e^0 \mathrm{Tr}(B'(0)) = \mathrm{Tr}(C)$. Thus $\mathrm{Tr}(C) = 0$ and $C \in \{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B,\ \mathrm{Tr}\, B = 0 \}$ and $T_I SU(2) \subseteq \{ B \in gl(2,\mathbb{C}) \mid B^\dagger = -B,\ \mathrm{Tr}\, B = 0 \}$.

Conversely, assume $B \in gl(2,\mathbb{C})$, $B^\dagger = -B$, $\mathrm{Tr}(B) = 0$. Let $\gamma(t) = \exp(tB)$. From the proof of Claim 1 we know $\gamma(t)^\dagger \gamma(t) = I$. But $\det \gamma(t) = \det(\exp(tB)) = e^{\mathrm{Tr}(tB)} = e^0 = 1$. So $\gamma(t) \in SU(2)$, $\forall t$ and $B = \gamma'(0)$, $\gamma(0) = I$. So $B \in T_I SU(2)$, $\therefore \{ B \mid B^\dagger = -B, \mathrm{Tr}\, B = 0 \} \subseteq T_I SU(2)$ $\square$

Theorem Let G be a Lie group and $v \in T_eG$. Define a mapping $X^v : G \to TG$ by

$$X^v(x) = d_e l_x(v).$$

Then X^v is a (smooth) vector field on G such that

$$d_x l_a(X^v(x)) = X^v(ax)$$

for all $a, x \in G$.

Proof. We first show X^v is smooth near the identity $e^2 = e \in G$. To do this we show that $X^v(f) \in C^\infty_{loc}(e)$ whenever $f \in C^\infty_{loc}(e)$. Let $(\tilde{U}, x)$ denote an arbitrary chart of G with $e \in \tilde{U}$. Let $U \subseteq G$ be open such that $U^2 \subseteq \tilde{U} \cap \mathrm{dom}(f)$ (such exists by continuity of the operation of G). Consider the local coordinate representative $X^v(f) \circ x^{-1}$ of $X^v(f)$:

$$\begin{aligned}
[X^v(f) \circ x^{-1}](r) &= X^v(f)(x^{-1}(r)) \\
&= X^v_{x^{-1}(r)}(f) \\
&= d_e l_{x^{-1}(r)}(v)(f) \\
&= df_{x^{-1}(r)}(d_e l_{x^{-1}(r)}(v)) \\
&= d_e(f \circ l_{x^{-1}(r)})(v) \\
&= v(f \circ l_{x^{-1}(r)}) \\
&= v^i \frac{\partial}{\partial x^i}\Big|_e (f \circ l_{x^{-1}(r)})
\end{aligned}$$

$$= v^i \frac{\partial}{\partial u^i}\left(f \circ l_{x^{-1}(r)} \circ x^{-1}\right)(0)$$

(where (u^i) are coordinates of $\mathbb{R}^m$, $m = \dim G$).
Since the mapping

$$(r, u) \longmapsto f\left(m(x^{-1}(r), x^{-1}(u))\right)$$

is smooth on $x(U) \times x(U) \subseteq \mathbb{R}^{2m}$ it follows that all its partials exist and are continuous. Thus

$$r \longmapsto \frac{\partial}{\partial u^i}\left[f\left(m(x^{-1}(r), x^{-1}(u))\right)\right]\Big|_{u=0}$$

has continuous partials on $x(U)$ and thus $(X^v(f) \circ x^{-1})(r) = v^i \frac{\partial}{\partial u^i}\left[f\left(m(x^{-1}(r), x^{-1}(u))\right)\right]\Big|_{u=0}$ is smooth on $x(U)$. So $X^v(f) \in C^\infty_{loc}(e)$.
Now let $p \in G$ be arbitrary. We show that X^v is smooth in a neighborhood of p. To do this we show that for $f \in C^\infty_{loc}(p)$ $X^v(f) \in C^\infty_{loc}(p)$. This will be shown by proving the identity

$$X^v(f) = X^v(f \circ l_p) \circ l_p^{-1}.$$

Since $f \circ l_p \in C^\infty_{loc}(e)$, $X^v(f \circ l_p) \in C^\infty_{loc}(e)$ and it will follow that $X^v(f)$ is the composite of smooth functions and therefore is smooth. So we have only to prove the identity. Let q denote any element of G near e, then

$$X^v(f \circ l_p)(q) = X^v_q(f \circ l_p) = d_e l_q(v)(f \circ l_p)$$

$$= d_q(f \circ l_p)(d_e l_q(v)) = d_e(f \circ l_p \circ l_q)(v)$$

$$= d_e(f \circ l_{pq})(v) = d_{pq} f\,(d_e l_{pq}(v))$$

$$= d_{pq}f\,(X^v_{pq}) = X^v_{pq}(f) = X^v(f)(pq)$$

$$= (X^v(f)\circ l_p)(q).$$

So

$$X^v(f\circ l_p) = X^v(f)\circ l_p \quad \text{or}$$

$$X^v(f) = X^v(f\circ l_p)\circ l_p^{-1}.$$

Finally we show that

$$d_x l_a(X^v(x)) = X^v(ax).$$

This follows trivially from

$$d_x l_a(X^v(x)) = d_x l_a(d_e l_x(v)) = d_e l_{(ax)}(v) = X^v(ax).$$

Definition A vector field X defined on a Lie group G is <u>left</u> (respectively, <u>right</u>) <u>invariant</u> iff for all $x, a \in G$

$$d_x l_a(X(x)) = X(ax) \quad \left(\text{respectively, } d_x r_a(X(x)) = X(xa)\right)$$

<u>Remark</u> Note that the set $\Gamma_{in}(G)$ of all left invariant vector fields is a subspace of the vector space $\Gamma(G)$ of all vector fields on a Lie group G. It is easy to see that the set of all vector fields on G is an infinite dimensional vector space, but it is a consequence of our next theorem that the subspace Γ_{in} of left-invariant vector fields is finite dimensional and has dimension $\dim(T_eG)$. Similar remarks may be made concerning the subspace of right-invariant vecto

fields on G.

Theorem Let G be a Lie group. The vector space $\Gamma_{inv}(G)$ of all left-invariant vector fields on G is vector space isomorphic to the vector space T_eG.

Proof Let $\Phi : T_eG \to \Gamma_{inv}(G)$ denote the mapping defined by $\Phi(v) = X^v$ where

$$X^v(x) = d_e l_x(v)$$

for all $x \in G$. First observe that for $v, w \in T_eG$

$$\Phi(v+w)(x) = X^{v+w}(x) = d_e l_x(v+w) = d_e l_x(v) + d_e l_x(w)$$
$$= X^v(x) + X^w(x) = \Phi(v)(x) + \Phi(w)(x) = [\Phi(v) + \Phi(w)](x)$$

for all $x \in G$. Thus $\Phi(v+w) = \Phi(v) + \Phi(w)$.
Similarly $\Phi(cv) = c\,\Phi(v)$ for $c \in \mathbb{R}$, $v \in T_eG$.
To see that Φ is injective assume that $\Phi(v) = 0$. Then

$$d_e l_x(v) = 0$$

for all $x \in G$. For $x = e$, $l_x = id_G$ and $d_e l_x(v) = v$. So $v = 0$ and $\ker(\Phi) = \{0\}$.
So Φ is a monomorphism. Let $X \in \Gamma_{inv}(G)$ and let $v = X(e)$. Then, for $x \in G$,

$$X^v(x) = d_e l_x(v) = d_e l_x(X(e)) = X(xe) = X(x)$$

So $X = X^v = \Phi(v)$ and Φ is an isomorphism from T_eG onto $\Gamma_{inv}(G)$.

It is the intent of the next few paragraphs to show that $\Gamma_{inv}(G)$ has a naturally defined algebraic structure defined on it called a *Lie algebra*. The last theorem assures that T_eG becomes a Lie algebra. Thus every Lie group induces a Lie algebra structure on its "linearization" T_eG at e. There is a partial converse to this result. If G is a simply connected (to be defined later) Lie group then G is uniquely determined by its Lie algebra. Thus there is a one-one correspondence between simply connected Lie groups and finite dimensional Lie algebras. An additional remarkable fact is that given *any Lie group* G there is a unique simply connected Lie group $\tilde{G}$ such that G is Lie-group isomorphic to $\tilde{G}/D$ for some normal discrete subgroup D of $\tilde{G}$. In particular, $\tilde{G}$ and G are "locally isomorphic".

First we show that the set of all vector fields of a manifold M is a Lie algebra (to be defined below). Let X and Y be vector fields on M. The Lie bracket $[X,Y]$ of X and Y is the vector field on M defined by

$$[X,Y]_x(f) = X_x(Y(f)) - Y_x(X(f))$$

for $x \in M$, $f \in C^\infty_{loc}(x)$. We must show that $[X,Y]_x$ is in fact a tangent vector of M at x and that the mapping $x \longmapsto [X,Y]_x$ is smooth