Definition 1.1 Let V and W be vector spaces. For $p \geq 1$, a mapping $\alpha : V^p = V \times V \times \cdots \times V \to W$ is called a W-valued p-form iff

(1) α is multi-linear; in the sense that for $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p \in V$ the mapping from V to W defined by

$$x \longrightarrow \alpha(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_p)$$

is linear,

(2) α is skew-symmetric; in the sense that for $(x_1, x_2, \ldots, x_p) \in V^p$ and $i < j$

$$\alpha(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_p) = -\alpha(x_1, \ldots, x_j, \ldots, x_i, \ldots x_p)$$

We also refer to α as a vector-valued p-form. Let $\Lambda^p(V^*, W)$ denote the set of all vector-valued p-forms and for $c \in \mathbb{R}$, $\alpha, \beta \in \Lambda^p(V^*, W)$ define

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x), \quad (c\alpha)(x) = c\alpha(x)$$

for all $x \in V^p$. Then $\Lambda^p(V^*, W)$ is a linear space over $\mathbb{R}$. When $W = \mathbb{R}$ we refer to $\alpha \in \Lambda^p(V^*, \mathbb{R})$ as simply a p-form and we write $\Lambda^p V^* = \Lambda^p(V, \mathbb{R})$. Notice that $\Lambda^1 V^* = V^*$. We define $\Lambda^0 V^* = \mathbb{R}$.

Remark The set of all multi-linear maps from a vector space V into a vector space W is denoted by $\bigotimes^p(V^*, W)$. If $W = \mathbb{R}$ we write $\bigotimes^p(V^*, \mathbb{R}) = \bigotimes^p V^*$. Clearly $\bigotimes^p(V, W)$

is also a linear space over $\mathbb{R}$ and $\Lambda^p(V,W)$ is a subspace if we use pointwise operations as defined above. Notice also that we can define

$$(\alpha\otimes\beta)(v_1,v_2,\ldots,v_p,v_{p+1},\cdots v_{p+q}) = \alpha(v_1,\cdots,v_p)\,\beta(v_{p+1},\cdots,v_{p+q})$$

for $\alpha\in\bigotimes^p V$, $\beta\in\bigotimes^q V$, and $v_1,v_2,\ldots,v_{p+q}\in V$.

It follows that $\alpha\in\bigotimes^p V$, $\beta\in\bigotimes^q V \Rightarrow \alpha\otimes\beta\in\bigotimes^{p+q}V$.

More generally if $\alpha_i\in\bigotimes^{p_i}V^*$, $1\le i\le r$

$$(\alpha_1\otimes\cdots\otimes\alpha_r)(v_1,\cdots,v_{p_1},v_{p_1+1},\cdots v_{p_1+p_2},\cdots v_{p_1+p_2+\cdots p_r})$$
$$:= \alpha_1(v_1,\cdots,v_{p_1})\,\alpha_2(v_{p_1+1},\cdots v_{p_1+p_2})\cdots\alpha_r(v_{p_1+p_2+\cdots+p_{r-1}+1},\cdots v_{p_1+\cdots+p_r})$$

defines an "r-ary operation" if $v_i\in V$ are arbitrary. In particular if $\alpha_1,\alpha_2,\ldots,\alpha_r\in\Lambda^1V^*=V^*$ $\alpha_1\otimes\alpha_2\otimes\cdots\otimes\alpha_r\in\bigotimes^r V^*$. It can be shown that if $\{v_i\}$ is a basis of V and $\{v^i\}$ is ~~the~~ basis of V^* dual to $\{v_i\}$ then $\alpha\in\bigotimes^r V^*$ implies that

$$\alpha = \alpha_{i_1 i_2\cdots i_r}(v^{i_1}\otimes v^{i_2}\otimes\cdots\otimes v^{i_r})$$

for $\alpha_{i_1\cdots i_r}\in\mathbb{R}$. Moreover $\{v^{i_1}\otimes\cdots\otimes v^{i_r}\}$ is a basis of $\bigotimes^r V^*$ so that $\dim(\bigotimes^r V^*)=n^r$ where $n=\dim(V)$.

<u>Definition 1.2</u> For $\alpha\in\Lambda^p V^*$, $\beta\in\Lambda^q V^*$, $p,q>0$ let $\alpha\wedge\beta\in\Lambda^{p+q}V^*$ be defined by:

$$(\alpha\wedge\beta)(v_1, v_2, \dots v_p, v_{p+1}, \dots, v_{p+q}) = \frac{1}{p!}\frac{1}{q!}\sum_{\sigma\in S(p+q)}(-1)^\sigma \alpha(v_1, v_2, \dots, v_p)\beta(v_{p+1}, v_{p+2}, \dots, v_{p+q})$$

where $v_i \in V$, $(-1)^\sigma$ is the sign of the permutation σ, and $S(p+q)$ is the set of all permutations of $\{1, 2, \dots, p+q\}$. It requires a little work to show that $\alpha\wedge\beta \in \Lambda^{p+q}V^*$ and that is left to the reader. We define, for $\alpha \in \Lambda^0 V^*$ (or $\beta \in \Lambda^0 V^*$)

$$\alpha\wedge\beta = \alpha\beta.$$

Remark Notice that if $\alpha \in \Lambda^p V^*$ and $\{e_i\}$ is a basis of V with dual basis $\{e^i\}$ of V^* then we can write

$$\alpha = \alpha_{i_1 i_2 \dots i_p}(e^{i_1}\otimes \cdots \otimes e^{i_p})$$

where $\alpha_{i_1 \dots i_k \dots i_l \dots i_p} = -\alpha_{i_1 \dots i_l \dots i_k \dots i_p}$ are skew-symmetric.

Definition 1.3 For $\omega^1, \omega^2, \dots, \omega^r \in \Lambda^1 V^* = V^*$ define

$$(\omega^1\wedge\cdots\wedge\omega^r) = \sum_{\sigma\in S(r)}(-1)^\sigma(\omega^{\sigma(1)}\otimes\cdots\otimes\omega^{\sigma(r)})$$

Using Definition 1.2 one can prove the following theorem and this result can be used to show that Definition 1.3 is compatible with Definition 1.2

Theorem 1.4 Assume that V is a finite dimensional vector space. Then

(1) $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$ $\quad \alpha\in\Lambda^p V^*, \beta\in\Lambda^q V^*, \gamma\in\Lambda^r V^*$

(2) $\alpha\wedge(\beta+\gamma) = (\alpha\wedge\beta)+(\alpha\wedge\gamma)$ $\quad \alpha\in\Lambda^p V^*, \beta,\gamma\in\Lambda^q V^*$

(3) $(c\alpha)\wedge\beta = c(\alpha\wedge\beta) = \alpha\wedge(c\beta)$ $\quad \alpha\in\Lambda^p V^*, \beta\in\Lambda^q V^*, c\in\mathbb{R}$

(4) $\alpha\wedge\beta = (-1)^{pq}(\beta\wedge\alpha)$ $\quad \alpha\in\Lambda^p V^*, \beta\in\Lambda^q V^*$

Theorem 1.5 If $\{V_1, V_2, \cdots, V_n\}$ is a basis of a vector space V then

$$\{ v^{i_1}\wedge v^{i_2}\wedge\cdots\wedge v^{i_r} \mid 1\le i_1 < i_2 < \cdots < i_r \le n\}$$

is a basis of $\Lambda^r V^*$. Thus

$$\dim(\Lambda^r V^*) = \binom{n}{r}$$

The proofs may be found in Spivak.

Definition 1.6 If V is a vector space we say g is a metric on V iff

(1) g is bilinear

(2) g is symmetric; $\quad g(x,y) = g(y,x)$ for all $x,y\in V$

(3) g is nondegenerate; $\quad g(x,y)=0$ for all $y\in V$ implies that $x\equiv 0$.

If g is a metric we define a mapping $g^b : V\to V^*$ by $g^b(x)(y) = g(x,y)$ for all $x,y\in V$

Notice that for each x, $g^b(x)$ is in fact linear in $y \in V$ and so $g^b(x) \in V$. Moreover it is clear that g^b is linear as a mapping from V to V^*. Finally observe that if

$$g^b(x) = 0$$

then $g^b(x)(y) = 0$ and $g(x,y) = 0$ for all $y \in V$. Thus $x = 0$ and g^b is one-one (its kernel is 0). If $\dim V = n$ is finite then the domain and range of g^b have the same dimension and g^b is necessarily onto as

$$\dim V = \dim(g^b(V)) + \dim(\operatorname{Ker} g^b)$$

and $n = \dim(g^b(V))$. Thus $g^b(V) = V^*$.

It follows that $V \cong V^*$ via a "coordinate independent" isomorphism *if* one has a metric.

<u>Remark</u> Assume that $\{v_i\}$ is a basis of V and define $G_{ij} = g(v_i, v_j)$, $1 \le i, j \le n$. Let $\{v^j\}$ be the basis of V^* dual to $\{v_i\}$. We claim that $\boxed{g^b(v_i) = G_{ij} v^j}$.

To see this let $w \in V$ and write $w = \lambda^j v_j$. We have

$$g^b(v_i)(w) = \lambda^j g^b(v_i)(v_j) = \lambda^j g(v_i, v_j) = G_{ij} \lambda^j$$

$$= G_{ij} v^j(w).$$

This holds for all $w \in V$ so
$$g^b(v_i) = G_{ij} v^j.$$

<u>Remark</u> If $\{v_i\}$ is a basis of a vector space V and $G_{ij} = g(v_i, v_j)$ ~~then~~ $G = (G_{ij})$ is nonsingular. Indeed $g^b : V \to V^*$ is an isomorphism and G is its matrix!

<u>Theorem 1.7</u> If V is an n-dimensional vector space and g is a metric on V then there exists a basis $\{e_i\}$ of V such that $g(e_i, e_j) = \pm \delta_{ij}$. Such a basis is called an <u>orthonormal</u> <u>basis</u>.

<u>Proof</u> Choose any basis $\{v_i\}$ of V and define $G_{ij} = g(v_i, v_j)$. Since $G = (G_{ij})$ is symmetric there is a matrix A such that $A^T = A^{-1}$ and $A^T G A = D$ is diagonal. Define
$$\bar{e}_i = A_i^j v_j.$$
Then
$$g(\bar{e}_i, \bar{e}_j) = A_i^k A_j^l \, g(v_k, v_l) = A_i^k A_j^l G_{kl}$$
$$= A^{T}{}^i_k G_{kl} A_j^l = (A^T G A)^i_j = D^i_j$$
where $(D^i_j) = D$, is diagonal. Thus
$$g(\bar{e}_i, \bar{e}_j) = d_j \delta_{ij}$$ and
$$g\left(\frac{\bar{e}_i}{\sqrt{|d_i|}}, \frac{\bar{e}_j}{\sqrt{|d_j|}}\right) = \frac{1}{\sqrt{|d_i|}} \frac{1}{\sqrt{|d_j|}} d_j \delta_{ij} = \frac{d_j}{|d_j|} \delta_{ij} = \pm \delta_{ij}$$

<u>Definition 1.8</u> Let V be a vector space of dimension n. To say that μ is a <u>volume</u> on V means μ is n n-form on V which is not identically zero. If g is a metric on V then μ is compatible with g iff there is a g-orthonormal basis $\{e_i\}$ of V such that $\mu(e_1, e_2, \ldots, e_n) = 1$. In such a case we often write $\mu = \mu_g$.

<u>Remark 1</u> If g is a metric on V and $\mu = \mu_g$ is a compatible volume and if $\{V_i\}$ is any basis of V ~~then~~ $\mu(V_1, V_2, \ldots, V_n) \neq 0$, $n = \dim V$.

<u>Proof</u> Suppose $\{V_i\}$ is a basis of V such that $\mu(V_1, V_2, \ldots, V_n) = 0$. For arbitrary $W_1, W_2, \ldots, W_n \in V$ write $W_i = \lambda_i^j V_j$ for $\lambda_i^j \in \mathbb{R}$. Then

$$\mu(W_1, W_2, \ldots, W_n) = (\lambda_1^{j_1}\lambda_2^{j_2}\cdots\lambda_n^{j_n})\,\mu(V_{j_1}, V_{j_2}, \ldots, V_{j_n}).$$

If $j_k = j_l$ for $k \neq l$ then $\mu(V_{j_1}, V_{j_2}, \ldots, V_{j_n}) = 0$. If $j_1, j_2, \ldots, j_n$ are distinct ~~then~~ ~~they~~ are a permutation of $\{1, 2, \ldots, n\}$ and ~~thus~~

$$\mu(V_{j_1}, V_{j_2}, \ldots, V_{j_n}) = \pm\,\mu(V_1, V_2, \ldots, V_n) = 0$$

since μ is skew symmetric. Thus $\mu(V_{j_1}, V_{j_2}, \ldots, V_{j_n}) = 0$ for all $j_1, j_2, \ldots, j_n$ and $\mu(W_1, W_2, \ldots, W_n) = 0$. Thus $\mu \equiv 0$ contrary to hypothesis.

<u>Remark 2</u> Given a metric g and a compatible volume μ_g as above there exists a basis $\{V_i\}$ of V such that $\mu(V_1, V_2, \ldots, V_n) = 1$,

Proof Let $\{\bar{V}_i\}$ be any basis of $\bar{V}$ and let $\lambda = \mu(\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_n)$. Then

$$\mu\left(\left(\tfrac{1}{\lambda}\right)\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_n\right) = 1$$

and $\{(\tfrac{1}{\lambda})\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_n\}$ is a basis of $\bar{V}$.

Remark 3 Assume that g is a metric on $\bar{V}$ and ~~that~~ $\mu = \mu_g$ is a compatible volume. If $\{V_i\}$ is any basis of $\bar{V}$ and $G_{ij} = g(V_i, V_j)$ then $\mu_g(V_1, V_2, \cdots, V_n) = \pm\sqrt{|\det G|}$ where $G = (G_{ij})$ is a matrix defined by g and $\{V_i\}$

Proof Let $\{V_i\}$ be any basis of $\bar{V}$ and $\{e_i\}$ an orthonormal basis. Write $V_i = A_i^j e_j$. Then

$$\mu_g(V_1, V_2, \cdots, V_n) = (A_1^{j_1} A_2^{j_2} \cdots A_n^{j_n})\, \mu_g(e_{j_1}, e_{j_2}, \cdots, e_{j_n})$$

$$= (-1)^{j}(A_1^{j_1} \cdots A_n^{j_n}) = \det A$$

Where A is the matrix (A_j^i) and $\{j_1, j_2, \cdots, j_n\}$ is a permutation of $\{1, 2, \cdots, n\}$ also denoted by j. There is an abuse of notation as initially $\mu_g(e_{j_1}, e_{j_2}, \cdots, e_{j_n})$ allow $j_k = j_l$ for $k = l$, but these terms are zero and may be discarded from the sum so ~~that~~ the only terms $\mu_g(e_{j_1}, \cdots, e_{j_n})$ which are possibly nonzero are ~~those~~ for which $j_1, j_2, \cdots, j_n$ is a permutation of $\{1, 2, \cdots, n\}$.

Now observe that $G_{ij} = g(V_i, V_j) = A_i^k A_j^l\, g(e_k, e_l)$ and $G_{ij} = A_i^k A_j^l D_{kl}$ where D_{kl} is 0 for $k \neq l$

and $D_{kk} = \pm 1$ for all k. Writing this equation in matrix notation we have

$$G^i_j = G_{ij} = A^k_i A^l_j D^k_l$$
$$= \bar{A}^i_k D^k_l A^l_j$$
$$= (\bar{A} D A)^i_j.$$

So $G = A^T D A$ and $\det G = (\det A)^2 (\det D)$

Thus $|\det G| = |\det A|^2$ and $\det A = \pm \sqrt{|\det G|}$

<u>Definition 1.9</u> If μ is a volume on an n-dimensional vector space V and $\{v_i\}$ is a basis of V then $\{v_i\}$ is <u>positively oriented</u> iff $\mu(v_1, v_2, \cdots, v_n) > 0$. It is <u>negatively oriented</u> iff $\mu(v_1, v_2, \cdots, v_n) < 0$. Clearly this definition depends on the choice of volume μ. Any volume thus produces a partition of the set of all bases of V into two equivalence classes, the class of positively oriented bases and the class of negatively oriented bases.

<u>Exercise 1.1</u> Let g be a metric on an n-dimensional vector space, μ_g a compatible volume, and $\{e_i\}$ a g-orthonormal basis of V. Show that

$$\mu_g = e^1 \wedge e^2 \wedge \cdots \wedge e^n = \varepsilon_{i_1 i_2 \cdots i_n} (e^{i_1} \wedge \cdots \wedge e^{i_n})$$

where
$$\varepsilon_{i_1 \cdots i_n} = \begin{cases} 0 & \text{if } i_k = i_l \quad k \neq l \\ 1 & \text{if } \{i_1, i_2, \cdots, i_n\} \text{ is a pos permutation } \{1, 2, \cdots\} \\ -1 & \text{if } \{i_1, i_2, \cdots, i_n\} \text{ is a neg permutation of } \{1, 2, \cdots\} \end{cases}$$

<u>Definition 1.1.0</u> Assume that W is a finite dimensional vector space with metric h. Recall that h^b is an isomorphism from W onto W^*. Define $h^\#: W^* \to W$ by $h^\# = (h^b)^{-1}$. Thus for $\alpha \in W^*$, $h^\#(\alpha) \in W$ such that $h^b(h^\#(\alpha)) = \alpha$ and therefore $\alpha(x) = h^b(h^\#(\alpha))(x) = h(h^\#(\alpha), x)$ for all $x \in W$.

If g is a metric on the finite dimensional vector space V define $\hat{g}$ on V^* by

$$\hat{g}(\alpha, \beta) = g(g^\#(\alpha), g^\#(\beta))$$

The reader is encouraged to show that $\hat{g}$ is a metric on V^*. In general, this metric $\hat{g}$ may be used to define a metric $\tilde{g}$ on $\Lambda^p V^*$ as follows. If $\alpha, \beta \in \Lambda^p V^*$,

$$\alpha = \alpha_{i_1 \cdots i_p}(v^{i_1} \otimes \cdots \otimes v^{i_p})$$

$$\beta = \beta_{j_1 \cdots j_p}(v^{j_1} \otimes \cdots \otimes v^{j_p})$$

relative to the basis $\{v^l\}$ of V^* dual to a given basis $\{v_k\}$ of V, then

$$\tilde{g}(\alpha, \beta) = \frac{1}{p!} \hat{g}(v^{i_1}, v^{j_1}) \cdots \hat{g}(v^{i_p}, v^{j_p}) \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_p}.$$

Relative to such choices of bases, one writes $g^{ij} = g(v^i, v^j)$ and

$$\tilde{g}(\alpha, \beta) = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_p}$$

$$= \frac{1}{p!} \alpha^{i_1 \cdots i_p} \beta_{i_1 \cdots i_p}$$

Exercise 1.2 Show that the mappings $\hat{g}$ and $\tilde{g}$ defined above are metrics on V^* and $\Lambda^p V^*$ respectively.

Exercise 1.3 If $\alpha \in \Lambda^p V^*$ and $\alpha \wedge \gamma = 0$ for all $\gamma \in \Lambda^{n-p} V^*$ show that $\alpha = 0$.

Theorem 1.11 For $\beta \in \Lambda^p V^*$ there exists a unique form $*\beta \in \Lambda^{n-p} V^*$ such that

$$\alpha \wedge (*\beta) = \tilde{g}(\alpha, \beta)\, \mu_g, \quad \text{for all } \alpha \in \Lambda^p V^*,$$

where V is an n-dimensional vector space, g is a metric on V, μ_g is a g-compatible volume on V, and $\tilde{g}$ is the metric on $\Lambda^p V^*$ induced by g. Moreover the mapping from $\Lambda^p V^*$ to $(\Lambda^{n-p} V^*)$ defined by $\beta \longmapsto *\beta$ is a vector space isomorphism.

Proof First we define, for each $\gamma \in \Lambda^{n-p} V^*$, a mapping φ_γ from $\Lambda^p V^*$ into $\mathbb{R}$ as follows: for $\alpha \in \Lambda^p V^*$, $\varphi_\gamma(\alpha)$ is the unique number such that $\alpha \wedge \gamma = \varphi_\gamma(\alpha)\, \mu_g$. Since

$$(\alpha_1 + \alpha_2) \wedge \gamma = \varphi_\gamma(\alpha_1 + \alpha_2)\, \mu_g$$

$$\alpha_1 \wedge \gamma = \varphi_\gamma(\alpha_1)\, \mu_g$$

$$\alpha_2 \wedge \gamma = \varphi_\gamma(\alpha_2)\, \mu_g .$$

We have that

$$\varphi_\gamma(\alpha_1 + \alpha_2)\mu_g = (\alpha_1 + \alpha_2) \wedge \gamma = (\alpha_1 \wedge \gamma) + (\alpha_2 \wedge \gamma) = (\varphi_\gamma(\alpha_1) + \varphi_\gamma(\alpha_2))\mu_g$$

and $\varphi_\gamma(\alpha_1+\alpha_2) = \varphi_\gamma(\alpha_1) + \varphi_\gamma(\alpha_2)$. Similarly, $\varphi_\gamma(c\alpha) = c\,\varphi_\gamma(\alpha)$. These identities hold for all $\alpha, \alpha_1, \alpha_2 \in \Lambda^p V^*$ and $c \in \mathbb{R}$. Thus $\varphi_\gamma \in (\Lambda^p V^*)^*$ for all $\gamma \in \Lambda^{n-p} V^*$.

Let $\hat{\varphi} : \Lambda^{n-p} V^* \longrightarrow (\Lambda^p V^*)^*$ be defined by $\hat{\varphi}(\gamma) = \varphi_\gamma$. Clearly, $\hat{\varphi}(\gamma_1+\gamma_2) = \hat{\varphi}(\gamma_1) + \hat{\varphi}(\gamma_2)$ and $\hat{\varphi}(c\gamma) = c\,\hat{\varphi}(\gamma)$ for $\gamma, \gamma_1, \gamma_2 \in \Lambda^{n-p} V^*$ and $c \in \mathbb{R}$. Notice that if $\gamma \in \text{Kernel}\,\hat{\varphi}$, then $\hat{\varphi}(\gamma) = 0$ and $\alpha \wedge \gamma = \varphi_\gamma(\alpha)\,\mu_g = 0$ for all $\alpha \in \Lambda^p V^*$. By Exercise 1.3, $\gamma = 0$. So Kernel $\hat{\varphi} = \{0\}$. But

$$\binom{n}{n-p} = \dim \Lambda^{n-p} V^* = \dim(\text{Kernel}\,\hat{\varphi}) + \dim \hat{\varphi}(\Lambda^{n-p} V^*)$$

and since $\hat{\varphi}(\Lambda^{n-p} V^*)$ has dimension $\binom{n}{n-p}$ and $\dim (\Lambda^p V^*)^* = \dim(\Lambda^p V^*) = \binom{n}{p} = \binom{n}{n-p}$ it follows that

$$\hat{\varphi}(\Lambda^{n-p} V^*) = (\Lambda^p V^*)^*.$$

Thus $\hat{\varphi}$ is a vector space isomorphism from $\Lambda^{n-p} V^*$ onto $(\Lambda^p V^*)^*$.

For each $\beta \in \Lambda^p V^*$ let $*\beta = \hat{\varphi}^{-1}(\tilde{g}^b(\beta))$ then the mapping $\beta \longrightarrow *\beta$ is just $\hat{\varphi}^{-1} \circ \tilde{g}^b$ and so is an isomorphism. Also

$$\tilde{g}(\alpha,\beta) = \tilde{g}(\beta,\alpha) = \tilde{g}\left(\tilde{g}^{\#}(\hat{\varphi}(*\beta)), \alpha\right)$$
$$= \hat{\varphi}(*\beta)(\alpha) = \varphi_{*\beta}(\alpha)$$

and $\alpha \wedge (*\beta) = \varphi_{*\beta}(\alpha)\mu_g = \tilde{g}(\alpha,\beta)\mu_g$.

Thus $*\beta$ satisfies the required identity.

Note also that if $\alpha \wedge (*\beta) = \tilde{g}(\alpha,\beta)\mu_g$ for all α then $\varphi_{*\beta}(\alpha) = \tilde{g}(\alpha,\beta) = \tilde{g}^{b}(\beta)(\alpha)$ and $\hat{\varphi}(*\beta) = \tilde{g}^{b}(\beta)$. Thus $*\beta = \hat{\varphi}^{-1}(\tilde{g}^{b}(\beta))$ and so $*\beta$ is uniquely defined.

<u>Lemma</u> Assume V is a vector space, g is a metric on V and $\{v_1, v_2, \ldots, v_n\}$ is a basis of V. If $\mu = \mu_g$ is a volume induced by g, then $\mu_g = \pm\, |\det(G)|^{\frac{1}{2}} (v^1 \wedge v^2 \wedge \cdots \wedge v^n)$ where $G = (G_{ij})$, $G_{ij} = g(v_i, v_j)$. The sign in the formula is positive when $\{v_1, v_2, \ldots, v_n\}$ is positively oriented and otherwise is negative.

<u>Proof</u> Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of V in the same orientation class as $\{v_1, v_2, \cdots, v_n\}$ (Why can you select such a basis?).

Write $v_i = a_i^k e_k$. Clearly $a = (a_j^i)$ is a nonsingular matrix and

$$v^i = (a^{-1})^i_l\, e^l.$$

Then
$$G_{ij} = g(v_i, v_j) = a^l_i\, a^k_j\, g(e_l, e_k) = a^l_i D_{lk} a^k_j$$
Where $D = (D_{lk})$ is diagonal and $D_{kk} = \pm 1$ for all k. Thus as matrices
$$G^i_j = (a^T)^i_l D^l_k a^k_j = (a^T D a)^i_j$$
and $G = a^T D a$. We have
$$\det G = (\det a)^2 (\det D) = \pm |\det a|^2$$
and $|\det G| = |\det a|^2$. Thus
$$v^1 \wedge v^2 \wedge \cdots \wedge v^n = \left[(a^{-1})^1_{j_1} (a^{-1})^2_{j_2} \cdots (a^{-1})^n_{j_n}\right](e^{j_1} \wedge \cdots \wedge e^{j_n})$$
$$= \sum_{\sigma \in S(n)} (a^{-1})^1_{\sigma 1} (a^{-1})^2_{\sigma 2} \cdots (a^{-1})^n_{\sigma n} (-1)^\sigma (e^1 \wedge e^2 \wedge \cdots \wedge e^n)$$
$$= \det(a^{-1})\, (e^1 \wedge e^2 \wedge \cdots \wedge e^n)$$
$$= \left(\frac{1}{\det a}\right) (e^1 \wedge e^2 \wedge \cdots \wedge e^n)$$
$$= \frac{1}{|\det G|^{\frac{1}{2}}} (e^1 \wedge e^2 \wedge \cdots \wedge e^n)$$
$$= \pm \frac{1}{|\det G|^{\frac{1}{2}}} \mu_g$$
Where we have used the fact that $\det a > 0$ since $\{v_i\}$ and $\{e_i\}$ are in the same orientation class and $\mu_g = \pm (e^1 \wedge e^2 \wedge \cdots \wedge e^n)$ depending on the orientation of $\{e_i\}$ and therefore on the orientation of $\{v_i\}$.

Theorem 1.12 Let V be a vector space, g a metric on V, and $\mu = \mu_g$ a compatible volume. If $\{v_1, v_2, \cdots, v_n\}$ is a basis of V, $\beta \in \Lambda^p V^*$, and

$$\beta = \beta_{|j_1 \cdots j_p|}(v^{j_1} \wedge v^{j_2} \wedge \cdots \wedge v^{j_p}),$$

then

$$*\beta = \pm |\det G|^{\frac{1}{2}} \beta^{|j_1 j_2 \cdots j_p|} \varepsilon_{j_1 \cdots j_p | s_1 s_2 \cdots s_{n-p}|} (v^{s_1} \wedge v^{s_2} \wedge \cdots \wedge v^{s_{n-p}})"$$

Proof Let $\alpha, \beta \in \Lambda^p V^*$ and write

$$\alpha = \alpha_{|i_1 \cdots i_p|}(v^{i_1} \wedge \cdots \wedge v^{i_p}), \quad \beta = \beta_{|j_1 j_2 \cdots j_p|}(v^{j_1} \wedge v^{j_2} \wedge \cdots \wedge v^{j_p})$$

Let $$\circledast\beta = \beta^{|j_1 j_2 \cdots j_p|} \varepsilon_{j_1 \cdots j_p | s_1 s_2 \cdots s_{n-p}|}(v^{s_1} \wedge v^{s_2} \wedge \cdots \wedge v^{s_{n-p}})$$

Note that the latter sum only gives nonzero terms when $s_1 < s_2 < \cdots < s_{n-p}$ are all distinct from each of the indices $j_1, j_2, \cdots, j_p$. Thus

$$\alpha \wedge \circledast\beta = \sum_{i,j,s} \alpha_{i_1 \cdots i_p} \beta^{j_1 \cdots j_p} \varepsilon_{j_1 \cdots j_p s_1 \cdots s_{n-p}} (v^{i_1} \wedge \cdots \wedge v^{i_p} \wedge v^{s_1} \wedge \cdots \wedge v^{s_{n-p}})$$

where the sum extends over increasing multi-indices

$$i = (i_1, i_2, \cdots, i_p) \quad j = (j_1, \cdots, j_p)$$
$$s = (s_1, s_2, \cdots, s_{n-p})$$

Moreover $(s_1, s_2, \cdots, s_{n-p})$ must be the complement of $(i_1, i_2, \cdots, i_p)$ in $(1, 2, \cdots, n)$ otherwise $v^{i_1} \wedge \cdots \wedge v^{i_p} \wedge v^{s_1} \wedge \cdots \wedge v^{s_{n-p}} = 0$

Similarly $(s_1, s_2, \cdots, s_{n-p})$ must be the complement of $(j_1, \dots, j_p)$ in $(1, 2, \dots, n)$ since otherwise $\varepsilon_{j_1 \cdots j_p s_1 \cdots s_{n-p}} = 0$. It follows that if $i_1 < i_2 < \cdots < i_p$ is any increasing p-tuple the only terms contributing to the sum must have $j = i$ and s the unique complement to i in $(1, 2, \dots, n)$. For each such i there is a permutation σ of $(i_1, i_2, \dots, i_p, s_1, \dots s_{n-p})$ which gives $\{1, 2, \dots, n\}$ and so

$$V^{i_1} \wedge V^{i_2} \wedge \cdots \wedge V^{i_p} \wedge V^{s_1} \wedge \cdots \wedge V^{s_{n-p}} = (-1)^\sigma (V^1 \wedge V^2 \wedge \cdots \wedge V^n)$$

Moreover since the only triples (i, j, s) which contribute in the sum are those for which $i = j$ we also have

$$\varepsilon_{j_1 \cdots j_p s_1 s_2 \cdots s_{n-p}} = (-1)^\sigma \varepsilon_{12 \cdots n}$$

Thus

$$\alpha \wedge \hat{*}\beta = \sum_{i_1 < i_2 < \cdots < i_p} \alpha_{i_1 \cdots i_p} \beta^{i_1 i_2 \cdots i_p} \varepsilon_{12 \cdots n} (V^1 \wedge V^2 \wedge \cdots \wedge V^n)$$

since $(-1)^\sigma (-1)^\sigma = 1$. Now

$$\tilde{g}(\alpha, \beta) = \alpha_{|i_1 \cdots i_p|} \beta^{i_1 \cdots i_p}$$

and

$$\alpha \wedge \hat{*}\beta = \tilde{g}(\alpha, \beta)(V^1 \wedge V^2 \wedge \cdots \wedge V^n).$$

It follows that

$$\begin{aligned} \alpha \wedge (|\det(G)|^{\frac{1}{2}} \hat{*}\beta) &= \tilde{g}(\alpha, \beta) |\det G|^{\frac{1}{2}} (V^1 \wedge V^2 \wedge \cdots \wedge V^n) \\ &= \pm \tilde{g}(\alpha, \beta) \mu_g = \pm \alpha \wedge (*\beta) \end{aligned}$$

So

$$\begin{aligned} *\beta &= \pm |\det G|^{\frac{1}{2}} \hat{*}\beta \\ &= \pm |\det G|^{\frac{1}{2}} \beta^{|j_1 \cdots j_p|} \varepsilon_{j_1 \cdots j_p s_1 \wedge \cdots \wedge s_{n-p}} (V^{s_1} \wedge V^{s_2} \wedge \cdots \wedge V^{s_{n-p}}) \end{aligned}$$

Theorem 1.13 Assume that V is an n-dimensional vector space, that g is a metric on V, and that $\mu = \mu_g$ is a g compatible volume on V. If $\alpha \in \Lambda^k V^*$ then $*(*\alpha) = (-1)^{(n-k)k+s}\alpha$ where s is the index of g.

Proof Let $\{e_1, e_2, \dots, e_n\}$ be a g-orthogonal positively oriented basis of V relative to $\mu = \mu_g$. Choose any fixed set of indices $1 \le i_1 < i_2 < \dots < i_k \le n$ and let

$$\alpha = e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k}.$$

We first prove the theorem for this type of $\alpha \in \Lambda^k V^*$. We have that $\alpha = \sum_{\sigma \in S(k)} (-1)^\sigma (e^{i_{\sigma 1}} \otimes e^{i_{\sigma 2}} \otimes \dots \otimes e^{i_{\sigma k}})$

For $1 \le j_a \le n$, $1 \le a \le k$, let $\alpha_{j_1 j_2 \dots j_k} = (-1)^\sigma$ if σ is a permutation such that $j_a = i_{\sigma(a)}$ for $1 \le a \le k$. Otherwise let $\alpha_{j_1 j_2 \dots j_k} = 0$. Thus

We have $\alpha = \alpha_{j_1 j_2 \dots j_k}(e^{j_1} \otimes \dots \otimes e^{j_k})$ $\quad (\star)$,

and also

$$\alpha = \alpha_{|j_1 j_2 \dots j_k|}(e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_k})$$

It now follows from Theorem 1.12 that

$$*\alpha = \alpha^{|j_1 j_2 \dots j_k|} \varepsilon_{j_1 \dots j_k | s_1 s_2 \dots s_{n-k}|} (e^{s_1} \wedge e^{s_2} \wedge \dots \wedge e^{s_{n-k}})$$

since $|\det G| = 1$ and $\{e_1, e_2, \dots, e_n\}$ is positively oriented. If $1 \le j_1, j_2, \dots, j_k \le n$ are arbitrary fixed numbers, then

$$\alpha^{j_1 j_2 \cdots j_k} = g^{j_1 l_1} g^{j_2 l_2} \cdots g^{j_k l_k} \alpha_{l_1 l_2 \cdots l_k} \text{ (sum over } l_a\text{'s)}$$

$$= g^{j_1 j_1} g^{j_2 j_2} \cdots g^{j_k j_k} \alpha_{j_1 j_2 \cdots j_k} \text{ (no sum)}$$

Let m denote the number of indices i_a for which $g_{i_a i_a} = -1$. This is the same as the number of indices i_a such that $g^{i_a i_a} = -1$. Since the only $\alpha_{j_1 j_2 \cdots j_k}$ summed over in (★) which are possibly nonzero are those for which $j_1, j_2, \cdots, j_k$ is a permutation of $i_1, i_2, \cdots, i_k$. We may assume that the $j_1, j_2, \cdots, j_k$ in the sum have the property that the number of $j_1, \cdots, j_k$ such that $g^{j_a j_a} = -1$ is also m. Thus for such $j_1, \cdots, j_k$

$$\alpha^{j_1 \cdots j_k} = (-1)^m \alpha_{j_1 \cdots j_k}$$

Moreover if $\alpha^{j_1 \cdots j_k}$ is nonzero and $j_1 < j_2 < \cdots < j_k$ then $j_a = i_a$ for $1 \leq i \leq k$.

Thus $\quad * \alpha = \alpha^{|j_1 \cdots j_k|} \varepsilon_{j_1 \cdots j_k | s_1 \cdots s_{n-k}|} (e^{s_1} \wedge \cdots \wedge e^{s_{n-k}})$

$$\overset{(\#)}{=} (-1)^m \varepsilon_{i_1 \cdots i_k | s_1 \cdots s_{n-k}|} (e^{s_1} \wedge \cdots \wedge e^{s_{n-k}})$$

But there is only one set of indices $s_1 < s_2 < \cdots < s_{n-k}$

complementary to $i_1, i_2, \cdots, i_k$ so the sum in ⊕ reduces to a single term! Choose any one sequence of indices $l_1, l_2, \cdots, l_{n-k}$ such that $i_1, i_2, \cdots, i_k, l_1, l_2, \cdots, l_{n-k}$ is an even permutation of $1, 2, \cdots, n$. When $s_1, s_2, \cdots, s_{n-k}$ is permuted into $l_1, l_2, \cdots, l_{n-k}$ both $\varepsilon_{i_1 \cdots i_k s_1 \cdots s_{n-k}}$ and $e^{s_1} \wedge e^{s_2} \wedge \cdots \wedge e^{s_{n-k}}$ change signs or both do not so ⊕ yields

$$(-1)^m (*\alpha) = \varepsilon_{i_1 i_2 \cdots i_k |s_1 \cdots s_{n-k}|} (e^{s_1} \wedge \cdots \wedge e^{s_{n-k}})$$

$$= \varepsilon_{i_1 i_2 \cdots i_k l_1 \cdots l_{n-k}} (e^{l_1} \wedge \cdots \wedge e^{l_{n-k}}) \text{ (no sum)}$$

$$= \varepsilon_{12 \cdots n} (e^{l_1} \wedge \cdots \wedge e^{l_{n-k}})$$

Thus
$$*\alpha = (-1)^m (e^{l_1} \wedge \cdots \wedge e^{l_{n-k}}). \quad (\heartsuit)$$

One consequence is that

$$\tilde{g}(\alpha, \alpha)\mu_g = \alpha \wedge (*\alpha) = (-1)^m (e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{l_1} \wedge \cdots \wedge e^{l_{n-k}})$$

$$= (-1)^m (e^1 \wedge e^2 \wedge \cdots \wedge e^n) = (-1)^m \mu_g$$

and $\tilde{g}(\alpha, \alpha) = (-1)^m$. Since the above calculat hold for every $i_1 < i_2 < \cdots < i_k$ and since $(\heartsuit)$ implies that

$$*\alpha = (-1)^t (e^{s_1} \wedge \cdots \wedge e^{s_{n-k}})$$

for some t we have

$$**\alpha = (-1)^t * (e^{s_1} \wedge \cdots \wedge e^{s_{n-k}}) = (-1)^q \alpha$$

for some q. Similarly we have

$$(*\alpha)\wedge *(*\alpha) = \tilde{g}(*\alpha,*\alpha)\,\mu_g = (-1)^{\bar{m}}\mu_g$$

where

m is the number of a such that $g^{i_a i_a} = -1$

$\bar{m}$ is the number of b such that $g^{s_b s_b} = -1$.

But

$$(-1)^{\bar{m}}\mu_g = (*\alpha)\wedge *(*\alpha) = (-1)^m (e^{l_1}\wedge\cdots\wedge e^{l_{n-k}})\wedge(-1)^q(e^{i_1}\wedge\cdots\wedge e^{i_k})$$

$$= (-1)^{m+q+(n-k)k}(e^{i_1}\wedge\cdots\wedge e^{i_k}\wedge e^{l_1}\wedge\cdots\wedge e^{l_{n-k}})$$

$$\overset{(M)}{=} (-1)^{(n-k)k+q}(\alpha\wedge *\alpha)$$

$$= (-1)^{(n-k)k+q}(-1)^m\mu_g$$

and $(-1)^{\bar{m}+m+(n-k)k} = (-1)^q$ or

$(-1)^q = (-1)^{s+(n-k)k}$ and so

$$\boxed{**\alpha = (-1)^{(n-k)k+s}\,\alpha}$$

in the case that $\alpha = (e^{i_1}\wedge\cdots\wedge e^{i_k})$. In general $\alpha = \alpha_{|i_1 i_2\cdots i_k|}(e^{i_1}\wedge\cdots\wedge e^{i_k})$ and

$$**\alpha = \alpha_{|i_1 i_2\cdots i_k|}\,**(e^{i_1}\wedge\cdots\wedge e^{i_k})$$

$$= \alpha_{|i_1 i_2\cdots i_k|}(-1)^{(n-k)k+s}(e^{i_1}\wedge\cdots\wedge e^{i_k})$$

$$= (-1)^{(n-k)k+s}\,\alpha.$$

The theorem follows.

Lemma If M is a paracompact manifold and $\{(U_\alpha, x_\alpha)\}_{\alpha\in A}$ is a family of admissible charts covering M, then there exists a family $\{(V_\tau, \rho_\tau)\}_{\tau\in T}$ such that

(1) $\{V_\tau\}_{\tau\in T}$ is a locally finite open covering of M, i.e. for each $x \in M$ there exists $\tau_0 \in T \ni x \in V_{\tau_0}$ and moreover there is a neighborhood N_x of x in M such that $N_x \cap V_\tau \neq \emptyset$ for at most finitely many $\tau \in T$.

(2) The covering $\{V_\tau\}_{\tau\in T}$ is subordinate to $\{U_\alpha\}_{\alpha\in A}$, i.e., for each $\tau \in T$ there exists $\alpha(\tau) \in A$ such that $V_\tau \subseteq U_{\alpha(\tau)}$;

(3) For each $\tau \in T$, ρ_τ is a smooth mapping from M to $\mathbb{R}$, moreover

(i) $\rho_\tau(x) \geq 0 \qquad \forall x \in M$

(ii) $\operatorname{supp} \rho_\tau \subseteq V_\tau \qquad \forall \tau \in T$

(iii) $\forall x \in M, \quad \sum_\tau \rho_\tau(x) = 1.$

Definition 1.14 Let M be a manifold. An atlas $\mathcal{A}$ of admissible charts of M is said to be an oriented atlas iff for each pair of charts $(U,x), (V,y)$ of $\mathcal{A}$ such that $U \cap V \neq \emptyset$ it follows that

$$\det\left(\frac{\partial y^i}{\partial x^j}\right)$$

is positive on $U \cap V$. The manifold M is orientable iff it possesses an oriented atlas. Two oriented atlases $\mathcal{A}$ and $\mathcal{B}$ of M are equivalent iff $\mathcal{A} \cup \mathcal{B}$ is an oriented atlas. This notion of equivalence is an equivalence relation and any one equivalence class of oriented atlases is called an orientation of M.

Definition 1.15 Assume M is an n-dimensional manifold. A volume on M is a nonzero section of the bundle $\Lambda^n M \xrightarrow{\pi} M$.

We explain this definition more fully. For $1 \leq p \leq n$ let

$$\Lambda^p M = \{(q,\alpha) \mid q \in M \text{ and } \alpha \in \Lambda^p(T_q M)\}.$$

Often we write α_q instead of the pair (q,α). For each chart (U,x) in the differentiable structure of M define a chart of $\Lambda^p M$ as follows: let $\tilde{U} = \{(q,\alpha) \in \Lambda^p M \mid q \in U\}$ and let $\tilde{x} : \tilde{U} \longrightarrow x(U) \times \Lambda^p \mathbb{R}^n$ be defined by

$$\tilde{x}(q,\alpha) = \left(x(q), \tfrac{1}{p!}(\alpha^x_{i_1 \ldots i_p})(r^{i_1} \wedge \cdots \wedge r^{i_p})\right)$$

where $\alpha = \frac{1}{p!}(\alpha^x_{i_1 \ldots i_p})(dx^{i_1}_q \wedge \cdots \wedge dx^{i_p}_q)$ and where

$r_i = (0,0,\ldots 0,1,0,\cdots 0)$ (with 1 in the i-th position).

The set of pairs $\{(\tilde{U},\tilde{x}) \mid (U,x) \in \mathcal{A}\}$ can be shown to be an atlas of $\Lambda^p M$ and thus $\Lambda^p M$ becomes a manifold relative to this atlas. The mapping $\pi_p : \Lambda^p M \to M$ defined by $\pi_p(q,\alpha) = q$ is smooth. To say that α is a section of $\pi_p : \Lambda^p M \to M$ means $\alpha : M \to \Lambda^p M$ is is a smooth mapping such that $\pi_p \circ \alpha = id_M$. Notice that α is a section implies that $\tilde{x} \circ \alpha \circ x^{-1}$ is smooth. If we write

$$\alpha(q) = \tfrac{1}{p!}\alpha_{i_1 \ldots i_p}(q)(dx^{i_1}_q \wedge \cdots \wedge dx^{i_p}_q)$$

it will follow that the maps $q \longmapsto \alpha_{i_1 \ldots i_p}(q)$ are all smooth and conversely if these maps are all smooth then $\bar{x}\circ\alpha\circ x^{-1}$ will be smooth for all $(U,x)\in\mathcal{a}$.

This all applies in the case $n=p$.

Given a chart (U,x) of M and a volume μ of M we see ~~that~~ $\mu(q)=(q,\mu_q)$ for $q\in U$ and $\mu_q\in\Lambda^n(T_qM)$.
Thus
$$\mu_q = f_x(q)(dx_q^1\wedge\cdots\wedge dx_q^n)$$
for some number $f(q)\in\mathbb{R}$ which changes as q changes. Thus one gets a function $f_x: U\to\mathbb{R}$ and the smoothness of μ guarantees f_x is smooth + vice-versa. Also f_x is never zero since μ_q is nonzero $\forall q$.

Theorem 1.16 If M is a manifold and $\{(U_\alpha,x_\alpha)\}_{\alpha\in A}$ is an oriented atlas on M ~~then~~ ~~there~~ is a volume μ of M such ~~that~~ $\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q,\cdots \frac{\partial}{\partial x_\alpha^n}\Big|_q\right)>0$ for each $\alpha\in A$ and all $q\in U_\alpha$. Notice that if μ is such a volume and $g: M\to\mathbb{R}^+$ is any smooth nonzero function ~~then~~ $g\mu$ is also a volume such that
$$g(q)\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q,\cdots,\frac{\partial}{\partial x_\alpha^n}\Big|_q\right)$$
for all $\alpha\in A$ and $q\in U_\alpha$. Conversely, if μ is a volume on M ~~then~~ ~~there~~ is an oriented atlas $\mathcal{a}$ of M such that $(U,x)\in\mathcal{a}$

implies ~~that~~ $\mu_q\left(\frac{\partial}{\partial x^1}\Big|_q, \cdots, \frac{\partial}{\partial x^n}\Big|_q\right) > 0 \;\forall q \in U$.

Proof First assume $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ is an oriented atlas. For each $\alpha \in A$ let

$$\mu_\alpha(q) = dx^1_{\alpha\,q} \wedge \cdots \wedge dx^n_{\alpha\,q}$$

for all $q \in U_\alpha$. Then μ_α is a volume on U_α. Choose a family $\{(V_\tau, \rho_\tau)\}_{\tau \in T}$ satisfying the conditions of the Lemma relative to the charts $\{(U_\alpha, x_\alpha)\}_\alpha$. If $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\mu_\beta = dx^1_\beta \wedge \cdots \wedge dx^n_\beta$$

$$= \left(\frac{\partial x^1_\beta}{\partial x^{i_1}_\alpha} \cdots \frac{\partial x^n_\beta}{\partial x^{i_n}_\alpha}\right)\left(dx^{i_1}_\alpha \wedge \cdots \wedge dx^{i_n}_\alpha\right)$$

$$= \sum_\sigma (-1)^\sigma \left(\frac{\partial x^1_\beta}{\partial x^{\sigma(1)}_\alpha} \cdots \frac{\partial x^n_\beta}{\partial x^{\sigma(n)}_\alpha}\right)\left(dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha\right)$$

$$= \det\left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right)\mu_\alpha$$

Since $\det\left(\frac{\partial x^i_\beta}{\partial x^j_\alpha}\right) > 0$ on $U_\alpha \cap U_\beta$ we see that μ_β and μ_α will agree on the orientation of any basis $v_1, v_2, \ldots, v_n$ of T_qM for $q \in U_\alpha \cap U_\beta$ i.e. $\mu_\alpha(v_1, \ldots v_n)$ and $\mu_\beta(v_1, \ldots v_n)$ will either both be positive or both will be negative.

*) See page for proof μ is a volume

Now *define* $\mu = \sum_{\tau \in T} \rho_\tau \mu_{\alpha(\tau)}$. For (U_α, x_α) in $\mathcal{A}$

$$\mu_q\left(\frac{\partial}{\partial x^1_\alpha}\Big|_q, \cdots, \frac{\partial}{\partial x^n_\alpha}\Big|_q\right) = \sum_{\tau \in T} \rho_\tau(q)\, \mu_{\alpha(\tau)}(q)\left(\frac{\partial}{\partial x^1_\alpha}\Big|_q, \cdots \frac{\partial}{\partial x^n_\alpha}\right.$$

Since $\sum_{\tau \in T} \rho_\tau(q) = 1$ there exist $\tau \ni \rho_\tau(q) > 0$ and since $\rho_{\tau'}(q) \geq 0 \;\forall \tau'$ and since

$$\mu_{\alpha(\tau)}(q)\left(\frac{\partial}{\partial x^1_\alpha}\Big|_q, \dots, \frac{\partial}{\partial x^n_\alpha}\Big|_q\right) =$$

$$= \left(d_q x^1_{\alpha(\tau)} \wedge \dots \wedge d_q x^n_{\alpha(\tau)}\right)\left(\frac{\partial}{\partial x^1_\alpha}\Big|_q, \dots, \frac{\partial}{\partial x^n_\alpha}\Big|_q\right)$$

$$= \det\left(\frac{\partial x^i_{\alpha(\tau)}}{\partial x^j_\alpha}\right) > 0 \qquad (\alpha, \alpha(\tau) \in A)$$

it ~~follows~~ that

$$\mu_q\left(\frac{\partial}{\partial x^1_\alpha}\Big|_q, \dots, \frac{\partial}{\partial x^n_\alpha}\Big|_q\right) > 0 \qquad \forall\, \alpha \in A,\ q \in U_\alpha.$$

Conversely let μ be any volume on M. Given any $q \in M$ and any admissible chart (U, x) of M with $q \in U$ we know $x(q)$ is in the open set $x(U) \subseteq \mathbb{R}^n$. So there is a ball B centered at $x(q)$ lying in $x(U)$. $V = x^{-1}(B)$ is open in U and $(V, x|V)$ is a chart about q with a ball as its image. Thus there is an atlas of admissible charts in M such that the domain of each of the charts is path connected.

if $u, v \in V$ choose a curve δ from $x(u)$ to $x(v)$ in the ball B. Then $\gamma = x^{-1} \circ \delta$ is a path from u to v in V

Notice that if (U, x) is an admissible chart of M and U is path connected then either

$$\mu_q\left(\frac{\partial}{\partial x^1}\Big|_q, \dots \frac{\partial}{\partial x^n}\Big|_q\right)$$

is positive for all $q \in U$ or

$$\mu_q\left(\frac{\partial}{\partial x^1}\Big|_q, \dots, \frac{\partial}{\partial x^n}\Big|_q\right)$$

is negative for all $q \in U$. Indeed if p and q are any two points of U ~~then~~ there is a path $\gamma : [a, b] \to U$ such ~~that~~ $\gamma(a) = p$, $\gamma($

and since $t \mapsto \mu_{\gamma(t)}\left(\frac{\partial}{\partial x^1}\Big|_{\gamma(t)}, \dots, \frac{\partial}{\partial x^n}\Big|_{\gamma(t)}\right)$ is a continuous function from $[a,b]$ to $\mathbb{R}$ which is never zero either

$$\mu_{\gamma(t)}\left(\frac{\partial}{\partial x^1}\Big|_{\gamma(t)}, \dots, \frac{\partial}{\partial x^n}\Big|_{\gamma(t)}\right)$$

is positive for all t or it is negative for all $t \in [a,b]$. Thus the map

$$q \longrightarrow \mu_q\left(\frac{\partial}{\partial x^1}\Big|_q, \dots, \frac{\partial}{\partial x^n}\Big|_q\right)$$

is positive throughout U or negative throughout U.

If we summarize what we have shown we may say ~~that~~ there exists an atlas of admissible charts $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ of M such that for each α

$$\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q, \dots, \frac{\partial}{\partial x_\alpha^n}\Big|_q\right) \text{ is positive } \forall\, q \in U_\alpha$$

or $\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q, \dots, \frac{\partial}{\partial x_\alpha^n}\Big|_q\right)$ is negative $\forall\, q \in U_\alpha$.

Given such an atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ let $\overline{U}_\alpha = U_\alpha$ and let $\overline{x}_\alpha = x_\alpha$ if $\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q, \dots, \frac{\partial}{\partial x_\alpha^n}\Big|_q\right) > 0$ $\forall\, q \in \overline{U}_\alpha$ and let

$$\overline{x}_\alpha^1 = x_\alpha^2, \quad \overline{x}_\alpha^2 = x_\alpha^1, \quad \overline{x}_\alpha^i = x_\alpha^i$$

for $i > 2$ if $\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q, \dots, \frac{\partial}{\partial x_\alpha^n}\Big|_q\right) < 0$ $\forall\, q \in \overline{U}_\alpha = U_\alpha$. Then $\tilde{\mathcal{A}} = \{(\overline{U}_\alpha, \overline{x}_\alpha)\}_{\alpha \in A}$ is a new atlas of admissible charts of M and

$$\mu_q\left(\frac{\partial}{\partial x_\alpha^1}\Big|_q, \dots, \frac{\partial}{\partial x_\alpha^n}\Big|_q\right) > 0$$

for all $q \in \overline{U}_\alpha$

Finally, we show $\{(\overline{U}_\alpha, \overline{x}_\alpha)\}_{\alpha \in A}$ is an oriented atlas of M. Assume $\overline{U}_\alpha \cap \overline{U}_\beta \neq \emptyset$ and ~~that~~ $q \in \overline{U}_\alpha \cap \overline{U}_\beta$. We show

$$\det\left(\frac{\partial \overline{x}_\beta^i}{\partial \overline{x}_\alpha^j}(q)\right) > 0.$$

Observe ~~that~~

$$\frac{\partial}{\partial \overline{x}_\alpha^i}\Big|_q = \frac{\partial \overline{x}_\beta^j}{\partial \overline{x}_\alpha^i}(q) \frac{\partial}{\partial \overline{x}_\beta^j}$$

and

$$\mu_q\left(\frac{\partial}{\partial \overline{x}_\alpha^1}\Big|_q, \ldots, \frac{\partial}{\partial \overline{x}_\alpha^n}\Big|_q\right) = \det\left(\frac{\partial \overline{x}_\beta^j}{\partial \overline{x}_\alpha^i}(q)\right)\mu_q\left(\frac{\partial}{\partial \overline{x}_\beta^1}\Big|_q, \ldots \frac{\partial}{\partial \overline{x}_\beta^n}\Big|_q\right)$$

Since $\mu_q\left(\frac{\partial}{\partial \overline{x}_\gamma^1}\Big|_q, \ldots, \frac{\partial}{\partial \overline{x}_\gamma^n}\Big|_q\right)$ is positive for $\gamma = \alpha, \beta$ and for all $q \in \overline{U}_\alpha \cap \overline{U}_\beta$ we have ~~that~~

$$\det\left(\frac{\partial \overline{x}_\beta^j}{\partial \overline{x}_\alpha^i}(q)\right) > 0 \quad \text{as required.}$$

Page 15 Continued

(*) We show $\mu = \sum_\tau \rho_\tau \mu_\tau$ is a volume. To do this we first show it is smooth. Let $p \in M$ and choose a neighborhood N_p of p such that $N_p \cap V_\tau \neq \emptyset$ for finitely many τ. Let $\tau_1, \tau_2, \ldots, \tau_N$ denote the set of τ's such that $N_p \cap V_\tau \neq \emptyset$.
Clearly $\mu_i \equiv \mu_{\alpha(\tau_i)} = dx^1_{\alpha(\tau_i)} \wedge \ldots \wedge dx^n_{\alpha(\tau_i)}$ is smooth on $U_{\alpha(\tau_i)}$. Moreover $\rho_i \equiv \rho_{\tau_i}$ is smooth on all of M and vanishes on the complement of $\operatorname{supp} \rho_i$ which is contained in $V_{\tau_i} \subseteq U_{\alpha(\tau_i)}$. So $\rho_i \mu_i$

may be regarded as a smooth form defined on all of M which is zero everywhere except on the "interior" of its support. Thus

$$\mu = \sum_\tau \rho_\tau \mu_\tau = \sum_{i=1}^N \rho_i \mu_i$$

is the sum of a finite number of smooth forms and so is smooth. To show μ is a volume we must also show it is nonzero at each point. Let $p \in M$ and choose notation as above. We show $\mu_p \neq 0$. Let $x_i = x_{\alpha(t_i)}$ for each i and consider

$$\mu\left(\frac{\partial}{\partial x_j^1}, \cdots, \frac{\partial}{\partial x_j^n}\right) = \sum_i \rho_i \mu_i\left(\frac{\partial}{\partial x_j^1}, \cdots, \frac{\partial}{\partial x_j^n}\right)$$

$$= \sum_i \rho_i \,(dx_i^1 \wedge \cdots \wedge dx_i^n)\left(\frac{\partial}{\partial x_j^1}, \cdots, \frac{\partial}{\partial x_j^n}\right)$$

$$= \sum_i \rho_i \det\left(\frac{\partial x_i}{\partial x_j}\right)$$

for each $1 \le j \le N$ (here $\frac{\partial x_i}{\partial x_j}$ is the matrix whose k,l components are $\left(\frac{\partial x_i^k}{\partial x_j^l}\right)$).

The determinants are positive at each point of $U_{\alpha(t_i)} \cap U_{\alpha(t_j)}$ and $\rho_i \ge 0$ on all of M. Moreover there exists $1 \le j \le N$ such that $\rho_j(p) > 0$ since $\sum_{i=1}^N \rho_i(p) = 1$. Thus

$$\mu_p\left(\frac{\partial}{\partial x_j^1}\Big|_p, \cdots, \frac{\partial}{\partial x_j^n}\Big|_p\right) > 0$$

and μ_p is not 0. So μ is indeed a volume.

In this section we assume M is a manifold + g a metric on M.

__Definition 1.17__ The codifferential δ is defined to be the mapping from $\Omega^p M$ to $\Omega^{p-1} M$, $1 \le p \le n$, defined by

$$\delta\alpha = (-1)^{n(p+1)+s+1} (* d(*\alpha))$$

for $\alpha \in \Omega^p M$

Notice that if we write $\delta\alpha = \pm * d(*\alpha)$ then one has

$$\delta(\delta\alpha) = \pm (* d *)(* d *)(\alpha)$$
$$= \pm * dd*\alpha = 0$$

regardless of the choices of signs. So $\delta^2 = 0$ and the sign choices are useful only for other reasons.

__Definition 1.8__ If $f : M \to \mathbb{R}$ is a smooth function and X is a vector field on M we define

$$\nabla f = g^{\#}(df)$$

$$\mathrm{Div}(X) = -\delta(g^{b}(X)).$$

Observe that ∇f satisfies the condition

$$d_p f(v) = g_p((\nabla f)(p), v)$$

for all $p \in M$, $v \in T_p M$.

<u>Remark</u> If (U,x) is a chart on M then

(1) $\nabla f = \left(g^{ij}\frac{\partial f}{\partial x^j}\right)\frac{\partial}{\partial x^i}$

(2) $\operatorname{Div} X = \sum_i \frac{\partial}{\partial x^i}\left(|\det G|^{\frac{1}{2}} X^i\right)$, $X = X^i\frac{\partial}{\partial x^i}$.

Here G is the matrix (g_{ij}) where $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.

To see that (1) holds note that if $\nabla f = a^i\frac{\partial}{\partial x^i}$ then

$$\frac{\partial f}{\partial x^j} = df\left(\frac{\partial}{\partial x^j}\right) = a^i g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = a^i g_{ij}.$$

Thus $a^i = g^{ij}\frac{\partial f}{\partial x^j}$ and

$$\nabla f = \left(g^{ij}\frac{\partial f}{\partial x^j}\right)\frac{\partial}{\partial x^i}.$$

Similarly if X is a vector field on M then

$$g^b(X) = g^b\left(X^i\frac{\partial}{\partial x^i}\right) = X^i g^b\left(\frac{\partial}{\partial x^i}\right)$$

and

$$\begin{aligned} g^b(X)\left(\frac{\partial}{\partial x^j}\right) &= X^i g^b\left(\frac{\partial}{\partial x^i}\right)\left(\frac{\partial}{\partial x^j}\right) \\ &= X^i g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ &= X^i g_{ij} \\ &= X^i g_{ik}\, dx^k\left(\frac{\partial}{\partial x^j}\right) \end{aligned}$$

This holds for all j, thus

$$g^b(X) = X^i g_{ik}\, dx^k.$$

We have

$$\begin{aligned} \operatorname{Div} X &= -\delta\left((X^i g_{ik})dx^k\right) \\ &= -(-1)^{2n+s+1}(*\,d\,*)\left(X^i g_{ik}\, dx^k\right) \end{aligned}$$

$$= (-1)^s * d\left[|\det G|^{\frac{1}{2}} g^{jk}(X^i g_{ik})\varepsilon_{j|s_1\cdots s_{n-1}|}(dx^{s_1}\wedge\cdots\wedge dx^{s_{n-1}})\right]$$

$$= (-1)^s * d\left[|\det G|^{\frac{1}{2}} X^j \varepsilon_{j|s_1\cdots s_{n-1}|}(dx^{s_1}\wedge\cdots\wedge dx^{s_{n-1}})\right]$$

$$= (-1)^s * \left(\frac{\partial}{\partial x^i}(|\det G|^{\frac{1}{2}} X^j)\right)\varepsilon_{j|s_1\cdots s_{n-1}|}(dx^i\wedge dx^{s_1}\wedge\cdots\wedge dx^{s_{n-1}})$$

$$= (-1)^s * \left[\left(\sum_i \frac{\partial}{\partial x^i}(|\det G|^{\frac{1}{2}} X^i)\right)\right]\varepsilon_{i|s_1\cdots s_{n-1}|}dx^i\wedge dx^{s_1}\wedge\cdots\wedge dx^{s_n}$$

$$= (-1)^s * \left[\sum_i \frac{\partial}{\partial x^i}(|\det G|^{\frac{1}{2}} X^i)\right]\varepsilon_{12\cdots n}(dx^1\wedge dx^2\wedge\cdots\wedge dx^n)$$

Where in the next to last equality we used the fact that the only nonzero terms in the sum over i and j occur when both i and j are complementary to $s_1 < s_2 < \cdots < s_{n-1}$ in $\{1, 2, \cdots, n\}$ and so have to be equal.

The last equality follows from the fact that $\varepsilon_{i|s_1\cdots s_n|} = \pm\,\varepsilon_{12\cdots n}$ and $dx^i\wedge dx^{s_1}\wedge\cdots\wedge dx^{s_{n-1}} = \pm\,(dx^1\wedge\cdots\wedge dx^n)$ have the same signs. Continuing we have

$$\operatorname{Div} X = (-1)^s(-1)^s \sum_i \frac{\partial}{\partial x^i}\left(|\det G|^{\frac{1}{2}} X^i\right)$$

$$= \sum_i \frac{\partial}{\partial x^i}\left(|\det G|^{\frac{1}{2}} X^i\right).$$

Now consider the curl of a vector field? We will see that $\tilde{g}^{\#}(* d(g^{b}(X)))$ reduces to the curl in restricted circumstances so let us calculate $\tilde{g}^{\#}(* d(g^{b}(X)))$ in local coordinates to determine those circumstances. We have shown that

$$g^{b}(X) = X^{i} g_{ij}\, dx^{j},$$

thus

$$d(g^{b}X) = \frac{\partial}{\partial x^{k}}(X^{i} g_{ij})(dx^{k} \wedge dx^{j})$$

$$= \sum_{k<j} \left[\frac{\partial}{\partial x^{k}}(X^{i} g_{ij}) - \frac{\partial}{\partial x^{j}}(X^{i} g_{ik})\right](dx^{k} \wedge dx^{j})$$

and

$$* d(g^{b}(X)) = |\det G|^{\frac{1}{2}} g^{ka} g^{jb} [\quad]_{kj}\, \varepsilon_{|ab||s_1 \cdots s_{n-2}|} (dx^{s_1} \wedge \cdots \wedge dx^{s_{n-2}})$$

Where $[\quad]_{kj} = \left[\frac{\partial}{\partial x^{k}}(X^{i} g_{ij}) - \frac{\partial}{\partial x^{j}}(X^{i} g_{ik})\right]$.

Now $\tilde{g}^{\#}$ converts $*(d(g^{b}(X)))$ to an element of $(\Lambda^{n-2}M)^{*}$ at each point of M, a space of dimension $\frac{1}{2}(n-1)n$ since

$$\dim(\Lambda^{n-2}M) = \binom{n}{n-2} = \frac{1}{2}(n-1)n.$$

The dual of $n-2$ forms are not differential forms in general but rather are contravariant tensor fields. In case $n=3$ the dimension of $\Lambda^{m-2}(T^*M) = (\Lambda^1 T^*M)^*$ (at each point of M) which can be identified with vector fields on M. If μ_g is a volume compatible with the metric g and (x^i) are coordinates of M positively oriented relative to μ_g then

$$\tilde{g}^\#(*d(g^b(X))) = \tilde{g}^\#\left(|\det G|^{\frac{1}{2}} g^{ka} g^{jb} [\ \ ]_{kj} \varepsilon_{|ab|s}\, dx^s\right)$$

$$= |\det G|^{\frac{1}{2}} g^{sc} g^{ka} g^{jb} [\ \ ]_{kj} \varepsilon_{|ab|s} \frac{\partial}{\partial x^s}$$

Assume that M is either $\mathbb{R}^3$ with the Euclidean metric then

$$g^\#(*d(g^b(X))) = \sum_{a,b,c} [\ \ ]_{ab}\, \varepsilon_{|ab|c} \frac{\partial}{\partial x^c}$$

$$= \sum_{a,b,c} \left[\frac{\partial}{\partial x^a}(X^b) - \frac{\partial}{\partial x^b}(X^a)\right] \varepsilon_{|ab|c} \frac{\partial}{\partial x^c}$$

The only nonzero terms involve a, b such that $\{a=1, b=2\}$, $\{a=2, b=3\}$, $\{a=1, b=3\}$

So

$$g^\#(*d(g^b(X))) = \left[\frac{\partial}{\partial x^1}(X^2) - \frac{\partial}{\partial x^2}(X^1)\right] \varepsilon_{12c} \frac{\partial}{\partial x^c}$$

$$+ \left[\frac{\partial}{\partial x^2}(X^3) - \frac{\partial}{\partial x^3}(X^2)\right] \varepsilon_{23c} \frac{\partial}{\partial x^c}$$

$$+ \left[\frac{\partial}{\partial x^1}(X^3) - \frac{\partial}{\partial x^3}(X^1)\right] \varepsilon_{13c} \frac{\partial}{\partial x^c}$$

$$= \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^3}{\partial x^1} - \frac{\partial X^1}{\partial x^3}\right)\frac{\partial}{\partial x^2} + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}$$

and so $\quad g^{\#}\left(* d\left(g^{b}(X)\right)\right) = \operatorname{Curl} X.$

Definition The Laplace-Beltrami operator on M is the mapping $\Delta^p : \Omega^p M \to \Omega^p M$ defined by
$$\Delta^p = \delta d + d\delta.$$

Note that if $f \in \Omega^0 M$ is a smooth function then δf is <u>defined to be zero</u> and
$$\Delta f = \delta(df) + d(\delta f) = \delta(df)$$
$$= \delta\left(g^b\left(g^{\#}(df)\right)\right) = -\operatorname{Div}\left(g^{\#}(df)\right)$$
and
$$\boxed{\Delta f = -\operatorname{Div}(\nabla f)}$$
If (x^i) are positively oriented relative to a volume μ_g which is compatible with a metric g, then
$$\Delta f = -\operatorname{Div}\left(g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}\right)$$
$$\boxed{\Delta f = -\frac{\partial}{\partial x^i}\left(|\det G|^{\frac{1}{2}} g^{ij}\frac{\partial f}{\partial x^j}\right)}$$

Finally if $g_{ij} = \delta_{ij}$ then
$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2}$$

Theorem Assume that g is a metric and μ_g is a compatible volume on a manifold M. If $\alpha \in \Omega^k M$, $\beta \in \Omega^{k+1} M$ then

$$\tilde{g}(d\alpha, \beta)\mu_g = \tilde{g}(\alpha, \delta\beta)\mu_g + d(\alpha \wedge *\beta)$$

Proof First note that $\delta\beta = (-1)^{n(k+2)+s+1}(* d * \beta)$.
Let $p = [n-(k+1)]+1 = n-k$, then $*\beta \in \Omega^{n-(k+1)} M$ and $d(*\beta) \in \Omega^p M$. Thus

$$** (d*\beta) = (-1)^{p(n-p)+s} d(*\beta)$$

and

$$*(\delta\beta) = (-1)^{n(k+2)+s+1} *(* d*\beta)$$
$$= (-1)^{n(k+2)+s+1}(-1)^{p(n-p)+s} d(*\beta).$$

But

$$n(k+2)+s+1+p(n-p)+s = nk+2n+s+1+(n-k)k+s$$
$$= 2n+2s+2nk-k^2+1$$

If k is even then $-k^2+1$ is odd. If k is odd then $-k^2+1$ is even, thus

$$\boxed{* \delta \beta = (-1)^{k+1} d(*\beta)}$$

Thus

$$\tilde{g}(d\alpha, \beta)\mu_g - \tilde{g}(\alpha, \delta\beta)\mu_g = d\alpha \wedge (*\beta) - \alpha \wedge (* \delta\beta)$$
$$= (d\alpha \wedge (*\beta)) - (-1)^{k+1}(\alpha \wedge d(*\beta))$$
$$= (d\alpha \wedge (*\beta)) + (-1)^k(\alpha \wedge d(*\beta))$$
$$= d(\alpha \wedge *\beta) \qquad \square$$

<u>Corollary</u> Let M be an oriented manifold of dimension n with metric g and compatible volume μ_g. If $\alpha \in \Omega^k M$, $\beta \in \Omega^{k+1} M$ have compact support then $\int_M \hat{g}(\alpha, \delta\beta)\mu_g = \int_M \hat{g}(d\alpha, \beta)\mu_g$.

<u>Proof</u>

$$\int_M \hat{g}(d\alpha, \beta)\mu_g = \int_M \hat{g}(\alpha, \delta\beta)\mu_g + \int_M \underbrace{d(\alpha \wedge *\beta)}_{0}$$

$$= \int_M \hat{g}(\alpha, \delta\beta)\mu_g \qquad \text{Stokes Thm.}$$

<u>Remark</u> It is left to the reader to show that if $\omega \in \Omega^p M$ and (U, x) is a chart on M, then $\omega = \frac{1}{k!}\, \omega_{i_1 \cdots i_p}(dx^{i_1} \wedge \cdots \wedge dx^{i_p})$ where

$$\omega_{i_1 \cdots i_p} = \omega\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial x^{i_2}}, \cdots, \frac{\partial}{\partial x^{i_p}}\right).$$

<u>Lemma</u> Let $\alpha \in \Omega^k M$, $\alpha = \frac{1}{k!}\alpha_{i_1 \cdots i_k}(dx^{i_1} \wedge \cdots \wedge dx^{i_k})$ in a local chart (U, x). Then

$$d\alpha = \frac{1}{(k+1)!}(d\alpha)_{j_1 \cdots j_{k+1}}(dx^{j_1} \wedge \cdots \wedge dx^{j_{k+1}})$$

where the nonzero terms of the sum are given by

$$(d\alpha)_{j_1 \cdots j_{k+1}} = d\alpha\left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \frac{(-1)^k}{k!}\sum_{\sigma \in S(k+1)} (-1)^\sigma \alpha_{j_{\sigma 1} \cdots j_{\sigma k}, j_{\sigma(k+1)}}$$

<u>Proof</u> By the Remark we have that

$$(d\alpha)_{j_1 \cdots j_{n+1}} = d\alpha\left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_{k+1}}}\right)$$

On the other hand we have that

$$d\alpha = \frac{1}{k!}\, d\left[\alpha_{i_1\cdots i_k}(dx^{i_1}\wedge\cdots\wedge dx^{i_k})\right]$$

$$= \frac{1}{k!}(\partial_j\alpha_{i_1\cdots i_k})(dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}).$$

Thus

$$d\alpha(\partial_{j_1},\cdots,\partial_{j_{k+1}}) = \frac{1}{k!}(\partial_j\alpha_{i_1\cdots i_k})(dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k})(\partial_{j_1},\cdots,\partial_{j_{k+1}})$$

$\boxed{i_{k+1}\equiv j}$

$$= \frac{(-1)^k}{k!}\alpha_{i_1\cdots i_k,i_{k+1}}(dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}})(\partial_{j_1},\cdots,\partial_{j_{k+1}})$$

$$= \frac{(-1)^k}{k!}\alpha_{i_1\cdots i_k,i_{k+1}}\,\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}}$$

where

$$\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}} = (dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}})(\partial_{j_1},\cdots,\partial_{j_{k+1}})$$

Now $\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}} = 0$ unless there is a permutation σ such that $i_a = j_{\sigma a}$ for all a. Observe that we may assume that $j_1, j_2, \ldots, j_{k+1}$ are all distinct as we are only interested in the nonzero terms in the expansion of $d\alpha$. Thus the only nonzero terms of the sum $\alpha_{i_1\cdots i_k,i_{k+1}}\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}}$ are those in which $i_a = j_{\sigma a}$ for some σ and all a.

1 On the other hand we have that

$$d\alpha = \frac{1}{k!}\, d\left[\alpha_{i_1\cdots i_k}(dx^{i_1}\wedge\cdots\wedge dx^{i_k})\right]$$

$$= \frac{1}{k!}\,(\partial_j\alpha_{i_1\cdots i_k})(dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}).$$

Thus

$$d\alpha(\partial_{j_1},\cdots,\partial_{j_{k+1}}) = \frac{1}{k!}(\partial_j\alpha_{i_1\cdots i_k})(dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k})(\partial_{j_1},\cdots,\partial_{j_{k+1}})$$

$\boxed{i_{k+1}\equiv j}$

$$= \frac{(-1)^k}{k!}\,\alpha_{i_1\cdots i_k, i_{k+1}}(dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}})(\partial_{j_1},\cdots,\partial_{j_{k+1}})$$

$$= \frac{(-1)^k}{k!}\,\alpha_{i_1\cdots i_k, i_{k+1}}\,\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}}$$

where

$$\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}} = (dx^{i_1}\wedge\cdots\wedge dx^{i_{k+1}})(\partial_{j_1},\cdots,\partial_{j_{k+1}})$$

Now $\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}} = 0$ unless there is a permutation σ such that $i_a = j_{\sigma a}$ for all a. Observe that we may assume that $j_1, j_2, \cdots, j_{k+1}$ are all distinct as we are only interested in the nonzero terms in the expansion of $d\alpha$. Thus the only nonzero terms of the sum $\alpha_{i_1\cdots i_k, i_{k+1}}\delta^{i_1\cdots i_{k+1}}_{j_1\cdots j_{k+1}}$ are those in which $i_a = j_{\sigma a}$ for some σ and all a, Thus

$$d\alpha(\partial_{j_1},\dots,\partial_{j_{k+1}}) = \frac{(-1)^k}{k!}\sum_\sigma \alpha_{j_{\sigma 1}\cdots j_{\sigma k}, j_{\sigma(k+1)}}\,\delta^{j_{\sigma 1}\cdots j_{\sigma(k+1)}}_{j_1\cdots j_{k+1}}.$$

But

$$\begin{aligned}
\delta^{j_{\sigma 1}\cdots j_{\sigma(k+1)}}_{j_1\cdots j_{k+1}} &= (dx^{j_{\sigma 1}}\wedge\cdots\wedge dx^{j_{\sigma(k+1)}})(\partial_{j_1},\dots,\partial_{j_{k+1}})\\
&= (-1)^\sigma (dx^{j_1}\wedge\cdots\wedge dx^{j_{k+1}})(\partial_{j_1},\dots,\partial_{j_{k+1}})\\
&= (-1)^\sigma \sum_\tau (-1)^\tau (dx^{j_{\tau 1}}\otimes\cdots\otimes dx^{j_{\tau(k+1)}})(\partial_{j_1},\dots,\partial_{j_{k+1}})\\
&= (-1)^\sigma \sum_\tau (-1)^\tau \delta^{j_{\tau 1}}_{j_1}\delta^{j_{\tau 2}}_{j_2}\cdots\delta^{j_{\tau(k+1)}}_{j_{k+1}}\\
&= (-1)^\sigma.
\end{aligned}$$

So $$d\alpha(\partial_{j_1},\dots,\partial_{j_{k+1}}) = \frac{(-1)^k}{k!}\sum_\sigma (-1)^\sigma \alpha_{j_{\sigma 1}\cdots j_{\sigma k}, j_{\sigma(k+1)}}$$

as required.

<u>Theorem 1.21</u> Assume that M is a manifold, that g is a metric on M, that μ_g is a volume on M compatible with g. If (U,x) is a chart on M positively oriented with respect to μ_g then the components of $\delta\beta$, for $\beta\in\Omega^{k+1}M$, are given by

$$(\delta\beta)^{j_1\cdots j_k} = \frac{(-1)^{k+1}}{k!\,|\det G|^{\frac12}}\frac{\partial}{\partial x^j}\left[|\det(G)|^{\frac12}\beta^{j_1\cdots j_k j}\right]$$

where

$$\beta = \frac{1}{(k+1)!}\beta_{i_1\cdots i_{k+1}}(dx^{i_1}\wedge\cdots\wedge dx^{i_k})$$

Proof If ρ and η are differential forms we write $\rho \approx \eta$ iff $\rho - \eta$ is exact. If $\beta \in \Omega^{k+1}M$ and $\alpha \in \Omega^k M$ such that $\operatorname{supp}\alpha \subseteq U$ is compact, then

$$\tilde{g}(\alpha, \delta\beta)\mu_g \approx \tilde{g}(d\alpha, \beta)\mu_g$$

$$= \frac{1}{(k+1)!}(d\alpha)_{j_1 \cdots j_{k+1}} \beta^{j_1 \cdots j_{k+1}} \mu_g$$

$$= \frac{(-1)^k}{k!(k+1)!} \sum_\sigma (-1)^\sigma \alpha_{j_{\sigma 1} \cdots j_{\sigma k}, j_{\sigma(k+1)}} \beta^{j_1 \cdots j_{k+1}} \mu_g$$

$j'_a \equiv j_{\sigma a}$

$$= \frac{(-1)^k}{k!(k+1)!} \sum_\sigma (-1)^\sigma \left[\alpha_{j'_1 \cdots j'_k, j'_{k+1}} \beta^{j'_{\sigma^{-1}1} \cdots j'_{\sigma^{-1}(k+1)}}\right] \mu_g$$

$$= \frac{(-1)^k}{k!(k+1)!} \sum_\sigma (-1)^\sigma \left[\alpha_{j'_1 \cdots j'_k, j'_{k+1}} (-1)^{\sigma^{-1}} \beta^{j'_1 \cdots j'_{k+1}}\right] \mu_g$$

$$= \frac{(-1)^k}{k!(k+1)!} \sum_\sigma \left[\alpha_{j'_1 \cdots j'_k, j'_{k+1}} \beta^{j'_1 \cdots j'_{k+1}}\right] \mu_g$$

$j' \to j$

$$= \frac{(-1)^k}{k!} \frac{\partial}{\partial x^{j_{k+1}}}\left(\alpha_{j_1 \cdots j_k}\right) \beta^{j_1 \cdots j_{k+1}} |\det G|^{\frac{1}{2}} d^n x$$

where $d^n x = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. Now let

$$\gamma^j = \frac{(-1)^k}{k!} \alpha_{j_1 \cdots j_k} \beta^{j_1 \cdots j_k j} |\det G|^{\frac{1}{2}}$$

and let $\bar{\delta}\beta$ denote the k-form defined on U by

$$(\bar{\delta}\beta)^{j_1 \cdots j_k} = \frac{(-1)^{k+1}}{k!\,|\det G|^{\frac{1}{2}}} \partial_j\left(|\det G|^{\frac{1}{2}} \beta^{j_1 \cdots j_k j}\right)$$

Now $\frac{\partial}{\partial x^j}\left(\alpha_{j_1 \cdots j_k}\right) \beta^{j_1 \cdots j_k j} |\det G|^{\frac{1}{2}} =$

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$$= \frac{\partial}{\partial x^j}\left(\alpha_{j_1\cdots j_R}\beta^{j_1\cdots j_R j}|\det G|^{\frac{1}{2}}\right) - \alpha_{j_1\cdots j_R}\frac{\partial}{\partial x^j}\left(\beta^{j_1\cdots j_R j}|\det G|^{\frac{1}{2}}\right)$$

"Moreover"

$$(\bar{\delta}\beta)^{j_1\cdots j_R}\mu_g = \frac{(-1)^{R+1}}{R!}\,\partial_j\left(|\det G|^{\frac{1}{2}}\beta^{j_1\cdots j_R j}\right)d^n x$$

so that

$$\tilde{g}(\alpha,\delta\beta)\mu_g \simeq (\partial_j \gamma^j)\,d^n x - \frac{(-1)^R}{R!}\alpha_{j_1\cdots j_R}\partial_j\left(\beta^{j_1\cdots j_R j}|\det G|^{\frac{1}{2}}\right)d^n x$$

$$= d\left(\sum_j \gamma^j (dx^1\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots\wedge dx^n)\right)$$

$$+ \alpha_{j_1\cdots j_R}(\bar{\delta}\beta)^{j_1\cdots j_R}\mu_g$$

$$= d\left(\gamma^j (d^n x)_j\right) + \tilde{g}(\alpha,\bar{\delta}\beta)\mu_g$$

Thus

$$\tilde{g}(\alpha,\delta\beta-\bar{\delta}\beta)\mu_g = d\left(\gamma^j (d^n x)_j\right).$$

γ^j has compact support in U so

$$\int_U \tilde{g}(\alpha,\delta\beta-\bar{\delta}\beta)\mu_g = \int_U d\left(\gamma^j (d^n x)_j\right) = 0$$

by Stokes formula. Thus

$$\int_U \tilde{g}(\alpha,\delta\beta)\mu_g = \int_U \tilde{g}(\alpha,\bar{\delta}\beta)\mu_g$$

for all α with compact support in U, so $\delta\beta = \bar{\delta}\beta$ on U since the metric $\hat{g}$ on $\Omega^R U$ defined by $\hat{g}(\alpha,\gamma) = \int_U \tilde{g}(\alpha,\gamma)\mu_g$ is nondegenerate.