Group Actions

Definition Let G be a group and S any set. To say that σ is a left-action of G on S means that σ is a function from $G \times S$ to S such that $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1 g_2, x)$ and $\sigma(e, x) = x$ for all $x \in S$ and $g_1, g_2 \in G$. Similarly, σ is a right-action of G on S if σ is a function from $S \times G$ to S such that $\sigma(x, g_1 g_2) = \sigma(\sigma(x, g_1), g_2)$ and $\sigma(x, e) = x$ for $x \in S$, $g_1, g_2 \in G$. When σ is a left action $\sigma(g, x)$ is denoted $g \cdot x$ but when it is a right-action $\sigma(x, g) = x \cdot g$. In this notation one has:

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \quad e \cdot x = x \quad \text{(left-action)}$$

$$x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2, \quad x \cdot e = x \quad \text{(right-action)}$$

In the case ~~that~~ S is a manifold and G is a Lie-group it is required that σ be smooth though this is often emphasized by saying that σ is a smooth action. Often the mapping σ itself is surpressed when $\sigma(g, x)$ is denoted by $g \cdot x$. In such a case the notation used for left and right translations in a Lie group is extended to actions of the Lie group on a manifold. Thus, for a given left-action we define $l_g : S \to S$ by $l_g(x) = g \cdot x$ for $g \in G$, $x \in S$. Similarly, for a given right-action we define $r_g : S \to S$ by $r_g(x) = x \cdot g$ for $x \in S$, $g \in G$.

<u>Remark</u> Observe that each left-action σ of a Lie group G on a manifold M defines a homomorphism $\hat{\sigma}$ from G into the group of diffeomorphisms of M. Indeed, if $\mathrm{Diff}(M)$ denotes the set of all diffeomorphisms of M then $\mathrm{Diff}(M)$ is obviously a group with respect to composition of diffeomorphisms. Moreover, for each $g \in G$ $l_g = \sigma \circ i_g$ where $i_g : M \to G \times M$ is defined by $i_g(x) = (g, x)$. Since i_g and σ are smooth so is l_g. It is clear that l_g is in fact a diffeomorphism since $l_{g^{-1}}$ is also smooth and $l_g \circ l_{g^{-1}} = id_M = l_{g^{-1}} \circ l_g$. So we may define $\hat{\sigma} : G \to \mathrm{Diff}(M)$ by $\hat{\sigma}(g) = l_g$. Clearly

$$\hat{\sigma}(g_1 g_2) = l_{g_1 g_2} = l_{g_1} \circ l_{g_2} = \hat{\sigma}(g_1) \circ \hat{\sigma}(g_2)$$

for $g_1, g_2 \in G$ so $\hat{\sigma}$ is a homomorphism.

Note that the corresponding mapping for right-actions fails to be a homomorphism but is instead an anti-homomorphism due to the fact that $r_{g_1 g_2} = r_{g_2} \circ r_{g_1}$ for $g_1, g_2 \in G$.

<u>Examples</u>

(1) Let $G \subseteq Gl(n, \mathbb{R})$ be any Lie subgroup and define $\sigma : G \times \mathbb{R}^n \to \mathbb{R}^n$ by $\sigma(g, x) = gx$ where gx is the product of the matrix $g \in Gl(n, \mathbb{R})$ and the vector $x \in \mathbb{R}$

(2) Let $G = Gl(n, \mathbb{R})$ and $M = T^0_2(\mathbb{R}^n)$. For $g \in G$, $b \in M$ define $g \cdot b$ by
$$(g \cdot b)(x, y) = b(g^{-1}x, g^{-1}y)$$
for $x, y \in \mathbb{R}^n$. It is easy to show that the mapping $\sigma : G \times M \to M$ defined by $\sigma(g, b) = g \cdot b$ is a left-action of G on M.

(3) A generalization of the last action may be obtained as follows. Recall that a tensor $\tau \in T^p_q \mathbb{R}^n$ is a multilinear mapping from $(\mathbb{R}^n \times \cdots \times \mathbb{R}^n) \times (\mathbb{R}^n)^* \times \cdots \times (\mathbb{R}^n)^*$ into $\mathbb{R}$ where there are p factors of $\mathbb{R}^n$ and q factors of $(\mathbb{R}^n)^*$. For $g \in Gl(n, \mathbb{R})$ and $\tau \in T^p_q(\mathbb{R}^n)$ define $g \cdot \tau$ by

$$(g \cdot \tau)(x_1, x_2, \cdots, x_p, \alpha_1, \cdots, \alpha_q) = \tau(g^{-1} \cdot x_1, \cdots, g^{-1} \cdot x_p, g^{-1} \cdot \alpha_1, \cdots g^{-1}\alpha_q)$$
where $x_i \in \mathbb{R}^n$, $\alpha_j \in (\mathbb{R}^n)^*$, $1 \leq i \leq p$, $1 \leq j \leq q$ and where for $\alpha \in (\mathbb{R}^n)^*$, $x \in \mathbb{R}^n$
$$(g \cdot \alpha)(x) = \alpha(g^{-1}x).$$
It is easy to show that $\sigma : G \times T^p_q \mathbb{R}^n \to T^p_q \mathbb{R}^n$ defined by $\sigma(g, \tau) = g \cdot \tau$ is a left action.

(4) Let X be a vector field on $\mathbb{R}^n$ such that each solution of the differential equation $\gamma'(t) = X(\gamma(t))$ exists for all $t \in \mathbb{R}$. If η is the flow of X, meaning that $\eta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the smooth mapping such that $\frac{d}{dt}(\eta(t, x)) = X(\eta(t, x))$, $\eta(0, x) = x$

action of $\mathbb{R}$ on $\mathbb{R}^n$. Indeed, $\eta : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ has the properties

$$\eta(t_1+t_2, x) = \eta(t_1, \eta(t_2, x))$$
$$\eta(0, x) = x$$

for $t_1, t_2 \in \mathbb{R}$ and $x \in \mathbb{R}^n$. These properties follow from the theorems on existence and uniqueness of solutions of differential equations so we surpress the details. An explicit example is the vector field X on $\mathbb{R}^2$ defined by

$$X(x,y) = y \frac{\partial}{\partial x}\Big|_{(x,y)} - x\frac{\partial}{\partial y}\Big|_{(x,y)}.$$

If $\eta(t,(x_0,y_0)) = (x(t), y(t))$ then

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -x$$

$x(0) = x_0$, $y(0) = y_0$. The reader may show that

$$x(t) = x_0 \cos t + y_0 \sin t$$
$$y(t) = -x_0 \sin t + y_0 \cos t$$

and so

$$\eta(t,(x_0,y_0)) = (x_0, y_0)\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

This defines a left-action $\mathbb{R}$ on $\mathbb{R}^2$ (in column notation)

(5) A Lie group acts on itself via left-translation

Definition If a Lie group G acts on the left of a manifold M then the *isotropy subgroup* of G at $x \in M$ is $\{g \in G \mid g\cdot x = x\}$. It is a subgroup of G and in fact is a Lie subgroup of G. If $x \in M$ then the subset $\{g\cdot x \mid g \in G\}$ of M

is called the orbit of the action through x or simply the orbit of x. The orbit through x is denoted $G\cdot x$. Clearly one has analogous definitions for right actions;

$$G_x = \{g \in G \mid x\cdot g = x\}$$
$$x\cdot G = \{x\cdot g \mid g \in G\}.$$

Finally an action of G on M is transitive iff M itself is an orbit of the action.

Example 1 Let $G = (\mathbb{R}, +)$ and let $M = \mathbb{R}^2$. Define a left action of G on M by

$$t\cdot(x,y) = (x,y)\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $t \in G$, $(x,y) \in M$. If $(x_0,y_0) \in \mathbb{R}^2$, $(x_0,y_0) \neq (0,0)$, then the orbit of (x_0,y_0) is the set of points of the form

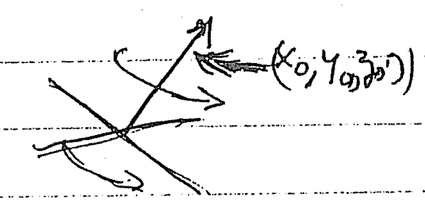
$$(x_0,y_0)\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = ((\cos t)x_0 + (\sin t)y_0, (-\sin t)x_0 + (\cos t)y_0)$$

Note that these points all lie on a circle of radius $\sqrt{x_0^2+y_0^2}$ and the orbit of the action is precisely this circle. The set of all non-trivial orbits ($(0,0)$ is an orbit) of this action is the family of all circles centered at the origin. Notice that the isotropy subgroup of (x_0,y_0) is the set of all $t \in \mathbb{R}$ such that

$$((\cos t)x_0 + (\sin t)y_0, (-\ \text{int})x_0 + (\cos t)y_0) = (x_0, y_0).$$

the point (x_0, y_0) through the angle t we see that $\mathbb{R}_{(x_0,y_0)} = \{ t \in \mathbb{R} \mid t = 2\pi n,\ n \in \mathbb{Z} \} = 2\pi\mathbb{Z}$ is clearly a subgroup of $(\mathbb{R}, +)$. Notice that the orbit through (x_0, y_0) is a circle and $\mathbb{R}/2\pi\mathbb{Z}$ may be identified with this circle. This illustrates a general fact stated below namely that the orbit through a point may be identified with the group modulo its isotropy subgroup (provided the orbit is actually a manifold as it is here).

Example 2 The last example may be generalized. $SO(3)$ acts on $\mathbb{R}^3$ via matrix multiplication: If $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $(x_0, y_0, z_0) \neq (0,0,0)$ then the orbit of (x_0, y_0, z_0) is the set of all vectors in $\mathbb{R}^3$ obtained by rotating (x_0, y_0, z_0) via a matrix $A \in SO(3)$. The set of all such vectors is a sphere of radius $\sqrt{x_0^2 + y_0^2 + z_0^2}$. The only matrices in $SO(3)$ which fix (x_0, y_0, z_0) are those which rotate each vector about the line through (x_0, y_0, z_0). This set of rotations



may be identified with the rotations of the plane orthogonal to (x_0, y_0, z_0) and so may be identified with $SO(2)$. So the isotropy subgroup of (x_0, y_0, z_0) is $SO(2)$. Notice that the orbit of $(0,0,0)$ is a single point, namely $(0,0,0)$. The isotropy subgroup of $(0,0,0)$ is all of $SO(3)$

Example 3 We note that the orbit of a group action may be a fairly complicated subset of the manifold on which the group acts. We describe such an action without writing out all the details. Let M be the torus $M = S^1 \times S^1 = \{ (e^{i\theta}, e^{i\varphi}) \in \mathbb{C} \times \mathbb{C} \}$,

$$M = \{ (e^{2\pi i \varphi}, e^{2\pi i \theta}) \mid \varphi, \theta \in \mathbb{R} \}$$

Define an action of the group $S^1 = \{ e^{2\pi i t} \mid t \in \mathbb{R} \}$ on M by

$$e^{2\pi i t} \cdot (e^{2\pi i \varphi}, e^{2\pi i \theta}) = (e^{2\pi i (t+\varphi)}, e^{2\pi i \alpha (t+\theta)})$$

where α is some fixed irrational number. Consider the orbit defined by $\varphi = 0$, $\theta = 0$. Then this orbit is given by

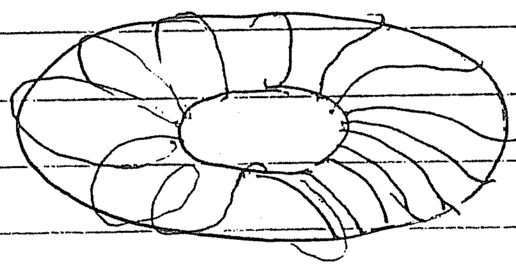
$$\{ e^{2\pi i t} \cdot (1,1) \mid t \in \mathbb{R} \} = \{ (e^{2\pi i t}, e^{2\pi i \alpha t}) \mid t \in \mathbb{R}$$

We claim that for $t_1 \neq t_2$, $(e^{2\pi i t_1}, e^{2\pi i \alpha t_1}) \neq (e^{2\pi i t_2}, e^{2\pi i \alpha t_2})$

Indeed if

$$(e^{2\pi i t_1}, e^{2\pi i \alpha t_1}) = (e^{2\pi i t_2}, e^{2\pi i \alpha t_2})$$

then it follows that $t_2 = t_1 + n$ and $\alpha t_2 = \alpha t_1 + m$ for integers m, n. Thus $t_2 - t_1 = n$ and $\alpha(t_2 - t_1) = m$. This implies that $\alpha = m/n$ a rational number. The orbit of $(1,1)$ on the torus is thus a copy of $\mathbb{R}$. In fact this orbit is <u>dense</u> in the torus and is not a submanifold of the torus.



This "winding" of $\mathbb{R}$ about the torus never closes on itself!

We state two theorems which are useful in constructing examples later but do not prove them here. Their proofs may be found: Warner [ ].

Theorem If H is a closed subgroup of a Lie group G, then

(1) H is a Lie subgroup of G, and

(2) there is one and only one manifold structure on the set G/H of coset of H in G for which the following conditions hold:

(i) the natural mapping η from G to G/H is a C^∞ mapping,

(ii) $\eta : G \to G/H$ is a fiber bundle in the sense that at each point p of G/H there exists a smooth local section of η defined on an open subset of G/H containing p.

Theorem If a Lie group G acts transitively on a manifold M and $x \in M$ then the mapping $\beta : G/G_x \to G \cdot x = M$ defined by $\beta(gG_x) = g \cdot x$ is a diffeomorphism. The result holds for both left and right actions.

Remark. Observe that G_x is indeed a Lie subgroup of G as it is easy to show G_x is closed in G and the result follows from part (1) of the first theorem quoted above. Also part (2) of the first theorem also guarantees that G/G_x is a well-defined manifold. Finally note that by Example 3 above $G \cdot x$ generally is not a manifold or at least not a submanifold of G. This is the rationale behind the assumption that G acts transitively on fact a group always acts transitively on any one of its orbits so if the orbit through x is a manifold then it is diffeomorphic to G/G_x

<u>Definition</u> Assume G is a Lie group which acts on the right of a manifold M. The action is called an <u>effective action</u> iff for $g \in G$ such that $x \cdot g = x$ for all $x \in M$ it follows that $g = e$. The action is called a <u>free action</u> iff for $g \in G$ one has that the existence of $x \in M$ such that $x \cdot g = x$ implies that $g = e$. One defines effective left actions and free left actions analogously:
(i) $g \in G$ such that $g \cdot x = x$ for <u>all</u> x implies $g = e$,
(ii) $g \in G$ such that $g \cdot x = x$ for <u>some</u> x implies $g = e$.

If M is a manifold then the set of all diffeomorphisms of M is denoted $\mathrm{Diff}(M)$. There is a differentiable structure on $\mathrm{Diff}(M)$ relative to which $\mathrm{Diff}(M)$ is a Lie group but the charts take their values in an open subset of an infinite-dimensional vector space. These structures are beyond the scope of the present work. Note that if $\gamma : (a,b) \to \mathrm{Diff}(M)$ is a curve in $\mathrm{Diff}(M)$ then $\gamma(t)$ is a function from M to M for each $t \in (a,b)$ and so for $x \in M$ $\gamma(t)(x)$ defines a curve $t \mapsto \gamma(t)(x)$ in M. The derivative of this curve gives tangent vectors to M. If $0 \in (a,b)$ one can define a vector field X_γ on M by $X_\gamma(x) = \frac{d}{dt}(\gamma(t)(x))\Big|_{t=0}$.

So each curve through the identity of Diff(M) defines a vector field X_γ. With more care one can show ~~that the~~ set ΓM of all vector fields on M is the Lie algebra of Diff(M). If G is an ordinary Lie group acting on M, $\varphi: G \times M \to M$ then for each $g \in G$ we have a mapping $\varphi_g : M \to M$ defined by

$$\varphi_g(x) = g \cdot x = \varphi(g, x).$$

Observe that φ_g is smooth and invertible with $(\varphi_g)^{-1} = \varphi_{g^{-1}}$ for each $g \in G$. So $\varphi_g \in \text{Diff}(M)$ for $g \in G$. Let $\hat{\varphi} : G \to \text{Diff}(M)$ be defined by $\hat{\varphi}(g) = \varphi_g$.

Then $\hat{\varphi}$ is a group homomorphism. With a more careful treatment of the manifold structure on Diff(M) $\hat{\varphi}$ is generically smooth. Note that $g \in \text{Ker}\,\hat{\varphi}$ iff $\hat{\varphi}(g) = \varphi_g$ is the identity mapping id_M and so $g \in \text{Ker}\,\hat{\varphi}$ iff $g \cdot x = x$ for all $x \in M$. Thus the kernel of $\hat{\varphi}$ is trivial iff the action φ is an <u>effective action</u>. In such a case G is embedded as a Lie subgroup of Diff(M). All of these statements are true under appropriate hypothesis but caution should be observed in general. We see that in case G is a Lie subgroup of Diff(M) the Lie algebra of G should be describable as a sub Lie-algebra of Diff(M), i.e. as a subalgebra of ΓM. This is in fact possible and is one motivati for the following definition.

Definition Assume the Lie group G acts on the right a manifold M. For each $A \in \mathfrak{g}$ in the Lie algebra of define a vector field δ_A of M by $\delta_A(x) = \frac{d}{dt}[x \cdot \exp(tA)$
The mapping $\delta : \mathfrak{g} \to \Gamma M$ defined by $\delta(A) = \delta_A$ will be called the <u>infinitesimal generator</u> of the <u>group action</u> or simply the <u>infinitesimal action</u>.

<u>Remark</u>. We formulated the definition in terms of a right action, but clearly there is an analogous concept for left actions. If $\varphi : M \times G \to M$ is an action of G on M then the mapping δ_A is clearly smooth as the mapping from $\mathbb{R} \times M$ to M defined by $(t, x) \longmapsto \varphi(x, \exp(tA))$ is smooth and $\delta_A(x)$ is the partial derivative $\frac{\partial}{\partial t}[\varphi(x, \exp(tA))]\big|_{t=0}$
Thus δ_A is in fact a vector field on M.

<u>Lemma 1</u> Assume that the Lie group G acts on the right of a manifold M and let $A \in \mathfrak{g}$. Define $\eta : \mathbb{R} \times M \to M$ by $\eta(t, x) = x \cdot \exp(tA)$. Then $\{\eta_t\}$ is the flow of δ_A.

<u>Proof</u>. Note that $\eta(0, x) = x \cdot \exp(0) = x$. Also

$$\delta_A(\eta(s,x)) = \frac{d}{dt}[\eta(s,x) \cdot \exp(tA)]\big|_{t=0}$$

$$= \frac{d}{dt}[x \cdot \exp(sA)\exp(tA)]\big|_{t=0}$$

$$= \frac{d}{dt}[x \cdot \exp((s+t)A)]\big|_{t=0}$$

$$= \frac{d}{dt}[x \cdot \exp(tA)]\big|_{t=s}$$

$$= \frac{d}{ds}[x \cdot \exp(sA)] = \frac{d}{ds}[\eta(s, x)]$$

Let M be any manifold and X and Y vector fields on M. If $\{\eta_t\}$ is the flow of X then the <u>Lie derivative of Y with respect to X</u> is defined to be the vector field $L_X Y$ where

$$(L_X Y)_x = \lim_{t \to 0} \frac{[d\eta_{-t}(Y(\eta(t,x))) - Y(x)]}{t}$$

Observe that one has a curve $t \to d\eta_{-t}(Y(\eta(t,x)))$ of ~~tangent~~ vectors all tangent to M at x and $(L_X Y)_x$ is the derivative of this curve at $t=0$.

<u>lemma 2</u> Let M be a manifold and X and Y vector fields on M. Then $L_X Y = [X, Y]$.

<u>Proof</u> Notice that if $f \in C^\infty_{loc}(x)$ then one has a real-valued function defined in an interval about 0 given by $t \longmapsto d\eta_{-t}(Y(\eta(t,x)))(f) = Y_{\eta(t,x)}(f \circ \eta_{-t})$. For each y we can expand the mapping $t \mapsto f(\eta_{-t}(y))$ using the Taylor formula to obtain

$$f(\eta_{-t}(y)) = f(y) + t\frac{d}{dt}(f(\eta_{-t}(y)))\Big|_{t=0} + t^2\frac{d^2}{dt^2}(f(\eta_{-t}(y)))\Big|_{t=t_*}$$

for some $t_* \in \mathbb{R}$. Let $g(y) = \frac{d^2}{dt^2}(f(\eta_{-t}(y)))\Big|_{t=t_*} = \frac{\partial^2}{\partial t}(f(\eta(-t,y)))\Big|_{t=t_*}$ and observe that g is smooth in a neighborhood of x. We have

$$(f \circ \eta_{-t})(y) = f(y) + t\, df\left(-\frac{d}{dt}(\eta_t(y))\Big|_{t=0}\right) + t^2 g(y)$$

or

$$f \circ \eta_{-t} = f - tX(f) + t^2 g.$$

We have

$$Y_{\eta(t,x)}(f\circ\eta_{-t}) = Y_{\eta(t,x)}f - t\,Y_{\eta(t,x)}(X(f)) + t^2(Y_{\eta(t,x)}g)$$
$$= Y(f)(\eta(t,x)) - t\,Y_{\eta(t,x)}(X(f)) + t^2(Y_{\eta(t,x)}g)$$

Note that

$$\frac{d}{dt}\left(Y(f)(\eta(t,x))\right) = d(Y(f))(X(\eta(t,x)))$$
$$\frac{d}{dt}\left[t\,Y_{\eta(t,x)}(X(f))\right] = t\,\frac{d}{dt}(Y_{\eta(t,x)}(X(f))) + Y_{\eta(t,x)}(X(f))$$
$$\frac{d}{dt}\left[t^2(Y_{\eta(t,x)}g)\right] = t\,\frac{d}{dt}\left[t(Y_{\eta(t,x)}g)\right] + t(Y_{\eta(t,x)}g).$$

Evaluate at $t=0$ to obtain

$$(L_X Y)_x(f) = d_xf((L_XY)_x)$$
$$= d_xf\left(\frac{d}{dt}\left(d\eta_{-t}(Y(\eta(t,x)))\right)\right)\Big|_{t=0}$$
$$= \frac{d}{dt}\left[d\eta_{-t}(Y(\eta(t,x)))(f)\right]\Big|_{t=0}$$
$$= \frac{d}{dt}\left[Y_{\eta(t,x)}(f\circ\eta_{-t})\right]\Big|_{t=0}$$
$$= d(Y(f))(X(x)) - Y_x(X(f))$$
$$= X_x(Y(f)) - Y_x(X(f))$$
$$= [X,Y]_x(f).$$

The lemma follows.

Theorem Assume that G is a Lie group which acts on the right of the manifold M. Then the infinitesimal generator $\delta : \mathfrak{g} \to \Gamma M$ of the action is a Lie algebra homomorphism which is injective when the group action is effective. More generally, when the group action is a free action one has that δ_A never vanishes for each $A \in \mathfrak{g} \setminus \{0\}$.

Proof. To show that δ is linear we first reformulate its definition slightly. For $x \in M$ let $\sigma_x : G \to M$ be defined by $\sigma_x(g) = x \cdot g$. Then $\sigma_x(\exp(tA)) = x \cdot \exp(tA)$ and

$$d_e\sigma_x(A) = \frac{d}{dt}[\sigma_x(\exp(tA))]\Big|_{t=0} = \frac{d}{dt}[x \cdot \exp(tA)]\Big|_{t=0} = \delta_A(x).$$

It follows that $\delta_{A+B}(x) = d_e\sigma_x(A+B) = d_e\sigma_x(A) + d_e\sigma_x(B) = (\delta_A + \delta_B)(x)$. Similarly $\delta_{cA}(x) = d_e\sigma_x(cA) = (c\delta_A)(x)$. So δ is a linear mapping from $\mathfrak{g}$ into ΓM. We show that δ is injective when the action is effective. In case $\delta_A = 0$ for $A \in \mathfrak{g}$ we see that $\frac{d}{dt}(\eta(t,x)) = 0$ where $\{\eta_t\}$ is the flow of δ_A. Thus $t \mapsto \eta(t,x) = x \cdot \exp(tA)$ is constant for each $x \in M$ and consequently $x \cdot \exp(tA) = x$ for all x and all t. Thus $\exp(tA) = e$ for all t. But $\exp(tA) = \exp(0A)$ implies $tA = 0$ for t sufficiently small, thus $A = 0$. Finally assume the action is a free action. Assume $\delta_A(x) = 0$ for some $x \in M$. Then $d_x r_{\exp(sA)}(\delta_A(x)) = 0$ for all $s \in \mathbb{R}$ and $d\,r \quad (x \cdot \exp(tA))\Big|\Big| \quad \quad (\delta \;(x)) \quad 0$

Thus $\frac{d}{ds}(x\cdot\exp(sA)) = \frac{d}{dt}(x\cdot\exp(t+s)A)\Big|_{t=0}$

$$= \frac{d}{dt}\left[r_{\exp(sA)}(x\cdot\exp(tA))\right]\Big|_{t=0}$$

$$= 0$$

Thus δ_A never vanishes if $A\neq 0$. and $x\exp(sA) = x$ for all s. It follows that $A = 0$.

It remains to show that for $A, B \in \mathfrak{g}$

$$\delta([A,B]) = [\delta(A), \delta(B)].$$

First observe that since $\eta_t(x) = x\cdot\exp(tA) = r_{\exp(tA)}(x)$ is the flow of

$$[A,B]_e = (L_A B)_e = \frac{d}{dt}\left[dr_{\exp(-tA)}(B(\exp(tA)))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[dr_{\exp(-tA)}(dl_{\exp(tA)}(B_e))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[dr_{\exp(-tA)}\left(dl_{\exp(tA)}\left(\frac{d}{ds}(\exp(sB))\Big|_{s=0}\right)\right)\right]_{t=0}$$

$$= \frac{d}{dt}\frac{d}{ds}\left[r_{\exp(-tA)}(l_{\exp(tA)}(\exp(sB)))\right]_{s=0,t=0}$$

$$= \frac{d}{dt}\frac{d}{ds}\left[\exp(tA)\exp(sB)\exp(-tA)\right]_{s=0,t=0}$$

Let $\{\eta_t\}$ be the flow of δ_A so that $\eta(t,u) = u\cdot\exp(tA)$.

But $[\delta_A, \delta_B]_u = L_{\delta_A}(\delta_B)_u$

$$= \frac{d}{dt}\left[d\eta_{-t}(\delta_B(\eta(t,u)))\right]_{t=0}$$

$$= \frac{d}{dt}\left[dr_{\exp(-tA)}(d\sigma_{\eta(t,u)}(B_e))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[dr_{\exp(-tA)}\left(\frac{d}{ds}[\sigma_{\eta(t,u)}(\exp(sB))]\Big|_{s=0}\right)\right]_{t=0}$$

$$= \frac{d}{dt}\frac{d}{ds}\left[r_{\exp(-tA)}(u\cdot\exp(tA)\exp(sB))\right]_{s=0,t=0}$$

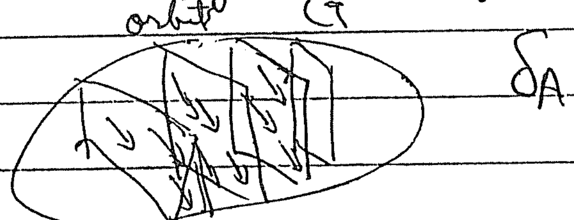
$$= \frac{d}{dt}\frac{d}{ds}\left(\sigma_u\left[\exp(tA)\exp(sB)\exp(-tA)\right]\right)_{s=0,\,t=0}$$

$$= \frac{d}{dt}\left(d\sigma_u\left(\frac{d}{ds}\left[\exp(tA)\exp(sB)\exp(-tA)\right]\right)_{s=0}\right)_{t=0}$$

$$= d_e\sigma_u\left([A,B]_e\right) = \delta([A,B])_u.$$

The theorem follows.

Remark. Observe that if G acts on the right of a manifold M and if the orbit of the action through some $x \in M$ is a submanifold $\mathcal{O}$ of M then for each $A \in \mathfrak{g}$ and $u \in \mathcal{O}$, the curve $t \mapsto u \cdot \exp(tA)$ is a curve which lies entirely in $\mathcal{O}$. Consequently $\delta_A(u) = \frac{d}{dt}[u \cdot \exp(tA)]|_{t=0}$ is tangent to $\mathcal{O}$ at u. Thus δ_A is everywhere tangent to orbits of the action.



Example Note that $SO(n+1)$ acts on the right of $\mathbb{R}^{n+1}$ via $(x,A) \mapsto x \cdot A$. If $x \in \mathbb{R}^{n+1}/\{0\}$ then the orbit through x is the set of all vectors $y = x \cdot A$ for $A \in SO(n+1)$. But each such y satisfies $\|y\|^2 = \langle x\cdot A, x\cdot A\rangle = \|x\|^2$ and so is on a sphere of radius $\|x\|$ in $\mathbb{R}^{n+1}$. Conversely if $\|y\| = \|x\|$ for some y then there exists $A \in SO(n+1)$ such that $y = xA$. Such a matrix may be constructed by finding orthonormal bases $\{x, e_1, \dots, e_n\}$ and $\{y, f_1, f_2, \dots f_n\}$ of $\mathbb{R}^{n+1}$, then by finding an appropriate

Thus the orbit $x \cdot SO(n+1)$ is precisely the n-sphere
$$S^n = \{ y \in \mathbb{R}^{n+1} \mid \|y\| = \|x\| \}.$$
Note that A belongs to the isotropy subgroup of x iff $x \cdot A = x$. Again choose an orthonormal basis $\{x, e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^{n+1}$. Then $\langle e_i A, e_j A \rangle = \langle e_i, e_j \rangle = \delta_{ij}$ and $\langle x, e_i A \rangle = \langle xA, e_i A \rangle = \langle x, e_i \rangle = 0$.
Thus
$$A = \begin{pmatrix} 1 & 0 \\ 0 & A_\perp \end{pmatrix}$$
where $A_\perp$ is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $e_i A_\perp = e_i A$ for all i. So $A_\perp \in SO(n)$ and $\det A_\perp = \det A = 1$. Thus $A \in SO(n+1)_x$ iff it is of the form $\mathbb{1} \oplus A_\perp$ where $A_\perp \in SO(n)$. We have
$$SO(n+1)/SO(n) \cong S^n$$
is diffeomorphic to the n-sphere. It follows that the mapping $\eta : SO(n+1) \to S^n$ defined by $\eta(A) = x \cdot A$ is a fiber bundle with fiber $SO(n)$.