Definition 2.1 A *fiber bundle* is a mapping $\pi$ from a manifold $E$ onto a manifold $M$ subject to the following properties:

(1) $\pi$ is smooth and surjective,

(2) there exist a manifold $F$ called the *fiber* of and an open cover $\mathcal{U}$ of $M$ along with a corresponding family of mappings $\psi_U : \pi^{-1}(U) \to U \times$ $U \in \mathcal{U}$ such ~~that~~

(a) $\psi_U$ is a diffeomorphism,

(b) if $\pi_U$ is the projection of $U \times F$ onto $U$ then $\pi_U(\psi_U(y)) = \pi(y)$ for all $y \in \pi^{-1}(U)$.

The latter condition, condition (b), is usually expressed by saying that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

is commutative. Moreover the mappings $\{\psi_U\}_U$ are said to be *local trivializing mappings* of ~~the~~ bundle.

Notice that if $u_0 \in U$, $U \in \mathcal{U}$, then $\pi^{-1}($ is a submanifold of $\pi^{-1}(U) \subseteq E$ which is diffeomorphic to the fiber $F$ of $\pi$. To see that this is so observe that $y \in \pi^{-1}($

~~maps to~~

is mapped to $u_0$ by $\pi$ iff $\pi_U(\psi_U(y)) = u_0$ and this is true iff $\psi_U(y) = (u_0, f)$ for some $f \in F$. Thus

$$\pi^{-1}(u_0) = \{\, y \in \pi^{-1}(U) \mid y = \psi_U^{-1}(u_0, f),\ f \in F \,\}$$

and

$$\pi^{-1}(u_0) = \psi_U^{-1}(\{u_0\} \times F).$$

<u>Definition 2.2</u> If $\pi : E \to M$ is a fiber bundle then $E$ is called the <u>bundle space</u> or total space of $\pi$, more simply $E$ is <u>the bundle</u> of $\pi$ and $M$ is called ~~the~~ <u>base space</u> or <u>base</u> of $\pi$.

<u>Definition 2.3</u> If $\pi : E \to M$ is a fiber bundle then $s$ is a <u>local section</u> of $\pi$ iff $s$ is a ~~smooth~~ mapping from some open subset $U \subseteq M$ into $E$ such that $\pi \circ s = id_U$. The ~~local~~ section $s$ is called a <u>global section</u> of $\pi$ iff $U = M$.

Exercise 2.1 Show that $s(U)$ is a submanifold of $E$ which intersects each fiber $\pi^{-1}(u)$ over points $u \in U$ in one and only one point.

More importantly $\forall m \in M$, $\forall y_0 \in \pi^{-1}(m)$, there is a local section $s$ of $\pi$ such that $m \in \mathrm{dom}\, s$ and $s(m) = y_0$

Observe that <u>every</u> point $m \in M$ is in ~~the~~ domain of some local section of $\pi$. To prove this choose a local trivializing mapping $\Psi_U : \pi^{-1}(U) \to U \times F$ such that $m \in U$. Let $f_0$ denote any element of $F$, the fiber of $\pi$ and define $s : U \to E$ by

$$s(x) = \Psi_U^{-1}(x, f_0)$$

for each $x \in U$. Clearly $s$ is smooth and $\pi_U(\Psi_U(s(x))) = \pi(x, f_0) = x$ and thus $\pi(s(x)) = x$ for all $x \in U$. Observe that if $y_0 \in \pi^{-1}(m)$ then $\Psi_U(y_0) = (m, f_0)$ for some $f_0 \in F$ and $s(m) = \Psi_U^{-1}(m, f_0) = y_0$

It follows that there is a family of local sections $\{s_U\}_{U \in \mathcal{U}}$ of $\pi$ whose domains cover the base space $M$. For many mappings $\pi : E \to M$ having such a family $\{s_U\}_{U \in \mathcal{U}}$ of local sections implies the existence of a family $\{\Psi_U\}_{U \in \mathcal{U}}$ of local trivializing mappings and thus implies ~~that~~ $\pi$ is a fiber bundle. This need not hold in general, however.

Definition 2.4 If $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ are fiber bundles with fibers $F_1$ and $F_2$, respectively then the pair of functions $(\Phi, \varphi)$ is a bundle isomorphism from $\pi_1$ to $\pi_2$ iff $\Phi$ is a diffeomorphism from $E_1$ to $E_2$, $\varphi$ is a diffeomorphism from $M_1$ to $M_2$ and the digram

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

is commutative. In this case the fiber $\pi_1^{-1}(x)$

of $E_1$ over $x \in M_1$ is mapped diffeomorphically by $\Phi$ onto the fiber $\pi_2^{-1}(\varphi(x))$ of $E_2$ over $\varphi(x)$. In particular $F_1$ is diffeomorphic to $F_2$. A fiber bundle $\pi: E \to M$ with fiber $F$ is said to be <u>trivial</u> iff it is bundle isomorphic to the product bundle $\pi_M : M \times F \to M$ (note that the product bundle possesses a single trivializing mapping with $U \equiv M$, $\mathcal{U} = \{U\}$, $\psi_U = id_{M\times F}$).

Finally observe that if $\pi: E \to M$ is any fiber bundle with local trivializing mappings $\{\psi_U\}_{U \in \mathcal{U}}$ then $\pi | \pi^{-1}(U) \to U$ is itself a fiber bundle with a single local trivializing mapping $\psi_U$ and in fact $\pi | \pi^{-1}(U)$ is isomorphic a trivial fiber bundle namely $\pi_U : U \times F \to U$. Moreover $(\psi_U, id_U)$ is a bundle isomorphism from $\pi | \pi^{-1}(U)$ to $\pi_U$. Thus <u>every fiber bundle is locally trivial in this sense</u> but most interesting fiber bundles are not trivial.

Examples

(1) If $M$ is a manifold then $\pi: TM \to M$ is a fiber bundle. To see this notice that if $(U, x)$ is any admissible chart of $M$ then

$$\pi^{-1}(U) = TU = \{(m,v) \mid m \in U,\ v \in T_m M\}$$

Let $dx : TU \to x(U) \times \mathbb{R}^n$ be the mapping defined by $dx(m,v) = (x(m), dx^i_m(v)\, r_i$

Local trivializing mappings $\{\psi_U\}$ may be defined in terms of these charts $(TU, dx)$ of $TM$ by $\psi_U = (x^{-1} \circ id_{\mathbb{R}^n}) \circ dx : \pi^{-1}(U) \to U \times \mathbb{R}^n$. Thus the diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "dx"] \arrow[dr, "\pi"'] & x(U) \times \mathbb{R}^n \arrow[r, "x^{-1} \times id_{\mathbb{R}^n}"] & U \times \mathbb{R}^n \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

is commutative and defines the bundle structure of $\pi$. Note that the fiber of the bundle is $\mathbb{R}^n$ where $n = \dim M$. Observe that if $M$ does not have a well-defined dimension (if it varies from component to component) then $TM$ is not a fiber bundle.

(2) If $M$ is a manifold and $T^*M$ is the cotangent bundle then the projection $\pi^*: T^*M \to M$ is a fiber bundle (assuming $M$ is $n$ dimensional). Its fiber is $(\mathbb{R}^n)^*$ and trivializing mappings $\psi_i$ may be defined by
$$\psi_U(m, \alpha) = \left(m, \alpha\left(\frac{\partial}{\partial x^i}\Big|_m\right) r^i\right)$$
where $(U, x)$ is an admissible chart of $M$.

(3) For each $1 \leq k \leq n$, $\pi : \Lambda^k M \to M$ is a fiber bundle. The fiber is $\Lambda^k \mathbb{R}^n$ and trivializing mappings are defined by

$$\psi_U(m,\alpha) = \left(m, \alpha\left(\frac{\partial}{\partial x^{i_1}}\Big|_m, \ldots, \frac{\partial}{\partial x^{i_k}}\Big|_m\right)(r^{i_1}\wedge\cdots\wedge r^{i_k})\right)$$

(4) Let $M$ be a manifold. We define a fiber bundle called the <u>frame bundle</u> of $M$. The bundle space is denoted $\mathcal{F}M$; it is the set of all ordered pairs $(m, \{e_i\})$ where $m \in M$ and $\{e_i\}$ is a basis of $T_m M$. Such a basis is called a <u>frame</u> at $m$ and thus $\mathcal{F}M$ is called a <u>bundle</u> <u>of</u> <u>frames</u> of $M$. The fiber bundle mapping is $\pi : \mathcal{F}M \to M$ defined by $\pi((m,\{e_i\})) = m$, it designates the point at which the frame $\{e_i\}$ is attached. We show that $\mathcal{F}M$ is a manifold and that $\pi : \mathcal{F}M \to M$ is a fiber bundle with fiber the group ~~of all~~ $Gl(\mathbb{R}^n)$ of all nonsingular $n\times n$ real matrices. We elaborate in some detail the structure of $\pi : \mathcal{F}M \to M$

First observe that if $m \in M$ then $\pi^{-1}(m)$ is the set of all frames at $m$. If $(m,\{e_i\})$ and $(m,\{f_i\})$ are two points in the fiber $\pi^{-1}(m)$ then they are related via a unique $n\times n$ matrix $A$ such that

$$f_j = A_j^{\,i} e_i.$$

This suggests that the fiber is $Gl(\mathbb{R}^n)$ and how to get charts and local trivializing mapping. Choose any admissible chart $(U,x)$ of $M$. Let

$$\mathcal{F}U = \{(m,\{e_i\}) \mid m \in U\}$$

and let $\mathcal{F}x : \mathcal{F}U \to x(U) \times \mathcal{G}l(\mathbb{R}^n)$ be defined by
$$(\mathcal{F}x)(m, \{e_i\}) = (x(m), (dx^j_m(e_i))).$$
Thus $(\mathcal{F}x)(m, \{e_i\}) = (x(m), A)$ where $A$ is the $n \times n$ matrix defined by
$$A_i^{\,j} = dx^j_m(e_i).$$
$A$ is invertible since both $\{e_i\}$ and $\{\frac{\partial}{\partial x^j}\big|_m\}$ are bases of $T_mM$ and
$$e_i = A_i^{\,j}\left(\frac{\partial}{\partial x^j}\Big|_m\right).$$
Moreover $\mathcal{F}x$ maps $\mathcal{F}U$ onto all of $x(U) \times \mathcal{G}l(\mathbb{R}^n)$.
[We leave it as an excercise to show that if
$\mathcal{A}_M$ is an admissible atlas of $M$ then
$$\mathcal{A} = \{(\mathcal{F}U, \mathcal{F}x) \mid (U, x) \in \mathcal{A}\}$$
is an atlas of $\mathcal{F}M$. Moreover $\mathcal{G}l(\mathbb{R}^n)$ is an open subset of $gl(\mathbb{R}^n)$ which may be identified with $\mathbb{R}^{n^2}$. Finally $\psi_U : \pi^{-1}(U) \to U \times \mathcal{G}l(\mathbb{R}^n)$ is a local trivializing mapping if we define it by
$$\psi_U = (x^{-1} \times id_{\mathcal{G}l(\mathbb{R}^n)}) \circ \mathcal{F}x.$$
Observe that $(\mathcal{F}x)^{-1}(a, A) = \left(x^{-1}(a), A^i_j\left(\frac{\partial}{\partial x^i}\Big|_{x^{-1}(a)}\right)\right)$. $(\mathcal{F}y \circ \mathcal{F}x^{-1})(a, A) = \left(y(x^{-1}(a)), \left(\frac{\partial y^k}{\partial x^i} A^i_j\right)\right)$

Move To (67)

~~**Definition 2.5** A fiber bundle $\pi : E \to M$ is called a vector bundle if~~
~~(1) the fiber of the bundle is a vector space~~
~~(2) there is a family of local trivializing mappings $\psi_U : \pi^{-1}(U) \to U \times V$, $U \in \mathcal{U}$, such that if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$ then~~

(5) Let $M$ be a manifold and $g$ a metric on $M$. Then $g$ is a type $\binom{0}{2}$ tensor field on $M$ which is symmetric, nondegenerate and which has constant index $k$. For each $m \in M$, $g_m$ is a metric on $T_mM$ and thus there is a $g$-orthonormal basis $\{e_i\}$ of $T_mM$ such that $\{j \mid g_m(e_j, e_j) = -1\}$ has $k$ elements in it.

Let $p = n-k$

By reordering this basis if necessary we obtain

$$g_m(e_i, e_j) = G_{ij}$$

where

$$G_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \quad 1 \leq i \leq p \\ -1 & i = j \quad p+1 \leq i \leq n \end{cases}$$

Define $O_gM = \{(m, \{e_i\}) \in FM \mid g_m(e_i, e_j) = G_{ij}\}$

We claim $O_gM \xrightarrow{\pi} M$, $\pi(m, \{e_i\}) = m$ is a fiber bundle. This is not difficult to prove given the following Theorem

Theorem 2.2 If $M$ is a manifold and $g$ is a metric on $M$ with index $k = n-p$ then for each $m_0 \in M$ there exists an open set $U$ about $m_0$ and vector fields $\{X_i\}$ on $U$ such that $g_m(X_i(m), X_j(m)) = G_{ij}$ for all $m \in U$.

We first show how to use the theorem to prove that $O_gM \xrightarrow{\pi} M$ is a fiber bundle, after which we will prove the Theorem.

Let $O(p,k) = \{A \in GL(\mathbb{R}^n) \mid A^T G A = G\}$.
We leave it as an exercise to be proven later that $O(p,k)$ is a manifold. We show that $\pi : O_g M \to M$ is locally trivial with fiber $O(p,k)$. By the Theorem there is an open cover $\mathcal{U}$ of $M$ such that for each $U \in \mathcal{U}$ there exist vector fields $\{\xi_i\}_{i=1}^{n}$ defined on $U$ such that

$$g_m(\xi_i(m), \xi_j(m)) = G_{ij}$$

for all $m \in U$. Define a mapping

We may also assume each $U$ is a chart domain

$$\psi_U : \pi^{-1}(U) \to U \times O(p,k)$$

by

$$\psi_U(m, \{e_i\}) = (m, (\xi^j(m)(e_i)))$$

where $\xi^j$ is the differential form defined on $U$ by

$$\xi^j(m)(\xi_i(m)) = \delta^j_i$$

To show that the matrix $\lambda$ whose components are defined by

$$\lambda_i^j = \xi^j(m)(e_i)$$

is actually in $O(p,k)$ observe that $e_i = \lambda_i^k \xi_k(m)$ and

$$G_{ij} = g_m(e_i, e_j) = \lambda_i^k \lambda_j^l g_m(\xi_k(m), \xi_l(m))$$

and

$$G^i_j = \sum_{k,l} (\lambda_i^k \lambda_j^l) G^k_l = (\lambda^T G \lambda)^i_j.$$

So $G = \lambda^T G \lambda$ and $\lambda \in O(p,k)$ as required. It follows that $\Psi_U$ maps $\pi^{-1}(U)$ onto $U \times O(p,k)$. Moreover $\Psi_U$ has an inverse and in fact

$$\Psi_U^{-1}(m, (\lambda_i^j)) = (m, \lambda_i^j \xi_j(m)) \quad , m \in U.$$

The mappings $\{\Psi_U\}_{U \in \mathcal{U}}$ have the formal requirements of local trivializing mappings but they must be smooth. One needs a differentiable structure on $O_g M$ relative to which the $\{\Psi_U\}$ are smooth. To obtain such a structure we use the product structures on the manifolds $\{U \times O(p,k)\}$ and induce structures on $\{\pi^{-1}(U)\}$ via $\{\Psi_U\}$. For this to be successful the structures on $\pi^{-1}(U \cap V)$ induced by the ones on $\pi^{-1}(U)$ and $\pi^{-1}(V)$ must agree. In order to compare these structures we must examine the maps $\Psi_U, \Psi_V$ in detail where $U, V \in \mathcal{U}$, $U \cap V \neq \phi$.

Consider, then the maps:

$$\psi_U : \pi^{-1}(U) \to U \times O(p,k)$$
$$\psi_V : \pi^{-1}(V) \to V \times O(p,k)$$

Let $\{\xi^U_i\}$ denote orthonormal vector fields on $U$ and $\{\xi^V_i\}$ orthonormal vector fields on $V$, $\{\xi^i_U\}$, $\{\xi^i_V\}$ are dual to $\{\xi^U_i\}$ and $\{\xi^V_i\}$ respectively. Define a mapping $T : U \cap V \to O(p,q)$ by $T(q) = A_q$ where $(A_q)$ is the matrix defined by $\xi^V_i(q) = (A_q)^j_i\, \xi^U_j(q)$. $T$ is called a <u>transition function</u>. Now observe that

$$\psi_V^{-1}(q, (\Lambda^i_j)) = (q, \Lambda^i_j \xi^V_i(q))$$

and
$$(\psi_U \circ \psi_V^{-1})(q, (\Lambda^i_j)) = \psi_U(q, \Lambda^i_j \xi^V_i(q))$$
$$= (q, (\xi^k_U(q)(\Lambda^i_j \xi^V_i(q))))$$
$$= (q, \xi^k_U(q)(\Lambda^i_j (A^l_i \xi^U_l(q))))$$
$$= (q, (\Lambda^i_j A^l_i \delta^k_l)) = (q, ((A\Lambda)^k_j))$$
$$= (q, (T(q)\Lambda)^k_l)$$

If $\varphi_{UV} = \psi_U \circ \psi_V^{-1}$ then $\psi_U = \varphi_{UV} \circ \psi_V$.

Charts on $\pi^{-1}(U)$ are $(x \times y) \circ \psi_U$ where $x, y$ are charts on $U$ and on $O(p,q)$ respectively. If $q_0 \in \pi^{-1}(U) \cap \pi^{-1}(V)$ then $\psi_U(q_0) \in U \times O(p,k)$, $\psi_V(q_0) \in V \times O(p,k)$ and one has charts $x, \bar{x}$ on $U$, $y, \bar{y}$ on $O(p,k)$ about $pr\,\psi_U(q_0)$, $Pr(\psi_V(q_0))$ and $Pr\,\psi_U(q_0) = T(q_0)(Pr(\psi_V(q_0)))$. We have

$$[(x \times y) \circ \psi_U] \circ [(\bar{x} \times \bar{y}) \circ \psi_V]^{-1} = (x \times y) \circ (\psi_U \circ \psi_V^{-1}) \circ (\bar{x}^{-1} \times \bar{y}^{-1}) =$$

which is

It follows that we can construct an explicit atlas on $O_g(M)$ as follows. Let $\mathcal{A}(p,k)$ be an atlas on $O(p,k)$ and for each $U \in \mathcal{U}$ choose a fixed chart $x = x_U$ on $U \subseteq M$. For each chart $y \in \mathcal{A}(p,k)$ let $U(y) = \psi_U^{-1}(U \times V_y)$ where $V_y \subseteq O(p,k)$ is the domain of the chart $y$. Define $\eta_y : U(y) \to x(U) \times y(V_y)$ by $\eta_y = (x \times y) \circ \psi_U$. The family

$$\{(U(y), \eta_y) \mid U \in \mathcal{U},\ y \in \mathcal{A}(p,k)\}$$

is an atlas on $O_g M$. It is clear that $O_g M$ is a Hausdorff space because $M$ and $O(p,k)$ are. The reader is invited to show that $\pi : O_g M \to M$ is smooth, but this is a direct consequence of the fact that $\pi \mid \pi^{-1}(U)$ is smooth for each $U$. Finally, to see that $\psi_U$ is smooth for each $U$ observe that if one chooses a point $q_0 \in \pi^{-1}(U)$ then $q_0 \in U(y) = \psi_U^{-1}(U \times V_y)$ for some chart $y : V_y \to y(V_y)$ of $O(p,k)$. We claim that $\psi_U \mid U(y)$ is smooth. Consider the "local coordinate representative" of $\psi_U \mid U(y)$. We see

~~$\pi^{-1}(U)$ for some $U \in \mathcal{U}$ then that point is in $U(y) = \psi_U^{-1}(U \times V_y)$ for some $V_y$ and one can show that $\psi_U$ restricted to $U(y)$ is smooth by considering its local representatives.~~ We see from the commutative diagram

$$\begin{tikzcd} U_y \arrow[r, "\psi_U"] \arrow[d, "\eta_y"'] & U \times V_y \arrow[d, "x \times y"] \\ x(U) \times y(V_y) \arrow[r, "\text{identity}"] & x(U) \times y(V_y) \end{tikzcd}$$

that the identity mapping is the local representative of $\psi_U$ relative to the charts $\eta_y$ and $x \times y$ and ~~some~~ $\psi_U$ is indeed smooth.

To complete the proof one needs to prove ~~theorem~~ Theorem 2.1 above.

Proof of Theorem 2.1. The proof requires a number of steps. Throughout the proof let $m_0 \in M$ and let $(W, \overline{x})$ denote an admissible chart of $M$ such that $m_0 \in W$.

<u>Step I</u> The chart $\overline{x}$ may be modified to obtain a new admissible chart $x$ defined on an open subset of $W$ such that $g_{m_0}\left(\frac{\partial}{\partial x^i}\Big|_{m_0}, \frac{\partial}{\partial x^j}\Big|_{m_0}\right) = G_{ij}$

To see this first choose any frame $\{e_i\}$ at $m_0$ such that $g_{m_0}(e_i, e_j) = G_{ij}$. Let $A$ be any matrix such that $\frac{\partial}{\partial \bar{x}^k}\Big|_{m_0} = A^i_k e_i$. Define $x^i = A^i_k \bar{x}^k$ on all of $W$, then

$$\frac{\partial}{\partial x^i}\Big|_{m_0} = \frac{\partial \bar{x}^k}{\partial x^i}\left(\frac{\partial}{\partial \bar{x}^k}\Big|_{m_0}\right) = \bar{A}^{-1\,k}_{\ \ i}\left(\frac{\partial}{\partial \bar{x}^k}\Big|_{m_0}\right) = e_i$$

and consequently

$$g_{m_0}\left(\frac{\partial}{\partial x^i}\Big|_{m_0}, \frac{\partial}{\partial x^j}\Big|_{m_0}\right) = G_{ij}.$$ This proves Step I.

Notice ~~that~~ a consequence of Step I is ~~that~~ $g_{m_0}\left(\frac{\partial}{\partial x^i}\Big|_{m_0}, \frac{\partial}{\partial x^j}\Big|_{m_0}\right) = \delta_{ij}$ for $1 \le i, j \le p$. We eventually show that this holds for all $m$ in some open set about $m_0$ and we characterize a maximal subset on which $g_m$ is positive definite.

<u>Step II</u> Let $T^+_m M = \left\{ \sum_{i=1}^{p} \lambda^i \left(\frac{\partial}{\partial x^i}\Big|_m\right) \Big| \ \lambda^i \in \mathbb{R} \right\}$ for each $m \in W$. We show that there is an open subset $\mathcal{O}_{m_0}$ of $m_0$ in $W$ such that for each $m \in \mathcal{O}_{m_0}$, $g_m$ restricted to $T^+_m M$ is positive definite.

<u>Proof of Step II</u>. Let $S$ denote the unit sphere in $\mathbb{R}^p$ thus $\vec{\lambda} \in S$ iff $\sum_{i=1}^{p} (\lambda^i)^2 = 1$. Define a function $H: S \times W \to \mathbb{R}$ by

$$H(\vec{\lambda}, m) = g_m\left(\sum_{i=1}^{p} \lambda^i \left(\frac{\partial}{\partial x^i}\Big|_m\right), \sum_{j=1}^{p} \lambda^j \left(\frac{\partial}{\partial x^j}\Big|_m\right)\right).$$

The mapping $H$ is continuous and

$$H(\vec{\lambda}, m_0) = g_m\left(\sum_{i=1}^{p} \lambda^i \frac{\partial}{\partial x^i}\Big|_{m_0}, \sum_{j=1}^{p} \lambda^j \frac{\partial}{\partial x^j}\Big|_{m_0}\right)$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} \lambda^i \lambda^j \delta_{ij} = \sum_{k=1}^{p} (\lambda^k)^2 = 1$$

For each $\vec{\lambda} \in S$ let $U_{\vec{\lambda}}$ be open about $\vec{\lambda}$ in $\Sigma$ and $O_{\vec{\lambda}}$ open about $m_0$ in $M$ such that $H$ is positive on $U_{\vec{\lambda}} \times O_{\vec{\lambda}}$. There exists a finite number of the sets $U_{\vec{\lambda}}$ which covers $S$, $U_{\vec{\lambda}_1}, \ldots, U_{\vec{\lambda}_N}$. Let

$$U_\alpha = U_{\vec{\lambda}_\alpha} \quad \text{and} \quad O_\alpha = O_{\vec{\lambda}_\alpha}.$$

Let $O_{m_0} = \bigcap_{\alpha=1}^{N} O_\alpha$. For $(\vec{\lambda}, m) \in S \times O_{m_0}$ we see that $\vec{\lambda} \in U_{\alpha_0}$ for some $\alpha_0$ and since $m \in O_\alpha$ for all $\alpha$ we see that $(\vec{\lambda}, m) \in U_{\alpha_0} \times O_{\alpha_0}$ and thus $H(\vec{\lambda}, m) > 0$. So $H$ is positive on $S \times O_{m_0}$. We claim $g_m$ is <u>positive definite</u> on $T^+_m M$ for all $m \in O_{m_0}$. To see this, let $m \in O_{m_0}$ and $V \in T^+_{m_0} M$ such that $V \neq 0$. Then $V = \sum_{i=1}^{p} \lambda^i \left(\frac{\partial}{\partial x^i}\Big|_{m_0}\right)$ and $\sum_{i=1}^{p} (\lambda^i)^2 \neq 0$. Let

$$\|\vec{\lambda}\| = \left[\sum_{i=1}^{p} (\lambda^i)^2\right]^{\frac{1}{2}}$$

and observe that

$$\frac{1}{\|\vec{\lambda}\|} V = \sum_{i=1}^{p} \left(\frac{\lambda^i}{\|\vec{\lambda}\|}\right)\left(\frac{\partial}{\partial x^i}\Big|_m\right)$$

where

$$\sum_{i=1}^{p} \left(\frac{\lambda^i}{\|\vec{\lambda}\|}\right)^2 = \sum_{i=1}^{p} \left[\frac{(\lambda^i)^2}{\sum_{j=1}^{p} (\lambda^j)^2}\right] = 1.$$

Thus $g_m\left(\frac{1}{\|\vec{\lambda}\|} V, \frac{1}{\|\vec{\lambda}\|} V\right) = \frac{1}{\|\vec{\lambda}\|^2} g_m(V, V)$ and

$\frac{1}{\|\vec{\lambda}\|^2} g(V, V) = H\left(\frac{\vec{\lambda}}{\|\vec{\lambda}\|}, m_0\right) > 0$. Thus $g_m(V, V) > 0$ as required. So $g_m$ is positive definite on $T^+_m M$

Notice that for each $m \in \mathcal{O}_{m_0}$, $\{\frac{\partial}{\partial x^i}|_m\}$ is a basis of $T_m^+M$. We may apply Gram-Schmidt orthogonalization to this basis to obtain a $g_m|(T_m^+M \times T_m^+M)$ orthogonal basis of $T_m^+M$. Let $\{\xi_i(m)\}$ denote this basis. An examination of the orthogonalization process shows that the resulting vector fields $\{\xi_i\}$ on $\mathcal{O}_{m_0}$ are in fact smooth and so one has vector fields $\{\xi_i\}_{i=1}^p$ on $\mathcal{O}_{m_0}$ such that

$$g_m(\xi_i(m), \xi_j(m)) = \delta_{ij}$$

for all $m \in \mathcal{O}_{m_0}$, $1 \le i, j \le p$.

Step III. Let $T_m^-M$ denote the $g_m$ othogonal complement of $T_m^+M$ in $T_mM$ for each $m \in \mathcal{O}$
We claim that $T_mM = T_m^+M \oplus T_m^-M$ for all $m \in \mathcal{O}_{m_0}$ and that the restriction of $g_m$ to $T_m^-M$ is negative definite.

Proof Let $V \in T_mM$, $m \in \mathcal{O}_{m_0}$. We show that $V = V^+ + V^-$ for some $V^+ \in T_m^+M$, $V^- \in T_m^-M$
Define $V^+$ by

$$V^+ = \sum_{j=1}^p g_m(V, \xi_j(m))\, \xi_j(m).$$

Note that

$$g_m(V - V^+, \xi_i(m)) = g_m(V, \xi_i(m)) - \sum_{j=1}^p g_m(V, \xi_j(m))\, g_m(\xi_j(m), \xi$$

$$= g_m(V, \xi_i(m)) - \sum_{j=1}^p g_m(V, \xi_j(m))\, \delta_{ji} =$$

Since this holds for all $\xi_i(m)$ and since $\{\xi_i(m)\}$ is a basis of $T_m^+M$ we see that $V - V^+$ is in the $g_m$-orthogonal complement of $T_m^+M$ in $T_mM$ and thus $V - V^+ \in T_m^-M$. If we let $V^- = V - V^+$ we have $V = V^+ + V^-$ as we require. To see that the sum is a direct sum note that if $V \in T_m^+M \cap T_m^-M$ then $V \in T_m^+M$ such that $g_m(V, V) = 0$ and since $g_m$ is positive definite on $T_m^+M$, $V = 0$. Thus $T_mM = T_m^+M \oplus T_m^-M$.

We now show that $g_m$ restricted to $T_m^-M$ is negative definite. Assume this is not so, then an orthonormal basis $\{f_j\}_{j=p+1}^n$ of $T_m^-M$ exists for which there is at least one $p+1 \leq j \leq n$ such that $g_m(f_j, f_j) = 1$. It follows that

$$\xi_1(m), \xi_2(m), \dots, \xi_p(m), f_{p+1}, f_{p+2}, \dots, f_n$$

is a $g_m$-orthonormal basis of $T_mM$ such that $g_m(\xi_i(m), \xi_i(m)) = 1$, $1 \leq i \leq p$, and $g_m(f_j, f_j) =$ 
This implies that the index of $g_m$ is less than $n-p$ contrary to hypothesis. It follows that $g_m$ restricted to $T_m^-M$ is negative definite.

<u>Proof of The Theorem itself</u>. Let $P_m : T_mM \to T_m^-$ denote the orthogonal projection of $T_mM$ onto $T_m^-$. Recall this may be defined by $P_m(V) = V^-$ where $V = V^+ + V^-$ is the decomposition in Step III. Let $W \in T_mM$ and write $W = W^+ + W^-$

Since $W^- \in T_m M$, $W^- = \sum_{i=1}^{n} \mu_i \left(\frac{\partial}{\partial x^i}\Big|_m\right)$ and

$$\rho_m(W^-) = \sum_{i=1}^{n} \mu_i \rho_m\left(\frac{\partial}{\partial x^i}\Big|_m\right) = \sum_{i=1}^{p} \mu_i \cancel{\rho_m}\left(\frac{\partial}{\partial x^i}\Big|_m\right) + \sum_{i=p+1}^{n} \mu_i \rho_m\left(\right.$$

So $$W^- = \rho_m(W^-) = \sum_{i=p+1}^{n} \mu_i \rho_m\left(\frac{\partial}{\partial x^i}\Big|_m\right)$$

The metric $-g_m$ is positive definite on $T_m^- M$ and so we can apply Gram–Schmit orthogonalizati to the vector fields

$$m \longmapsto \rho_m\left(\frac{\partial}{\partial x^i}\Big|_m\right), \quad p+1 \le i \le n$$

on $\mathcal{O}_{m_0}$. We obtain vector fields $\xi_{p+1}, \dots, \xi_n$ on $\mathcal{O}_{m_0}$ such that

$$(-g_m)(\xi_i(m), \xi_j(m)) = \delta_{ij}, \quad p+1 \le i, j \le n.$$

for all $m \in \mathcal{O}_{m_0}$. Thus we have vector fields $\xi_1, \xi_2, \dots, \xi_n$ on $\mathcal{O}_{m_0}$ such that

$$g_m(\xi_i(m), \xi_j(m)) = G_{ij}$$

for all $m \in \mathcal{O}_{m_0}$

<u>Definition 2.5</u> A fiber bundle $\pi : E \to M$ is called a vector bundle iff

(1) the fiber of the bundle is a vector space $V$,

(2) there is a family of local trivializing mapping $\varphi_U : \pi^{-1}(U) \to U \times V$, $U \in \mathcal{U}$ such that if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \ne \phi$, then

for each $m \in U_1 \cap U_2$ the mapping from $V$ to $V$ defined by

$$x \longmapsto \pi_V\left(\psi_{U_2}\left(\psi_{U_1}^{-1}(m,x)\right)\right)$$

is a vector space isomorphism. Observe that in this case there exist well-defined continuous operations + and · on each fiber $\pi^{-1}(m)$, $m \in M$. These operations are defined by

$$v + w = \psi_U^{-1}\left(m, \pi_V(\psi_U(v)) + \pi_V(\psi_U(w))\right)$$

$$cv = \psi_U^{-1}\left(m, c \cdot \pi_V(\psi_U(v))\right)$$

Exercise 2.2 Show that $TM, T^*M, \Lambda^k M$ are vector bundles.

<u>Definition</u> 2.6 Two vector bundles $(E_1, M_1, \pi_1)$ and $(E_2, M_2, \pi_2)$ are <u>vector bundle</u> isomorphic iff there exists a fiber bundle isomorphism $(\Phi, \varphi)$ from $\pi_1$ to $\pi_2$ such that for each $m \in M$ the restriction of $\Phi$ to $\pi_1^{-1}(m)$ is a vector space isomorphism from $\pi_1^{-1}(m)$ onto $\pi_2^{-1}(\varphi(x))$.

Examples.

(1) Let $S$ denote <u>Newtonian space</u>, i.e. $S$ is a manifold with an atlas $\mathcal{A}_N$ such that

(a) if $x, y \in \mathcal{A}_N$ then $y \circ x^{-1}$ is a rigid motion of $\mathbb{R}^3$

(b) if $x \in \mathcal{A}_N$ and $\varphi$ is a rigid motion of $\mathbb{R}^3$ then $\varphi \circ x \in \mathcal{A}_N$.

Let $N = \mathbb{R} \times S$ denote the bundle space of the trivial bundle $\pi_T : N \to \mathbb{R}$, $\pi_T(t,x) = t$. Observe that trajectories of objects in Newtonian space are described by local sections of this bundle: $\hat{\gamma}(t) = (t, \gamma(t))$ where $\gamma(t) \in N$ is the position of the object at time $t$. The velocity of the object is $\frac{d}{dt}\pi_S(\hat{\gamma}(t)) = \frac{d}{dt}(\gamma(t))$. We thus see ~~that~~ Newtonian spacetime is a fiber bundle over time-axis but Minkowski spacetime is not.

Example 2 Let $Q$ be the configuration space of a system of particles. The time evolution of the system is a section of the trivial bundle $\mathbb{R} \times TQ \to \mathbb{R}$.

Example 3 Let $M$ denote Minkowski spacetime. The electromagnetic field tensor is a section of the bundle $\Lambda^2 M \to M$, a ~~trivial~~ fiber bundle which is not obviously trivial. Similarly vector potentials are sections of the bundle $\Lambda^1 M \to M$

Example 4 Let $M$ denote Minkowski space and $\psi : M \to \mathbb{C}^2$ a spin field. Note that this defines a section $\hat{\psi}(x) = (x, \psi(x))$ of the trivial bundle $M \times \mathbb{C}^2 \to M$.

These examples show that most dynamical fields in physics may be viewed as a (local) section of some fiber bundle.

It is our intent to formulate a theory in which all Lagrangians have domain an appropriate fiber bundle.

<u>Definition 2.7</u> If $\pi: E \to M$ is a fiber bundle with fiber $F$ and $(U, \bar{y})$ is an admissib[le] chart of $E$ then we say ~~that this~~ chart is <u>adapted to the bundle</u> $\pi$ iff $\pi(U)$ is open in $M$ and there is a chart $\bar{x}$ of $M$ defined on $\pi(U)$ such that $\bar{y}^\mu = \bar{x}^\mu \circ \pi$ for $1 \leq \mu \leq n = \dim M$. In this case we often write $x^\mu = \bar{x}^\mu \circ \pi$, $1 \leq \mu \leq n$ and $y^a = \bar{y}^{a+n}$ for $1 \leq a \leq N$ where $N = \dim F$.

<u>xercise</u>
f $\pi: E \to M$
s a fiber
undle
nd $y \in E$ then
There is an
adapted
coordinate
system at $y$.

Note that if $u \in E$ and $W \in T_u E$ such that $d_u\pi(W) = 0$ then

$$W = \sum_{a=1}^{N} W^a \left(\frac{\partial}{\partial y^a}\Big|_w\right).$$

Indeed, in general, $W = \sum_{\mu=1}^{n} W^\mu \left(\frac{\partial}{\partial x^\mu}\Big|_w\right) + \sum_{a=1}^{N} W^a \left(\frac{\partial}{\partial y^a}\right.$

But $d_u x^\mu(W) = d\bar{x}^\mu(d_u\pi(W)) = 0$ and also

$$d_u x^\mu(W) = dx^\mu\left(\sum_\nu W^\nu \left(\frac{\partial}{\partial x^\nu}\Big|_w\right) + \sum_a W^a \left(\frac{\partial}{\partial y^a}\Big|_w\right)\right) = W^\mu$$

Thus $W^\mu = 0$ for $1 \leq \mu \leq n$ and

$$W = \sum^{N} W^a \left(\frac{\partial}{\partial y^a}\Big|_w\right)$$

as asserted.

<u>Definition 2.8</u> If $\pi : E \to M$ is a fiber bundle then a tangent vector $w \in T_u E$ at $u \in E$ is <u>vertical</u> iff $d\pi_u(w) = 0$. A curve $\gamma : I \to E$ in $E$ is <u>vertical</u> iff $\gamma'(t) \in T_{\gamma(t)} E$ is vertical for all $t \in I$.

Exercise Show that a curve $\gamma : I \to E$ is vertical iff the image of $\gamma$ lies in a <u>single fiber</u> of $E$.

<u>Definition 2.9</u> If $\pi : E \to M$ is a fiber bundle and $y_0 \in E$ then $J_{y_0} E$ denotes the set of all linear mappings $\gamma : T_{\pi(y_0)} M \to T_{y_0} E$ such that

$$d\pi_{y_0} \circ \gamma = id_{(T_{\pi(y_0)} M)}.$$

If $JE = \{ (y, \gamma) \mid y \in E \text{ and } \gamma \in J_y E \}$ ~~then~~ we will show ~~that~~ the mapping $\pi_E : JE \to E$ defined by $\pi_E(y,\gamma) = y$ defines a fiber bundle structure. This fiber bundle is called the <u>first order jet bundle of the fiber bundle</u> $\pi : E \to M$.

<u>Theorem 2.3</u> If $\pi : E \to M$ is a fiber bundle and $\gamma \in J_y E$, then $T_y E = \operatorname{Im}\gamma \oplus \operatorname{Ker} d\pi_y$. Moreover ~~there~~ is a local section $s : U \to E$ of $\pi$ such that $s(x) = y$ and $\gamma = ds_x$ where $x = \pi(y)$.

In the proof that $O_g M$ is a fiber bundle we saw that it was critical to understand how to compare the differentiable structures imposed on $\pi^{-1}(U \cap V)$ by $\Psi_U$ and $\Psi_V$ (when $U \cap V \neq \phi$). This required an analysis of the mappings $\Psi_U \circ \Psi_V^{-1}$ for $U, V \in \mathcal{U}$ such that $U \cap V \neq \phi$. These mappings are crucial to understanding how the local trivializations on the submanifolds $\{\pi^{-1}(U)\}$ "glue" together on an arbitrary manifold.

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<u>Remark</u> If $\Psi_U : \pi^{-1}(U) \to U \times F$, $\Psi_V : \pi^{-1}(V) \to V \times F$ are local trivializing maps of a fiber bundle $\pi : E \to M$ with fiber $F$ and if $U \cap V \neq \phi$, then there is a smooth mapping

$$\varphi_{UV} : U \cap V \to \mathrm{Diff}(F)$$

such that

$$(\Psi_U \circ \Psi_V^{-1})(m, f) = (m, \varphi_{UV}(m)(f))$$

for $(m, f) \in (U \cap V) \times F$. If $U, V, W$ are open subsets of $M$ with local trivializing maps $\Psi_U : \pi^{-1}(U) \to U \times F$, $\Psi_V : \pi^{-1}(V) \to V \times F$, $\Psi_W : \pi^{-1}(W) \to W \times F$ and $U \cap V \cap W \neq \phi$ then

$$\varphi_{WU}(m) \circ \varphi_{UV}(m) = \varphi_{WV}(m)$$

for all $m \in U \cap V \cap W$. In particular,

$$\varphi_{UV}(m)^{-1} = \varphi_{VU}(m), \quad \varphi_{UU}(m) = id_F$$

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To prove the Remark first observe that if $Pr_F$ is the projection of $U \times F$ onto $F$, then $\Psi_U(u) = (\pi(u), Pr_F(\Psi_U(u)))$ and

$$(\Psi_U \circ \Psi_V^{-1})(m,f) = (m, \tilde{\varphi}_{UV}(m,f))$$

where $\tilde{\varphi}_{UV}(m,f) \equiv Pr_F((\Psi_U \circ \Psi_V^{-1})(m,f))$. Clearly $\tilde{\varphi}_{UV} : (U \cap V) \times F \to F$ is smooth. <u>Define</u> $\varphi_{UV} : U \cap V \to \mathrm{Maps}(F,F)$ by $\varphi_{UV}(m)(f) = \tilde{\varphi}_{UV}(m,f)$. <u>Generally</u> if $\Phi$ is any mapping from a manifold $Q$ into a space of mappings $\mathrm{Maps}(F,F)$ we will say $\Phi$ is smooth iff the mapping from $Q \times F$ to $F$ defined by $(q,f) \mapsto \Phi(q)(f)$ is smooth. Thus our mapping $\varphi_{UV}$ is smooth <u>by</u> <u>definition</u>. Notice that if $U, V, W$ are chosen as in the statement of the Remark above then

$$[(\Psi_W \circ \Psi_U^{-1}) \circ (\Psi_U \circ \Psi_V^{-1})](m,f) = (\Psi_W \circ \Psi_V^{-1})(m,f)$$

implies that $(m, \varphi_{WV}(m)(f)) = (\Psi_W \circ \Psi_U^{-1})(m, \varphi_{UV}(m)(f))$
$= (m, \varphi_{WU}(m)(\varphi_{UV}(m)(f)))$ and consequently,

$\boxed{\varphi_{WU}(m) \circ \varphi_{UV}(m) = \varphi_{WV}(m)}$. It is easy to show that for open sets $P, Q$, $\varphi_{PP}(m) = id_F$, $\varphi_{QQ}(m) = id_F$. For $U = P$ and $W = V = Q$ the condition in the box implies that $\varphi_{QP}(m)\circ\varphi_{PQ}(m) = \varphi_{QQ}(m)$. Similarly, $Q=U$, $W=V=P$ implies that $\varphi_{PQ}(m)\circ\varphi_{QP}(m) = \varphi_{PP}(m)$. It follows that $\varphi_{PQ}(m)$ has an inverse and that $\varphi_{PQ}(m)^{-1} = \varphi_{QP}(m)$. Moreover the mapping $m \longmapsto \varphi_{PQ}(m)^{-1}$ is smooth since $\varphi_{QP}$ is smooth. Thus $\varphi_{QP}(m) \in \mathrm{Diff}(F)$. The Remark follows.

It turns out that having mappings such as $\{\varphi_{UV}\}$ is also sufficient to build a fiber bundle.

<u>Definition</u> Assume that $M$ and $F$ are manifolds, that $\mathcal{U}$ is an open cover of $M$ and that $\varphi_{UV} : U \cap V \to \mathrm{Diff}(F)$ are mappings defined for all pairs $(U,V)$ such that $U, V \in \mathcal{U}$ and $U \cap V \neq \phi$. We say that the family of mappings $\{\varphi_{UV}\}$ <u>satisfy the (Čech) cocyle condition</u> iff for $U, \cdots, W \in \mathcal{U}$ such that $U \cap V \cap W \neq \phi$,

$$\varphi_{WU}(m)\circ\varphi_{UV}(m) = \varphi_{WV}(m)$$

for all $m \in U \cap V \cap W$. As in the proof of the Remark above one can show that this condition implies that $\varphi_{UU}(m) = id_F$ for each $U \in \mathcal{U}$, $m \in U$ and that for $U, V \in \mathcal{U}$, $U \cap V \neq \phi$,

$$\varphi_{UV}(m)^{-1} = \varphi_{VU}(m) \qquad \forall\, m \in U \cap V.$$

Theorem Assume that $M$ and $F$ are manifolds, that $\mathcal{U}$ is an open cover of $M$ and that for $U, V \in \mathcal{U}$ such that $U \cap V \neq \phi$ one has a smooth mapping $\varphi_{UV} : U \cap V \to \mathrm{Diff}(F)$.
If the family of mappings $\{\varphi_{UV}\}$ satisfies the Čech cocyle condition, then there is a fiber bundle $\pi : E \to M$ with fiber $F$ and local trivializing mappings $\Psi_U : \pi^{-1}(U) \to U \times F$ such that, for $U, V \in \mathcal{U}$ with $U \cap V \neq \phi$,

$$(\Psi_U \circ \Psi_V^{-1})(m, f) = (m, \varphi_{UV}^{(m)}(f))$$

for all $m \in U \cap V$, $f \in F$.

Proof. For convenience let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ and $\varphi_{\alpha\beta} = \varphi_{U_\alpha U_\beta}$ for some index set $A$. Let $S$ denote the disjoint union of the family $\{U_\alpha \times F\}_{\alpha \in A}$. One way to think of this is to define $S$ to be the set of all ordered triples $(\alpha, x, f)$ such that $\alpha \in A$, $x \in U_\alpha$, $f \in F$. Let $\mathcal{a}_\alpha$ be an atlas of admissible charts on $\{\alpha\} \times (U_\alpha \times F)$ and $\mathcal{a} = \bigcup_{\alpha \in A} \mathcal{a}_\alpha$. If $(O_\alpha, x_\alpha) \in \mathcal{a}_\alpha$ and $(O_\beta, x_\beta) \in \mathcal{a}_\beta$ are charts and $\alpha \neq \beta$, then $O_\alpha \cap O_\beta = \phi$ and the two charts are compatible by default. Thus $\mathcal{A}$ is an atlas on $S$ and for each $\alpha \in A$ $\{\alpha\} \times (U_\alpha \times F)$ is an open submanifold of $S$.

Define a relation $\sim$ on $S$ by

$(\alpha, x, f_1) \sim (\beta, y, f_2)$ iff $x, y \in U_\alpha \cap U_\beta$, $x = y$, and $f_2 = \varphi_{\beta\alpha}(x)(f_1)$. Since $\varphi_{\alpha\alpha}(x) = id_F$, $\varphi_{\alpha\beta}(x)^{-1} = \varphi_{\beta\alpha}(x)$ and $\varphi_{\gamma\beta}(x) \circ \varphi_{\beta\alpha}(x) = \varphi_{\gamma\alpha}(x)$ for appropriate $x \in M$, the relation $\sim$ is an equivalence relation. Let $[\alpha, x, f]$ denote the equivalence class containing $(\alpha, x, f) \in S$.

Define a mapping $\hat{\pi} : S \to M$ by $\hat{\pi}(\alpha, x, f) = x$. Since $S_\alpha$ is open in $S$ and $\hat{\pi}|S_\alpha$ is essentially the projection $U_\alpha \times F \to U_\alpha$ we see that $\hat{\pi}$ is smooth and $\hat{\pi}^{-1}(U_\alpha) = S_\alpha$.

We claim that $\hat{\pi}$ restricted to an equivalence class is constant. Indeed if $(\beta, y, f_2) \in [\alpha, x, f_1]$ then $x, y \in U_\alpha \cap U_\beta$ and $x = y$ so $\hat{\pi}(\beta, y, f_2) = y = x$. Thus $\hat{\pi}|[\alpha, x, f_1] \equiv x$. So $\exists$ a well-defined mapping $\pi : S/\sim \to M$.

Let $E = S/\sim$. We define a mapping $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ as follows. First observe that if $[\beta, y, f_2] \in \pi^{-1}(U_\alpha)$ then $\pi([\beta, y, f_2]) \in U_\alpha$ and thus $y \in U_\alpha$. Moreover $[\beta, y, f_2] = [\alpha, y, \varphi_{\beta\alpha}(y)(f_1)]$ for some $f_1 \in F$.

So we define $\psi_\alpha([\beta, y, f_2]) = (y, \varphi_{\beta\alpha}(y)(f_1))$.

Thus what we do is first write the equivalence class $[\beta, y, f_2]$ so that $\alpha$ appears in the first coordinate and then project to the pair in the second and third positions. Now observe ~~that~~ if

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

$$\psi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times F$$

then

$$(\psi_\beta \circ \psi_\alpha^{-1})((x, f)) = \psi_\beta([\alpha, x, f])$$

$$= \psi_\beta([\beta, x, \varphi_{\beta\alpha}(x)(f)])$$

$$= (x, \varphi_{\beta\alpha}(x)(f))$$

is a smooth mapping and in fact is a diffeomorphism since $(\psi_\alpha \circ \psi_\beta^{-1})(y, f) = (y, \varphi_{\alpha\beta}(y)(f))$

We may assume that each member of $\mathcal{U}$ is a chart domain in $M$. Choose a chart $x_\alpha$ of $M$ with domain $U_\alpha$.

Let $\mathcal{A}_F$ denote an atlas of $F$ and let

$\mathcal{B} = \{ (x_\alpha \times z) \circ \psi_\alpha \mid \alpha \in A,\ z \text{ is a chart on } F \}$.

Let $U_\alpha(z)$ denote the domain of $(x_\alpha \times z) \circ \psi_\alpha$ for $z$ a chart on $F$. Given any point of $E$, say $w \in E$, it lies in $\pi^{-1}(U_\gamma)$ for some $\gamma \in A$ and $\psi_\gamma(w)$ is in $U_\gamma \times F$. Let $z$ be a chart of $F$ with $\mathrm{Pr}_F(\psi_\gamma(w)) \in F$ in its domain. Then $w \in U_\gamma(z)$. Thus every point of $E$ is in the domain of an element of $\mathcal{B}$. Moreover for $z_1, z_2 \in \mathcal{A}_F$ such that $U_\alpha(z_1) \cap U_\beta(z_2) \neq \phi$, we have

$$[(x_\beta \times z_2) \circ \psi_\beta] \circ [(x_\alpha \times z_1) \circ \psi_\alpha]^{-1}(s, t)$$

$$= [(x_\beta \times z_2) \circ (\psi_\beta \circ \psi_\alpha^{-1})](x_\alpha^{-1}(s), z_1^{-1}(t))$$

$$= (x_\beta \times z_2)(x_\alpha^{-1}(s), \varphi_{\beta\alpha}(x_\alpha^{-1}(s))(z_1^{-1}(t)))$$

$$= (x_\beta(x_\alpha^{-1}(s)), z_2(\underline{\varphi_{\beta\alpha}(x_\alpha^{-1}(s))(z_1^{-1}(t))}))$$

$$= ((x_\beta \circ x_\alpha^{-1})(s), z_2(\tilde{\varphi}_{\beta\alpha}(x_\alpha^{-1}(s), z_1^{-1}(t))))$$

$$= ((x_\beta \circ x_\alpha^{-1})(s), [z_2 \circ \tilde{\varphi}_{\beta\alpha} \circ (x_\alpha^{-1} \times z_1^{-1})](s, t))$$

which is clearly smooth (here we write $\tilde{\varphi}_{\beta\alpha}(p, q)$ in place of $\varphi_{\beta\alpha}(p)(q)$).

The fact that $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is smooth for each $\alpha$ follows from the fact that its local coordinate representatives are identity maps:

$$\begin{tikzcd}
U_\alpha(z) \arrow[r, "\psi_\alpha"] \arrow[d, "(x_\alpha \times z)\circ\psi_\alpha"'] & U_\alpha \times (\mathrm{dom}\, z) \arrow[d, "x_\alpha \times z"] \\
U_\alpha \times z(\mathrm{dom}\, z) \arrow[r, "\mathrm{id}"] & x_\alpha(U_\alpha) \times z(\mathrm{dom}\, z)
\end{tikzcd}$$