Proof $e \notin \mathbb{Q}$ (due to Fourier)

Let us suppose $e = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$.
Define then (we suppose $a, b > 0$)

$$x = b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right)$$

$$= b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right)$$

$$= a(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!}$$

$$= \underbrace{a(b-1)!}_{\text{integer}} - \underbrace{\frac{b!}{0!} - \frac{b!}{1!} - \cdots - \frac{b!}{b!}}_{\text{integer}} \quad \therefore \underline{x \in \mathbb{Z}}.$$

Next, we prove $0 < x < 1$, we assume that

$$e \overset{\text{def}^n}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \quad \text{Thus,}$$

all terms in sum are strictly positive

$$x = b!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!}\right) = \sum_{n=b+1}^{\infty} \frac{b!}{n!} > 0$$

Thus $0 < x$. Next we show $x < 1$, note for all terms with $n \geq b+1$ we have the "upper estimate"

$$\frac{b!}{n!} = \frac{1}{(b+1)(b+2)\cdots(b+(n-b))} \leq \frac{1}{(b+1)^{n-b}} \qquad (*)$$

where the inequality is $\underbrace{\text{strict}}$ for all terms with $n \geq b+2$

(certainly the sum to ∞ includes such terms $\Rightarrow <$ next page)

Change the index of summation to $k = n - b$, but 1st use (*) for comparison,

$$x = \sum_{n=b+1}^{\infty} \frac{b!}{n!} < \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}}$$

($k = n - b$ so $n = b+1$ gives $k = b+1-b = 1$)

$$= \sum_{k=1}^{\infty} \frac{1}{(b+1)^k}$$

$$= \frac{1}{(b+1)} + \frac{1}{(b+1)^2} + \cdots$$

$$= \left(\frac{1}{b+1}\right)\left(\frac{1}{1 - \frac{1}{b+1}}\right)$$

$$= \frac{1}{b+1-1}$$

$$= \frac{1}{b} \leq 1 \quad \therefore \quad \underline{x < 1}.$$

Consequently, $x \in \mathbb{Z}$ and $0 < x < 1$ which is impossible $\therefore \nexists\, a, b \in \mathbb{Z}$ s.t. $e = \frac{a}{b}$ and we conclude $e \notin \mathbb{Q}$.//