Corollary

If G is a group of order p^2, where p is a prime, then G is Abelian.

We mention in passing that if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, then G can be expressed as the internal direct product of subgroups isomorphic to $H_1, H_2, \ldots, H_n$. For example, if $G = H_1 \oplus H_2$, then $G = \overline{H_1} \times \overline{H_2}$, where $\overline{H_1} = H_1 \oplus \{e\}$ and $\overline{H_2} = \{e\} \oplus H_2$.

The topic of direct products is one in which notation and terminology vary widely. Many authors use $H \times K$ to denote both the internal direct product and the external direct product of H and K, making no notational distinction between the two products. A few authors define only the external direct product. Many people reserve the notation $H \oplus K$ for the situation where H and K are Abelian groups under addition and call it the *direct sum* of H and K. In fact, we will adopt this terminology in the section on rings (Part 3), since rings are always Abelian groups under addition.

The U-groups provide a convenient way to illustrate the preceding ideas and to clarify the distinction between internal and external direct products. It follows directly from Theorem 8.3, its corollary, and Theorem 9.6 that if $m = n_1 n_2 \cdots n_k$, where $\gcd(n_i, n_j) = 1$ for $i \neq j$, then

$$U(m) = U_{m/n_1}(m) \times U_{m/n_2}(m) \times \cdots \times U_{m/n_k}(m)$$
$$\approx U(n_1) \oplus U(n_2) \oplus \cdots \oplus U(n_k).$$

Let us return to the examples given following Theorem 8.3.

$$U(105) = U(15 \cdot 7) = U_{15}(105) \times U_7(105)$$
$$= \{1, 16, 31, 46, 61, 76\} \times \{1, 8, 22, 29, 43, 64, 71, 92\}$$
$$\approx U(7) \oplus U(15),$$
$$U(105) = U(5 \cdot 21) = U_5(105) \times U_{21}(105)$$
$$= \{1, 11, 16, 26, 31, 41, 46, 61, 71, 76, 86, 101\}$$
$$\times \{1, 22, 43, 64\} \approx U(21) \oplus U(5),$$
$$U(105) = U(3 \cdot 5 \cdot 7) = U_{35}(105) \times U_{21}(105) \times U_{15}(105)$$
$$= \{1, 71\} \times \{1, 22, 43, 64\} \times \{1, 16, 31, 46, 61, 76\}$$
$$\approx U(3) \oplus U(5) \oplus U(7).$$

Exercises

The heart of mathematics is its problems.

Paul Halmos

1. Let $H = \{(1), (12)\}$. Is H normal in S_3?
2. Prove that A_n is normal in S_n.

3. In D_4, let $K = \{R_0, R_{90}, R_{180}, R_{270}\}$. Write HR_{90} in the form xH, where $x \in K$. Write DR_{270} in the form xD, where $x \in K$. Write $R_{90}V$ in the form Vx, where $x \in K$.
4. Write (12)(13)(14) in the form $\alpha(12)$, where $\alpha \in A_4$. Write (1234)(12)(23), in the form $\alpha(1234)$, where $\alpha \in A_4$.
5. Show that if G is the internal direct product of $H_1, H_2, \ldots, H_n$ and $i \neq j$ with $1 \leq i \leq n$, $1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$. (This exercise is referred to in this chapter.)
6. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbf{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbf{R})$?
7. Let $G = GL(2, \mathbf{R})$ and let K be a subgroup of $\mathbf{R}^*$. Prove that $H = \{A \in G \mid \det A \in K\}$ is a normal subgroup of G.
8. Viewing $\langle 3 \rangle$ and $\langle 12 \rangle$ as subgroups of Z, prove that $\langle 3 \rangle / \langle 12 \rangle$ is isomorphic to Z_4. Similarly, prove that $\langle 8 \rangle / \langle 48 \rangle$ is isomorphic to Z_6. Generalize to arbitrary integers k and n.
9. Prove that if H has index 2 in G, then H is normal in G. (This exercise is referred to in Chapters 24 and 25 and this chapter.)
10. Let $H = \{(1), (12)(34)\}$ in A_4.
    a. Show that H is not normal in A_4.
    b. Referring to the multiplication table for A_4 in Table 5.1 on page 105, show that, although $\alpha_6 H = \alpha_7 H$ and $\alpha_9 H = \alpha_{11} H$, it is not true that $\alpha_6 \alpha_9 H = \alpha_7 \alpha_{11} H$. Explain why this proves that the left cosets of H do not form a group under coset multiplication.
11. Prove that a factor group of a cyclic group is cyclic.
12. Prove that a factor group of an Abelian group is Abelian.
13. Let H be a normal subgroup of a finite group G and let a be an element of G. Complete the following statement: The order of the element aH in the factor group G/H is the smallest positive integer n such that a^n is ______________.
14. What is the order of the element $14 + \langle 8 \rangle$ in the factor group $Z_{24}/\langle 8 \rangle$?
15. What is the order of the element $4U_5(105)$ in the factor group $U(105)/U_5(105)$?
16. Recall that $Z(D_6) = \{R_0, R_{180}\}$. What is the order of the element $R_{60}Z(D_6)$ in the factor group $D_6/Z(D_6)$?
17. Let $G = Z/\langle 20 \rangle$ and $H = \langle 4 \rangle / \langle 20 \rangle$. List the elements of H and G/H.
18. What is the order of the factor group $Z_{60}/\langle 15 \rangle$?
19. Determine all normal subgroups of D_n of order 2.
20. List the elements of $U(20)/U_5(20)$.
21. Prove that an Abelian group of order 33 is cyclic. Does your proof hold when 33 is replaced by pq where p and q are distinct primes?
22. Determine the order of $(Z \oplus Z)/\langle (2, 2) \rangle$. Is the group cyclic?

23. Let G_1 and G_2 be finite groups. If H_1 is a normal subgroup of G_1 and H_2 is a normal subgroup of G_2 give a formula for $|G_1/H_1 \oplus G_2/H_2|$ in terms of $|G_1|$, $|G_2|$, $|H_1|$ and $|H_2|$.

24. The group $(Z_4 \oplus Z_{12})/\langle(2, 2)\rangle$ is isomorphic to one of Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. Determine which one by elimination.

25. Let $G = U(32)$ and $H = \{1, 15\}$. The group G/H is isomorphic to one of Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. Determine which one by elimination.

26. Let $H = \{1, 17, 41, 49, 73, 89, 97, 113\}$ under multiplication modulo 120. Write H as a external direct product of groups of the form Z_{2^k}. Write H as an internal direct product of nontrivial subgroups.

27. Let $G = U(16)$, $H = \{1, 15\}$, and $K = \{1, 9\}$. Are H and K isomorphic? Are G/H and G/K isomorphic?

28. Let $G = Z_4 \oplus Z_4$, $H = \{(0, 0), (2, 0), (0, 2), (2, 2)\}$, and $K = \langle(1, 2)\rangle$. Is G/H isomorphic to Z_4 or $Z_2 \oplus Z_2$? Is G/K isomorphic to Z_4 or $Z_2 \oplus Z_2$?

29. Explain why a non-Abelian group of order 8 cannot be the internal direct product of proper subgroups.

30. Express $U(165)$ as an internal direct product of proper subgroups in four different ways.

31. Let $\mathbf{R}^*$ denote the group of all nonzero real numbers under multiplication. Let $\mathbf{R}^+$ denote the group of positive real numbers under multiplication. Prove that $\mathbf{R}^*$ is the internal direct product of $\mathbf{R}^+$ and the subgroup $\{1, -1\}$.

32. If N is a normal subgroup of G and $|G/N| = m$, show that $x^m \in N$ for all x in G.

33. Let H and K be subgroups of a group G. If $G = HK$ and $g = hk$, where $h \in H$ and $k \in K$, is there any relationship among $|g|$, $|h|$, and $|k|$? What if $G = H \times K$?

34. In Z, let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $Z = HK$. Does $Z = H \times K$?

35. Let $G = \{3^a 6^b 10^c \mid a, b, c \in Z\}$ under multiplication and $H = \{3^a 6^b 12^c \mid a, b, c \in Z\}$ under multiplication. Prove that $G = \langle 3 \rangle \times \langle 6 \rangle \times \langle 10 \rangle$, whereas $H \neq \langle 3 \rangle \times \langle 6 \rangle \times \langle 12 \rangle$.

36. Determine all subgroups of $\mathbf{R}^*$ (nonzero reals under multiplication) of index 2.

37. Let G be a finite group and let H be a normal subgroup of G. Prove that the order of the element gH in G/H must divide the order of g in G.

38. Prove that for every positive integer n, Q/Z has an element of order n.

39. Let H be a subgroup of a group G with the property that for all a and b in G, $aHbH = abH$. Prove that H is a normal subgroup of G.

40. Let in S_3 let $H = \{(1), (12)\}$. Show that $(13)H(23)H \neq (13)(23)H$. (This proves that when H is not a normal subgroup of a group G, the product of two left cosets of H in G need not be a left coset of H in G.)

41. Show that Q, the group of rational numbers under addition, has no proper subgroup of finite index.

42. An element is called a *square* if it can be expressed in the form b^2 for some b. Suppose that G is an Abelian group and H is a subgroup of G. If every element of H is a square and every element of G/H is a square, prove that every element of G is a square. Does your proof remain valid when "square" is replaced by "nth power," where n is any integer?

43. Show, by example, that in a factor group G/H it can happen that $aH = bH$ but $|a| \neq |b|$.

44. Verify that the mapping defined at the end of the proof of Theorem 9.6 is an isomorphism.

45. Let p be a prime. Show that if H is a subgroup of a group of order $2p$ that is not normal, then H has order 2.

46. Show that D_{13} is isomorphic to $\text{Inn}(D_{13})$.

47. Let H and K be subgroups of a group G. If $|H| = 63$ and $|K| = 45$, prove that $H \cap K$ is Abelian.

48. If G is a group and $|G: Z(G)| = 4$, prove that $G/Z(G) \approx Z_2 \oplus Z_2$.

49. Suppose that G is a non-Abelian group of order p^3, where p is a prime, and $Z(G) \neq \{e\}$. Prove that $|Z(G)| = p$.

50. If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

51. Let H be a normal subgroup of G and K a subgroup of G that contains H. Prove that K is normal in G if and only if K/H is normal in G/H.

52. Let G be an Abelian group and let H be the subgroup consisting of all elements of G that have finite order. Prove that every nonidentity element in G/H has infinite order.

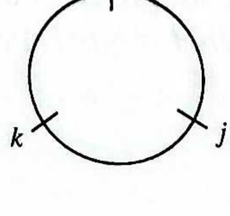
53. Determine all subgroups of $\mathbf{R}^*$ that have finite index.

54. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = -1$, $-i = (-1)i$, $1^2 = (-1)^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$.

a. Show that $H = \{1, -1\} \lhd G$.

b. Construct the Cayley table for G/H. Is G/H isomorphic to Z_4 or $Z_2 \oplus Z_2$?

(The rules involving i, j, and k can be remembered by using the circle below.



Going clockwise, the product of two consecutive elements is the third one. The same is true for going counterclockwise, except that we obtain the negative of the third element. This group is called the *quaternions*. It was invented by William Hamilton in 1843. The quaternions are used to describe rotations in three-dimensional space, and they are used in physics. The quaternions can be used to extend the complex numbers in a natural way).

55. In D_4, let $K = \{R_0, D\}$ and let $L = \{R_0, D, D', R_{180}\}$. Show that $K \triangleleft L \triangleleft D_4$, but that K is not normal in D_4. (Normality is not transitive.)

56. Show that the intersection of two normal subgroups of G is a normal subgroup of G. Generalize.

57. Give an example of subgroups H and K of a group G such that HK is not a subgroup of G.

58. If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

59. Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G. (This exercise is referred to in Chapter 24.)

60. Without looking at inner automorphisms of D_n, determine the number of such automorphisms.

61. Let H be a normal subgroup of a finite group G and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$. (This exercise is referred to in Chapter 25.)

62. Let G be a group and let G' be the subgroup of G generated by the set $S = \{x^{-1}y^{-1}xy \mid x, y \in G\}$.

a. Prove that G' is normal in G.

b. Prove that G/G' is Abelian.

c. If G/N is Abelian, prove that $G' \leq N$.

d. Prove that if H is a subgroup of G and $G' \leq H$, then H is normal in G.

63. Prove that the group $\mathbf{C}^*/\mathbf{R}^*$ has infinite order.

64. Suppose that a group G has a subgroup of order n. Prove that the intersection of all subgroups of G of order n is a normal subgroup of G.

65. If G is non-Abelian, show that Aut(G) is not cyclic.

66. Let $|G| = p^n m$, where p is prime and $\gcd(p, m) = 1$. Suppose that H is a normal subgroup of G of order p^n. If K is a subgroup of G of order p^k, show that $K \subseteq H$.

67. Suppose that H is a normal subgroup of a finite group G. If G/H has an element of order n, show that G has an element of order n. Show, by example, that the assumption that G is finite is necessary.

68. Prove that A_4 is the only subgroup of S_4 of order 12.

69. If $|G| = 30$ and $|Z(G)| = 5$, what is the structure of $G/Z(G)$? What is the structure of $G/Z(G)$ if $|Z(G)| = 3$? Generalize to the case that $|G| = 2pq$ where p and q are distinct odd primes.

70. If H is a normal subgroup of G and $|H| = 2$, prove that H is contained in the center of G.

71. Prove that A_5 cannot have a normal subgroup of order 2.

72. Let G be a group and H an odd-order subgroup of G of index 2. Show that H contains every element of G of odd order.

Suggested Readings

Tony Rothman, "Genius and Biographers: The Fictionalization of Évariste Galois," *The American Mathematical Monthly* 89 (1982): 84–106.

The author argues that many popular accounts of Galois's life have been greatly embroidered.

Paul F. Zweifel, "Generalized Diatonic and Pentatonic Scales: A Group-theoretic Approach," *Perspectives of New Music* 34 (1996): 140–161.

The author discusses how group theoretic notions such as subgroups, cosets, factor groups, and isomorphisms of Z_{12} and Z_{20} relate to musical scales, tuning, temperament, and structure.

problem is solved. Clearly, the mapping from $U(8) \oplus Z$ onto $U(8)$ given by $\phi(a, b) = a$ is such a mapping, and therefore $U(8) \oplus Z$ is the union of the proper subgroups $\phi^{-1}(H)$, $\phi^{-1}(K)$ and $\phi^{-1}(L)$. ∎

Although an isomorphism is a special case of a homomorphism, the two concepts have entirely different roles. Whereas isomorphisms allow us to look at a group in an alternative way, homomorphisms act as investigative tools. The following analogy between homomorphisms and photography may be instructive.† A photograph of a person cannot tell us the person's exact height, weight, or age. Nevertheless, we *may* be able to decide from a photograph whether the person is tall or short, heavy or thin, old or young, male or female. In the same way, a homomorphic image of a group gives us *some* information about the group.

In certain branches of group theory, and especially in physics and chemistry, one often wants to know all homomorphic images of a group that are matrix groups over the complex numbers (these are called *group representations*). Here, we may carry our analogy with photography one step further by saying that this is like wanting photographs of a person from many different angles (front view, profile, head-to-toe view, close-up, etc.), as well as x-rays! Just as this composite information from the photographs reveals much about the person, several homomorphic images of a group reveal much about the group.

Exercises

The greater the difficulty, the more glory in surmounting it. Skillful pilots gain their reputation from storms and tempests.

Epicurus

1. Prove that the mapping given in Example 2 is a homomorphism.
2. Prove that the mapping given in Example 3 is a homomorphism.
3. Prove that the mapping given in Example 4 is a homomorphism.
4. Prove that the mapping given in Example 11 is a homomorphism.
5. Let $\mathbf{R}^*$ be the group of nonzero real numbers under multiplication, and let r be a positive integer. Show that the mapping that takes x to x^r is a homomorphism from $\mathbf{R}^*$ to $\mathbf{R}^*$ and determine the kernel. Which values of r yield an isomorphism?

†"All perception of truth is the detection of an analogy." Henry David Thoreau, *Journal*.

6. Let G be the group of all polynomials with real coefficients under addition. For each f in G, let $\int f$ denote the antiderivative of f that passes through the point (0, 0). Show that the mapping $f \to \int f$ from G to G is a homomorphism. What is the kernel of this mapping? Is this mapping a homomorphism if $\int f$ denotes the antiderivative of f that passes through (0, 1)?
7. If ϕ is a homomorphism from G to H and σ is a homomorphism from H to K, show that $\sigma\phi$ is a homomorphism from G to K. How are Ker ϕ and Ker $\sigma\phi$ related? If ϕ and σ are onto and G is finite, describe [Ker $\sigma\phi$:Ker ϕ] in terms of $|H|$ and $|K|$.
8. Let G be a group of permutations. For each σ in G, define

$$\text{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this homomorphism allow you to conclude that A_n is a normal subgroup of S_n of index 2? Why does this prove Exercise 23 of Chapter 5?
9. Prove that the mapping from $G \oplus H$ to G given by $(g, h) \to g$ is a homomorphism. What is the kernel? This mapping is called the *projection* of $G \oplus H$ onto G.
10. Let G be a subgroup of some dihedral group. For each x in G, define

$$\phi(x) = \begin{cases} +1 & \text{if } x \text{ is a rotation,} \\ -1 & \text{if } x \text{ is a reflection.} \end{cases}$$

Prove that ϕ is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this prove Exercise 26 of Chapter 3?
11. Prove that $(Z \oplus Z)/(\langle(a, 0)\rangle \oplus \langle(0, b)\rangle)$ is isomorphic to $Z_a \oplus Z_b$.
12. Suppose that k is a divisor of n. Prove that $Z_n/\langle k\rangle \approx Z_k$.
13. Prove that $(A \oplus B)/(A \oplus \{e\}) \approx B$.
14. Explain why the correspondence $x \to 3x$ from Z_{12} to Z_{10} is not a homomorphism.
15. Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and Ker $\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9.
16. Prove that there is no homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$.
17. Prove that there is no homomorphism from $Z_{16} \oplus Z_2$ onto $Z_4 \oplus Z_4$.
18. Can there be a homomorphism from $Z_4 \oplus Z_4$ onto Z_8? Can there be a homomorphism from Z_{16} onto $Z_2 \oplus Z_2$? Explain your answers.
19. Suppose that there is a homomorphism ϕ from Z_{17} to some group and that ϕ is not one-to-one. Determine ϕ.

20. How many homomorphisms are there from Z_{20} onto Z_8? How many are there to Z_8?

21. If ϕ is a homomorphism from Z_{30} onto a group of order 5, determine the kernel of ϕ.

22. Suppose that ϕ is a homomorphism from a finite group G onto $\overline{G}$ and that $\overline{G}$ has an element of order 8. Prove that G has an element of order 8. Generalize.

23. Let ϕ be a homomorphism from a finite group G to $\overline{G}$. If H is a subgroup of $\overline{G}$ give a formula for $|\phi^{-1}(H)|$ in terms of $|H|$ and ϕ.

24. Suppose that $\phi: Z_{50} \to Z_{15}$ is a group homomorphism with $\phi(7) = 6$.
   a. Determine $\phi(x)$.
   b. Determine the image of ϕ.
   c. Determine the kernel of ϕ.
   d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.

25. How many homomorphisms are there from Z_{20} onto Z_{10}? How many are there to Z_{10}?

26. Determine all homomorphisms from Z_4 to $Z_2 \oplus Z_2$.

27. Determine all homomorphisms from Z_n to itself.

28. Suppose that ϕ is a homomorphism from S_4 onto Z_2. Determine Ker ϕ. Determine all homomorphisms from S_4 to Z_2.

29. Suppose that there is a homomorphism from a finite group G onto Z_{10}. Prove that G has normal subgroups of indexes 2 and 5.

30. Suppose that ϕ is a homomorphism from a group G onto $Z_6 \oplus Z_2$ and that the kernel of ϕ has order 5. Explain why G must have normal subgroups of orders 5, 10, 15, 20, 30, and 60.

31. Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and that Ker $\phi = \{1, 11\}$. If $\phi(7) = 7$, find all elements of $U(30)$ that map to 7.

32. Find a homomorphism ϕ from $U(30)$ to $U(30)$ with kernel $\{1, 11\}$ and $\phi(7) = 7$.

33. Suppose that ϕ is a homomorphism from $U(40)$ to $U(40)$ and that Ker $\phi = \{1, 9, 17, 33\}$. If $\phi(11) = 11$, find all elements of $U(40)$ that map to 11.

34. Prove that there is no homomorphism from A_4 onto Z_2.

35. Prove that the mapping $\phi: Z \oplus Z \to Z$ given by $(a, b) \to a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$ (that is, all elements that map to 3).

36. Suppose that there is a homomorphism ϕ from $Z \oplus Z$ to a group G such that $\phi((3, 2)) = a$ and $\phi((2, 1)) = b$. Determine $\phi((4, 4))$ in terms of a and b. Assume that the operation of G is addition.

37. Let $H = \{z \in C^* \mid |z| = 1\}$. Prove that C^*/H is isomorphic to $\mathbf{R}^+$, the group of positive real numbers under multiplication.

(Recall $|a + bi| = \sqrt{a^2 + b^2}$.)

38. Let α be a homomorphism from G_1 to H_1 and β be a homomorphism from G_2 to H_2. Determine the kernel of the homomorphism γ from $G_1 \oplus G_2$ to $H_1 \oplus H_2$ defined by $\gamma(g_1, g_2) = (\alpha(g_1), \beta(g_2))$.

39. Prove that the mapping $x \to x^6$ from $\mathbf{C}^*$ to $\mathbf{C}^*$ is a homomorphism. What is the kernel?

40. For each pair of positive integers m and n, we can define a homomorphism from Z to $Z_m \oplus Z_n$ by $x \to (x \bmod m, x \bmod n)$. What is the kernel when $(m, n) = (3, 4)$? What is the kernel when $(m, n) = (6, 4)$? Generalize.

41. (Second Isomorphism Theorem) If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to KN/N.

42. (Third Isomorphism Theorem) If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$. Think of this as a form of "cancelling out" the N in the numerator and denominator.

43. Prove that the only homomorphism from A_4 to a finite group with order not divisible by 3 is the trivial mapping that takes every element to the identity.

44. Let k be a divisor of n. Consider the homomorphism from $U(n)$ to $U(k)$ given by $x \to x \bmod k$. What is the relationship between this homomorphism and the subgroup $U_k(n)$ of $U(n)$?

45. Determine all homomorphic images of D_4 (up to isomorphism).

46. Let N be a normal subgroup of a finite group G. Use the theorems of this chapter to prove that the order of the group element gN in G/N divides the order of g.

47. Suppose that G is a finite group and that Z_{10} is a homomorphic image of G. What can we say about $|G|$? Generalize.

48. Suppose that Z_{10} and Z_{15} are both homomorphic images of a finite group G. What can be said about $|G|$? Generalize.

49. Suppose that for each prime p, Z_p is the homomorphic image of a group G. What can we say about $|G|$? Give an example of such a group.

50. (For students who have had linear algebra.) Suppose that x is a particular solution to a system of linear equations and that S is the entire solution set of the corresponding homogeneous system of linear equations. Explain why property 6 of Theorem 10.1 guarantees that $x + S$ is the entire solution set of the nonhomogeneous system. In particular, describe the relevant groups and the homomorphism between them.

51. Let N be a normal subgroup of a group G. Use property 7 of Theorem 10.2 to prove that every subgroup of G/N has the form H/N, where H is a subgroup of G. (This exercise is referred to in Chapter 11 and Chapter 24.)

52. Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

53. Use the First Isomorphism Theorem to prove Theorem 9.4.

54. Determine all homomorphisms from D_5 onto $Z_2 \oplus Z_2$. Determine all homomorphisms from D_5 to $Z_2 \oplus Z_2$.

55. Let $Z[x]$ be the group of polynomials in x with integer coefficients under addition. Prove that the mapping from $Z[x]$ into Z given by $f(x) \to f(3)$ is a homomorphism. Give a geometric description of the kernel of this homomorphism. Generalize.

56. Prove that the mapping from **R** under addition to $SL(2, \mathbf{R})$ that takes x to

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

is a group homomorphism. What is the kernel of the homomorphism?

57. Suppose there is a homomorphism ϕ from G onto $Z_2 \oplus Z_2$. Prove that G is the union of three proper normal subgroups.

58. If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \oplus G/K$.

59. If ϕ is a homomorphism from G onto H, prove that $\phi(Z(G)) \subseteq Z(H)$.

60. Suppose that ϕ is a homomorphism from D_{12} onto D_3. What is $\phi(R_{180})$?

61. Prove that every group of order 77 is cyclic.

62. Determine all homomorphisms from Z onto S_3. Determine all homomorphisms from Z to S_3.

63. Let G be an Abelian group. Determine all homomorphisms from S_3 to G.

64. If m and n are positive integers prove that the mapping from Z_m to Z_n given by $\phi(x) = x \bmod n$ is a homomorphism if and only if n divides m.

65. Prove that the mapping from $\mathbf{C}^*$ to $\mathbf{C}^*$ given by $\phi(x) = x^2$ is a homomorphism and that $\mathbf{C}^*/\{1, -1\}$ is isomorphic to $\mathbf{C}^*$. What happens if $\mathbf{C}^*$ is replaced by $\mathbf{R}^*$?

66. Let p be a prime. Determine the number of homomorphisms from $Z_p \oplus Z_p$ into Z_p.

Computer Exercise

A computer exercise for this chapter is available at the website:

http://www.d.umn.edu/~jgallian

Suggested Readings

A. Crans, T. Fiore, and R. Satyendra, "Musical Actions of Dihedral Groups," *The American Mathematical Monthly* 116 (2009):479-495. Available at http://arxiv.org/abs/0711.1873

In this award winning article the authors illustrate how music theorists have modeled works of music as diverse as Hindemith and the Beatles using the dihedral group of order 24.

Jeremiah W. Johnson, "The Number of Group Homomorphisms from D_m into D_n," *The College Mathematics Journal* 44(2013): 190–192. Available at http://arxiv.org/pdf/1201.2363

In this article the author gives a formula for the number of group homomorphisms between any two dihedral groups using elementary group theory only.