GRAM-SCHMIDT EXAMPLE

Find orthonormal basis for span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
where $\vec{v}_1 = \langle 1,1,1\rangle$, $\vec{v}_2 = \langle 1,0,1\rangle$, $\vec{v}_3 = \langle 2,3,4\rangle$.
Use Gram-Schmidt procedure.

1.) Let $\vec{u}_1 = \frac{1}{v_1}\vec{v}_1 = \underline{\frac{1}{\sqrt{3}}\langle 1,1,1\rangle = \hat{u}_1}$.

2.) Calculate $\vec{v}_2 \cdot \hat{u}_1 = \frac{1}{\sqrt{3}}\langle 1,0,1\rangle \cdot \langle 1,1,1\rangle = \frac{2}{\sqrt{3}}$

$$
\begin{aligned}
\vec{v}_2{}' &= \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1 \\
&= \langle 1,0,1\rangle - \frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\langle 1,1,1\rangle \\
&= \langle 1,0,1\rangle - \frac{2}{3}\langle 1,1,1\rangle \\
&= \langle 1/3, -2/3, 1/3\rangle \\
&= \frac{1}{3}\langle 1,-2,1\rangle
\end{aligned}
$$

$\Rightarrow \vec{v}_2{}'$ normalizes to $\underline{\hat{u}_2 = \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle}$.

3.) $\vec{v}_3 \cdot \hat{u}_1 = \langle 2,3,4\rangle \cdot \frac{1}{\sqrt{3}}\langle 1,1,1\rangle = \frac{9}{\sqrt{3}}$.

$\vec{v}_3 \cdot \hat{u}_2 = \langle 2,3,4\rangle \cdot \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle = \frac{2-6+4}{\sqrt{6}} = 0$.

$$
\begin{aligned}
\vec{v}_3{}' &= \vec{v}_3 - (\vec{v}_3 \cdot \hat{u}_1)\hat{u}_1 - (\vec{v}_3 \cdot \hat{u}_2)\hat{u}_2 \\
&= \langle 2,3,4\rangle - \frac{9}{\sqrt{3}}\frac{1}{\sqrt{3}}\langle 1,1,1\rangle \\
&= \langle 2,3,4\rangle - 3\langle 1,1,1\rangle \\
&= \langle -1,0,1\rangle \implies \underline{\hat{u}_3 = \frac{1}{\sqrt{2}}\langle -1,0,1\rangle}.
\end{aligned}
$$

Therefore span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ has $\left\{\frac{1}{\sqrt{3}}\langle 1,1,1\rangle, \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle, \frac{1}{\sqrt{2}}\langle -1,0,1\rangle\right\}$
as an orthonormal basis.