$\boxed{10.9}$

(b) $I(t) = q_0\,\delta(t)$

$$\vec{A} = \frac{\mu_0}{4\pi}\hat{z}\int_{-\beta}^{\beta}\frac{q_0\,\delta(t_r)}{\mathscr{r}}\,dz \qquad t_r = t - \mathscr{r}/c$$

$$\mathscr{r}^2 = s^2 + z^2$$

$$\vec{\mathscr{r}} = \sqrt{s^2+z^2}$$

$$\frac{d\mathscr{r}}{dz} = \frac{d}{dz}\left(\sqrt{s^2+z^2}\right) = \frac{1}{2}\,\frac{\partial z}{\sqrt{s^2+z^2}}$$

$$d\mathscr{r} = \frac{z}{\mathscr{r}}\,dz \implies \frac{dz}{\mathscr{r}} = \frac{d\mathscr{r}}{z}$$

$$z = \sqrt{\mathscr{r}^2 - s^2}$$

$$\vec{A} = \frac{q_0\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{c\,\delta(ct-\mathscr{r})\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-s^2}}$$

$$ct = \mathscr{r}$$

$$\boxed{\vec{A} = \frac{q_0\mu_0}{4\pi}\hat{z}\,\frac{c}{\sqrt{(ct)^2-s^2}}}$$

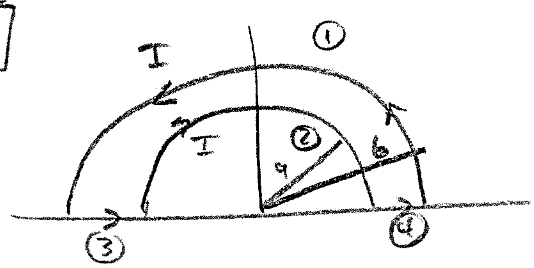
$$E = -\frac{\partial\vec{A}}{\partial t} = -\frac{q_0\mu_0 c}{4\pi}\hat{z}\,\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{(ct)^2-s^2}}\right) = \delta\left(\frac{-\tfrac{1}{2}(c^2\,\partial t)}{((ct)^2-s^2)^{+3/2}}\right)$$

$$\vec{E} = \frac{q_0\mu_0 c^3}{4\pi}\,\frac{t}{((ct)^2-s^2)^{+3/2}}\hat{z}$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\varphi} = \frac{-c\,q_0\mu_0}{4\pi}\hat{\varphi}\,\frac{\partial s}{((ct)^2-s^2)^{3/2}}\,\frac{-1}{2}$$

$$\vec{B} = \frac{q_0\mu_0 c}{4\pi}\,\frac{s}{((ct)^2-s^2)^{3/2}}\hat{\varphi}$$

$\boxed{10.10}$



$$I(t) = kt$$
$$t_r = t - \frac{r}{c}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,d\tau'}{r} = \frac{\mu_0}{4\pi}\int \frac{I(t_r)\,d\vec{l}'}{r} = \frac{\mu_0 k}{4\pi}\oint\left(\frac{t}{r} - \cancel{\frac{1}{c}}\right)d\vec{l}'$$

closed loop

$$\vec{A} = \frac{\mu_0 k}{4\pi}\oint \frac{t}{r}\,d\vec{l}'$$

$$③, ④ \Rightarrow \quad \vec{A}_{3,4} = \frac{\mu_0 k t}{4\pi}\, 2\int_a^b \frac{1}{x}\,dx = \frac{\mu_0 k t}{2\pi}\ln\frac{b}{a}\,\hat{x}$$

$$\vec{A}_1 = \underbrace{\frac{\mu_0 k}{4\pi}}_{\delta}\int \frac{d\vec{l}'}{r} = -\frac{\delta}{b}\,2b\,\hat{x}$$

$$\vec{A}_2 = \delta\int \frac{d\vec{l}'}{r} = \frac{\delta}{a}\,2a\,\hat{x}$$

$$\vec{A} = \vec{A}_{3,4} + \cancel{\vec{A}_2 + \vec{A}_1} = \boxed{\frac{\mu_0 k t}{2\pi}\ln\left(\frac{b}{a}\right)\hat{x} = \vec{A}}$$

$$\vec{E} = -\underbrace{\cancel{\nabla V}}_{0} - \frac{\partial \vec{A}}{\partial t} = \frac{-\partial}{\partial t}\left(\frac{\mu_0 k t}{2\pi}\ln\left(\frac{b}{a}\right)\right) = \boxed{-\frac{\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)\hat{x} = \vec{E}}$$

$\boxed{10-9}$ PY 415, DR. CHUNG

JAMES COOK

19/20

(a) Suppose the wire in Ex 10.2 carries $I(t) = kt$ for $t > 0$. Find $\vec{E}, \vec{B}$. First Calculate $\vec{A}$,

$$\vec{A} = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{I(t_r)}{\imath}dz \qquad t_r = t - \frac{\imath}{c}$$

only segment of z with $|z| \leq \sqrt{(ct)^2 - s^2} \equiv \beta$ matters, else $I(t_r) = 0$.

$$\vec{A} = \frac{\mu_0}{4\pi}\hat{z}\int_{-\beta}^{\beta}\frac{kt_r}{\imath}dz = \frac{\mu_0}{4\pi}\hat{z}\int_{-\beta}^{\beta}\frac{k}{\imath}\left(t - \frac{\imath}{c}\right)dz$$

$$= \frac{\mu_0}{4\pi}\hat{z}\left[\int_{-\beta}^{\beta}\left(\frac{kt}{\imath}\right)dz - \int_{-\beta}^{\beta}\frac{k}{c}dz\right]$$

$$= \frac{\mu_0}{4\pi}\hat{z}\left[2kt\int_0^{\beta}\frac{dz}{\sqrt{z^2+s^2}} - 2\int_0^{\beta}\frac{k}{c}dz\right]$$

$$= \left\{\frac{\mu_0 kt}{2\pi}\left[\ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)\right] - \frac{\mu_0}{2\pi}\left[\frac{k}{c}\sqrt{(ct)^2 - s^2}\right]\right\}\hat{z}$$

$$= \frac{\mu_0 k}{2\pi}\hat{z}\left(t\ln\left(\frac{ct+\beta}{s}\right) - \frac{1}{c}\beta\right)$$

$$\frac{d\beta}{dt} = \frac{d}{dt}\sqrt{(ct)^2 - s^2} = \frac{1}{2}\frac{2c^2t}{\beta} = \frac{c^2t}{\beta}$$

$$\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 k}{2\pi}\hat{z}\left(\ln\left(\frac{ct+\beta}{s}\right) + t\left(\frac{s}{ct+\beta}\right)\frac{d}{dt}\left(\frac{ct+\beta}{s}\right) - \frac{1}{c}\frac{c^2t}{\beta}\right)$$

$$\frac{d}{dt}\left(\frac{ct+\beta}{s}\right) = \frac{c}{s} + \frac{\dot{\beta}}{s}$$

$$= \frac{\mu_0 k}{2\pi}\hat{z}\left(\ln\left(\frac{ct+\beta}{s}\right) + t\left(\frac{s}{ct+\beta}\right)\left(\frac{c}{s} + \frac{c^2t}{\beta s}\right) - \frac{ct}{\beta}\right)$$

$$= \frac{\mu_0 k}{2\pi}\hat{z}\left(\ln\left(\frac{ct+\beta}{s}\right) + t\left(\frac{s}{ct+\beta}\right)\left(\frac{c\beta + c^2t}{\beta \cancel{s}}\right) - \frac{ct}{\beta}\right)$$

$$= \frac{\mu_0 k}{2\pi}\hat{z}\left(\ln\left(\frac{ct+\beta}{s}\right) + t\left(\frac{c(\beta+ct)}{\beta(\beta+ct)}\right) - \frac{ct}{\beta}\right)$$

$$= \frac{\mu_0 k}{2\pi}\hat{z}\left(\ln\left(\frac{ct+\beta}{s}\right) + \cancel{\frac{ct}{\beta}} - \cancel{\frac{ct}{\beta}}\right)$$

$$\boxed{\vec{E} = -\frac{\mu_0 k}{2\pi}\ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)\hat{z}}$$

$\boxed{10-9}$

$$\vec{B} = \nabla \times \vec{A}$$

$$= \nabla \times \left(\frac{\mu_0 k}{2\pi}\left[t \ln\left(\frac{ct+\beta}{s}\right) - \frac{\beta}{c}\right]\right)\hat{z}$$

$$= \left[\frac{1}{s}\frac{\partial}{\partial \varphi}(\cancel{A_z})_{\to 0} - \frac{\partial \cancel{A_\varphi}}{\partial z}_{\to 0}\right]\hat{s} + \left[\frac{\partial \cancel{A_s}}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\varphi} \qquad \text{(use cylindricals)}$$

$$= -\frac{\partial}{\partial s}\left(\frac{\mu_0 k}{2\pi}\left[t \ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right) - \frac{1}{c}\sqrt{(ct)^2-s^2}\,\right]\hat{\varphi}\right)$$

$$= \frac{-\mu_0 k}{2\pi}\left(t\,\frac{s}{ct+\beta}\,\frac{\partial}{\partial s}\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right) - \frac{1}{2c}\,\frac{(-2s)}{\sqrt{(ct)^2-s^2}}\right)\hat{\varphi}$$

$$\frac{\partial}{\partial s}\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right) = \frac{1}{s}\left(\frac{-s}{\sqrt{(ct)^2-s^2}}\right) - \frac{1}{s^2}\left(ct+\sqrt{(ct)^2-s^2}\right)$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{st}{ct+\beta}\left(\frac{-1}{\beta} - \frac{1}{s^2}(ct+\beta)\right) + \frac{s}{c\beta}\right)\hat{\varphi}$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{-st}{\beta(ct+\beta)} - \frac{\cancel{s}t}{s^{\cancel{2}}} + \frac{s}{c\beta}\right)\hat{\varphi}$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{-s^2 t - t(\beta(ct+\beta)) + \frac{s}{c}s(ct+\beta)}{\beta s(ct+\beta)}\right)\hat{\varphi}$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{\cancel{-s^2 t} - \beta c t^2 - t\beta^2 + \cancel{s^2 t} + \frac{1}{c}s^2\beta}{\beta s(ct+\beta)}\right)\hat{\varphi}$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{\cancel{\beta}\left(-ct^2 - t\beta + s^2/c\right)}{\cancel{\beta}s(ct+\beta)}\right)\hat{\varphi}$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{-t\cancel{(ct+\beta)}}{s\cancel{(ct+\beta)}} + \frac{s^2}{cs(ct+\beta)}\right)\hat{\varphi}$$

$$= \frac{-\mu_0 k}{2\pi}\left(\frac{-t}{s} + \frac{s}{c(ct+\beta)}\right)\hat{\varphi}$$

$$\boxed{\vec{B} = \frac{\mu_0 k}{2\pi}\left(\frac{t}{s} - \frac{s}{c\left(ct+\sqrt{(ct)^2-s^2}\right)}\right)\hat{\varphi}}$$

$\boxed{10.9}$

$$\int f(x)\,\delta(x)\,dx = f(0)$$

(b) Let $I(t) = q_0\,\delta(t)$

$$\vec{A} = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}dz = \frac{\mu_0}{4\pi}\hat{z}\int_{-\beta}^{\beta}\frac{q_0\,\delta(t_r)}{\mathscr{r}}dz \qquad t_r = t - \frac{\mathscr{r}}{c}$$

$$\vec{A} = \frac{q_0\mu_0}{4\pi}\hat{z}\int_{-\beta}^{\beta}\frac{1}{\mathscr{r}}\,\delta\left(t-\frac{\mathscr{r}}{c}\right)dz$$

$$\xi$$

$$\mathscr{r} = \sqrt{s^2+z^2}$$

$$\frac{d\mathscr{r}}{dz} = \frac{d}{dz}\left(\sqrt{s^2+z^2}\right) = \frac{1}{2}\frac{2z}{\sqrt{s^2+z^2}}$$

$$\vec{A} = \xi\hat{z}\int_{-\beta}^{\beta}\delta\left(\frac{1}{c}(ct-\mathscr{r})\right)\frac{d\mathscr{r}}{z}$$

$$d\mathscr{r} = \frac{z}{\sqrt{s^2+z^2}}dz$$

$$d\mathscr{r} = \frac{z}{\mathscr{r}}dz$$

$$\vec{A} = \xi\hat{z}\int_{-\beta}^{\beta}c\,\delta(ct-\mathscr{r})\frac{d\mathscr{r}}{\sqrt{\mathscr{r}^2-s^2}}$$

$$\frac{dz}{\mathscr{r}} = \frac{d\mathscr{r}}{z}$$

$$\vec{A} = \xi\hat{z}\,c\,\frac{1}{\sqrt{\mathscr{r}^2-s^2}}\Bigg|^{\mathscr{r}=ct}$$

$$\mathscr{r}^2 = s^2+z^2$$

$$\vec{A} = \frac{q_0\mu_0}{2\,4\pi}\hat{z}\,c\,\frac{1}{\sqrt{(ct)^2-s^2}}.$$

$$z = \sqrt{\mathscr{r}^2-s^2}$$

$\boxed{10.9}$

(b) $\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{q_0 \mu_0}{4\pi} \frac{c}{\sqrt{(ct)^2 - s^2}} \right) \hat{z}$

$$\vec{E} = -\frac{q_0 \mu_0 c}{4\pi} \left(\frac{-1}{2}\right)\left(\frac{2c^2 t}{[(ct)^2 - s^2]^{3/2}}\right) \hat{z}$$

$$\boxed{\vec{E} = \frac{q_0 \mu_0 c^3 t}{2\cancel{4}\pi\,((ct)^2 - s^2)^{3/2}} \hat{z}}$$

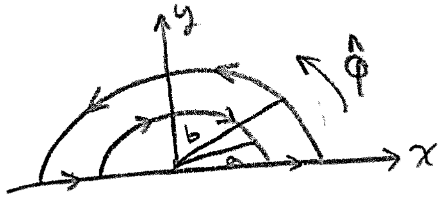
$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A_z}{\partial s} \hat{\phi}$ as $\vec{A} = A_z \hat{z}$ with $A_\phi, A_s = 0$ and $A_z = A_z(s)$ only.

$$\vec{B} = -\frac{\partial}{\partial s}\left(\frac{q_0 \mu_0}{4\pi} c \frac{1}{\sqrt{(ct)^2 - s^2}} \right) \hat{\phi}$$

$$= -\frac{q_0 \mu_0 c}{4\pi} \left(\frac{-1}{2} \frac{-2s}{[(ct)^2 - s^2]^{3/2}} \right) \hat{\phi}$$

$$\boxed{\vec{B} = -\frac{q_0 \mu_0 c}{2\cancel{4}\pi} \frac{s}{((ct)^2 - s^2)^{3/2}} \hat{\phi}}$$

10-10



$I(t) = kt$ $\qquad t_r = t - \frac{r}{c}$

$$\vec{A} = \underset{\substack{\uparrow \\ \delta}}{\frac{\mu_0}{4\pi}} \int \frac{\vec{I}}{r}\, dl' = \frac{\mu_0}{4\pi}\left[\int_{-b}^{-a} \frac{I(t_r)}{r} d\hat{x} + \int_0^{\pi} \frac{I(t_r)}{r}(-a\, d\hat{\varphi}) + \int_b^{a} \frac{I(t_r)}{r} d\hat{x} + \int_0^{\pi} \frac{I(t_r)}{r} b\, d\hat{\varphi}\right]$$

$$(\vec{A})_{\hat{\varphi}} = \delta \int_0^{\pi} \frac{k t_r}{a}(-a\, d\hat{\varphi}) + \delta \int_0^{\pi} \frac{k t_r}{b} b\, d\hat{\varphi}$$

$$\frac{\mu_0 I}{4R}$$

$$= \delta \int_0^{\pi} -k\left(t - \frac{a}{c}\right) d\hat{\varphi} + \delta \int_0^{\pi} k\left(t - \frac{b}{c}\right) d\hat{\varphi}$$

$$= \delta\left[\pi k\left(\cancel{-t} + \frac{a}{c}\right) + \pi k\left(\cancel{t} - \frac{b}{c}\right)\right]\hat{\varphi} = \frac{\mu_0 k}{4c}(a-b)\,\hat{\varphi}$$

$$\int_{-a}^{-b} \frac{k\left(t - \frac{x}{c}\right)}{x} dx = kt \ln\left(\frac{b}{a}\right) + \left(\frac{b-a}{c} k\right)$$

$$\int_a^b \frac{k\left(t - \frac{x}{c}\right)}{x} dx = kt \ln\left(\frac{b}{a}\right) + \left(\frac{a-b}{c} k\right)$$

$$\vec{A} = \left(\frac{\mu_0 k}{4c}(a-b)\right)\hat{\varphi} + \left(\frac{\mu_0 k}{2\pi} \ln\left(\frac{b}{a}\right) t\right)\hat{x} = \boxed{\frac{\mu_0 k}{4}\left(\frac{a-b}{c}\hat{\varphi} + \frac{2\ln(b/a)}{\pi} t\, \hat{x}\right) = \vec{A}}$$

10-10

$$\vec{A} = \frac{\mu_0 k}{4c}(a-b)\,\hat{\varphi} + \left[\frac{\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)t\right]\hat{x}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)t\,\hat{x}\right) + 0.$$

$$\boxed{\vec{E} = \frac{-\mu_0 k}{2\pi}\ln\left(\frac{b}{a}\right)\hat{x}}$$

$\vec{B} = \nabla\times\vec{A} =$ means very little as we have calculated $\vec{A}$ (at origin) not the general coordinate dependence of $\vec{A}$, Thus I cannot take the curl of $\vec{A}$ till I do a <u>whole</u> lot more work.

PY 415, Dr. Chang

James Cook

20/20

10.18 Suppose a point charge q to move along x-axis. Show that the fields at points on the axis to the right of the charge are given by,

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\eta^2}\left(\frac{c+v}{c-v}\right)\hat{x} \quad , \quad \vec{B} = 0.$$

What are the fields on the axis to the left of the charge?

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\eta}{(\vec{\eta}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\eta}\times(\vec{u}\times\vec{a})\right]$$

$$u = c\hat{\eta} - \vec{v}$$

$$\vec{v} = v\hat{\eta}$$

As particle is confined to x axis $\Rightarrow$ $\vec{u} = u\hat{x}$ and $\vec{a} = a\hat{x}$

So we see $\vec{\eta}\times(\underbrace{\cancel{u\hat{x}\times a\hat{x}}}_{0}) = \vec{0}$, Note $\vec{u} = c\hat{\eta} - \vec{v} = c\hat{x} - v\hat{x} = (c-v)\hat{x}$

$\uparrow$ v must be along x axis, along $\hat{\eta}$.

$$\vec{\eta}\cdot\vec{u} = \eta\hat{x}\cdot(c-v)\hat{x} = (c-v)\eta.$$

$$\vec{u} = (c-v)\hat{x}$$

Thus
$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\eta}{[(c-v)\eta]^3}\left[(c^2-v^2)(c-v)\hat{x}\right]$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\hat{x}}{\eta^2}\left[\frac{c^2-v^2}{(c-v)^2}\right] = \frac{q}{4\pi\epsilon_0}\frac{\hat{x}}{\eta^2}\left[\frac{(c+v)(c-v)}{(c-v)^2}\right]$$

$$\therefore \boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\eta^2}\left(\frac{c+v}{c-v}\right)\hat{x}}$$

$\longleftarrow$ right of particle

$$\vec{B} = \frac{1}{c}(\hat{\eta}\times\vec{E}) = \frac{1}{c}(\hat{x}\times E\hat{x}) = \frac{E}{c}\hat{x}\times\hat{x} = \boxed{\vec{0} = \vec{B}}$$

$\boxed{10.18}$ What about to the <u>left</u> of particle? Now $\hat{\mathscr{r}}$ points left. $\hat{\mathscr{r}} = -\hat{x}$

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times\cancel{(\vec{u}\times\vec{a})}\right]$$

0 by the same argument as before.

$$\vec{u} = c\hat{\mathscr{r}} - \vec{v} = -c\hat{x} - v\hat{x}$$

$$\vec{\mathscr{r}}\cdot\vec{u} = -\mathscr{r}\hat{x}\cdot(-c-v)\hat{x} = \mathscr{r}(c+v)$$

$$\vec{u} = c\hat{\mathscr{r}} - \vec{v} = (c+v)\hat{\mathscr{r}}$$

$$E(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\mathscr{r}(c+v))^3}(c^2-v^2)(c+v)\hat{\mathscr{r}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{c^2-v^2}{(c+v)^2}\hat{\mathscr{r}}$$

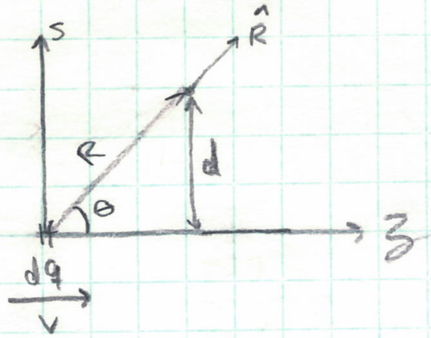
$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)}{(c+v)^2}\hat{\mathscr{r}} \quad , \hat{\mathscr{r}} = -\hat{x}$$

$$\boxed{E(\vec{r},t) = \frac{-1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{x}} \leftarrow \text{left of particle}$$

$$\vec{B} = \frac{1}{c}(\hat{\mathscr{r}}\times\vec{E}) = \frac{1}{c}\left(-\hat{x}\times\cancel{\left(\frac{-1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{x}\right)}\right) = \boxed{\vec{0} = \vec{B}}$$

0 $\quad \hat{x}\times\hat{x} = 0!$

10.19



$$\sin\theta = \frac{d}{\sqrt{d^2+z^2}} \qquad \sin^2\theta = \frac{d^2}{d^2+z^2}$$

$$R^2 = z^2 + d^2$$

$$\hat{R} = \cos\theta\,\hat{z} + \sin\theta\,\hat{s}$$

$$\gamma^2 = \frac{1}{1-v^2/c^2} \qquad \beta^2 = v^2/c^2$$

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} \frac{\lambda}{4\pi\epsilon_0} \frac{1-v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \frac{\hat{R}}{R^2}\,dz$$

$$= \int_{-\infty}^{\infty} \frac{\lambda}{4\pi\epsilon_0} \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \left[\frac{z}{\sqrt{d^2+z^2}}\hat{z} + \frac{d}{\sqrt{d^2+z^2}}\hat{s}\right] \frac{1}{z^2+d^2}\,dz$$

$$\hat{z}: \quad \frac{\lambda}{4\pi\epsilon_0}(1-\beta^2)\int_{-\infty}^{\infty} \frac{1}{\left(1-\beta^2\left(\frac{d^2}{d^2+z^2}\right)\right)^{3/2}} \frac{z}{(d^2+z^2)^{3/2}}\,dz$$

the integrand is odd, $f(z) = -f(z)$ thus the integral goes to zero as is seen easily from the symetry here anyway.

Thus all that remains is the $\hat{s}$ component,

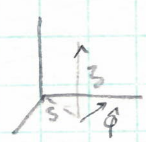
$$\vec{E}(\vec{r},t) = \frac{\lambda}{4\pi\epsilon_0}(1-\beta^2)\int_{-\infty}^{\infty} \frac{1}{(1-\beta^2\sin^2\theta)^{3/2}} \frac{d}{(z^2+d^2)^{3/2}}\,dz$$

$$(1-\beta^2\sin^2\theta)(z^2+d^2) = z^2+d^2 - \beta^2\left(\frac{d^2}{d^2+z^2}\right)(z^2+d^2) = z^2+d^2-\beta^2 d^2 = z^2+d^2(1-\beta^2)$$

$$\vec{E}(\vec{r},t) = \frac{\lambda d}{4\pi\epsilon_0}(1-\beta^2)\int_{-\infty}^{\infty} \frac{dz}{(z^2+\underbrace{d^2(1-\beta^2)}_{\delta^2})^{3/2}} = \frac{\lambda d}{4\pi\epsilon_0}(1-\beta^2)\lim_{t\to\infty}\left[\frac{t}{\sqrt{t^2+\delta^2}\,\delta^2} + \frac{t}{\sqrt{t^2+\delta^2}\,\delta^2}\right]$$

$$\vec{E}(\vec{r},t) = \frac{\lambda d}{4\pi\epsilon_0\delta^2}(1-\beta^2)\lim_{t\to\infty}\left(\frac{2t}{\sqrt{t^2+\delta^2}}\right) = \frac{\lambda d(1-\beta^2)}{2\pi\epsilon_0(d^2(1-\beta^2))} = \frac{\lambda}{2\pi\epsilon_0 d}$$

$$\boxed{\vec{E}(\vec{r},t) = \frac{\lambda}{2\pi\epsilon_0 d}\hat{s}}$$



(b)

$$\vec{B} = \frac{1}{c}(\hat{r}\times\vec{E}) = \frac{1}{c^2}(\vec{v}\times\vec{E}) = \frac{1}{c^2}Ev(\hat{z}\times\hat{s}) = \frac{1}{c^2}Ev\,\hat{\varphi}$$

$$\boxed{\vec{B} = \frac{\lambda v}{c^2 2\pi\epsilon_0 d}\hat{\varphi}} = \boxed{\vec{B} = \frac{\mu_0\lambda v}{2\pi d}\hat{\varphi}}$$

I box both, you pick the aesthetically right one.

10.11 PY 415, Dr. Chung. JAMES COOK

20/20

Let $\vec{J}(\vec{r})$ be constant in time, so $\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)\,t$.

Show $\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$.

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} + \frac{\dot{J}(\vec{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'$$

$\vec{J}(\vec{r})$ constant in time $\Rightarrow \dot{\vec{J}} = 0$. Also note $\dot{\rho}(\vec{r}',t) = \dot{\rho}(\vec{r}',0)$ then $\ddot{\rho}(\vec{r}',t) = 0$.

$t_r = t - \frac{\mathscr{r}}{c}$, hmmm, I drop the $\vec{r}'$ dependence too much writing.

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(t_r)}{\mathscr{r}^2} + \frac{\dot{\rho}(t_r)}{c\mathscr{r}}\right]\hat{\mathscr{r}}\,d\tau' \quad \text{as} \quad \dot{\vec{J}} = 0.$$

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(t-\mathscr{r}/c)}{\mathscr{r}^2} + \frac{\dot{\rho}(t-\mathscr{r}/c)}{c\mathscr{r}}\right]\hat{\mathscr{r}}\,d\tau'$$

These next 2 lines are taylor expansions of ρ and $\dot{\rho}$ about the point $-\mathscr{r}/c$ these are exact because higher order ρ and $\dot{\rho}$ vanish that is $\ddot{\rho} = \dddot{\rho} = \cdots = 0$!

$$\rho(t-\mathscr{r}/c) = \rho(t) + \dot{\rho}(t)\left[-\mathscr{r}/c\right] = \rho(t) - \mathscr{r}/c\,\dot{\rho}(t)$$

$$\dot{\rho}(t-\mathscr{r}/c) = \dot{\rho}(t) + \cancel{\ddot{\rho}(t)[t-\mathscr{r}/c]}$$

0

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(t)}{\mathscr{r}^2} - \frac{\mathscr{r}}{c}\frac{\dot{\rho}(t)}{\mathscr{r}^2} + \frac{\dot{\rho}(t)}{c\mathscr{r}}\right]\hat{\mathscr{r}}\,d\tau'$$

$$\therefore \quad \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'}$$

$\boxed{10.14}$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\underbrace{\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}}_{\star}} \Leftrightarrow V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-v^2/c^2\sin^2\theta}}$$

$$\begin{aligned} \vec{r} &= \vec{R}+\vec{v}t \\ r_x &= R\cos\theta + vt \\ r_y &= R\sin\theta \\ V_x &= V \\ V_y &= 0 \end{aligned} \qquad \begin{aligned} r^2 &= (R\cos\theta - vt)^2 + (R\sin\theta)^2 = R^2 + 2Rvt\cos\theta + v^2t^2 \\ \vec{r}\cdot\vec{v} &= (\vec{R}+\vec{v}t)\cdot(\vec{v}) = \vec{R}\cdot\vec{v} + tv^2 \\ \vec{r}\cdot\vec{v} &= (R\cos\theta + vt)(v) + tv^2 \\ \vec{r}\cdot\vec{v} &= Rv\cos\theta + tv^2 \end{aligned}$$

$$\begin{aligned} (c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2) &= \cancel{c^4t^2} - 2c^2t(\vec{r}\cdot\vec{v}) + (\vec{r}\cdot\vec{v})^2 + c^2r^2 - \cancel{c^4t^2} - v^2r^2 + v^2c^2t^2 \\ &= (\vec{r}\cdot\vec{v})^2 - 2c^2t(\vec{r}\cdot\vec{v}) + c^2r^2 - v^2r^2 + v^2c^2t^2 \end{aligned}$$

thus

$$\star \to (Rv\cos\theta + tv^2)^2 - 2c^2t(Rv\cos\theta + tv^2) + (c^2-v^2)(R^2+2Rvt\cos\theta+v^2t^2) + v^2c^2t^2$$

$$\begin{aligned} \to\ & R^2v^2\cos^2\theta + \cancel{2Rv^3t\cos\theta} + \cancel{t^2v^4} - \cancel{2c^2Rvt\cos\theta} - \cancel{2c^2t^2v^2} \\ & + c^2R^2 + \cancel{2c^2Rvt\cos\theta} + \cancel{c^2v^2t^2} - R^2v^2 - \cancel{2Rv^3t\cos\theta} - \cancel{v^4t^2} + \cancel{v^2c^2t^2} \end{aligned}$$

$$\to R^2v^2\cos^2\theta + R^2c^2 - R^2v^2$$

$$\to R^2(v^2\cos^2\theta + c^2 - v^2)$$

thus $$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{R^2(v^2\cos^2\theta + c^2 - v^2}} = \frac{qc}{4\pi\epsilon_0 R\sqrt{v^2(1-\sin^2\theta)+c^2-v^2}}$$

that is $$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{R\sqrt{c^2-v^2\sin^2\theta}} = \frac{1}{4\pi\epsilon_0}\frac{q\cancel{c}}{R\cancel{c}\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$

$$\therefore \quad \boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}}$$