PY 414, ELECTRODYNAMICS, DR. CHUNG HW #1 JAMES COOK

$\boxed{1.2}$ IS THE CROSS PRODUCT ASSOCIATIVE? NO

30/30

LET,

$$\vec{P} = i$$
$$\vec{Q} = i + j$$
$$\vec{R} = j$$

THEN

$$\vec{P} \times \vec{Q} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -j \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -j$$

$$\vec{Q} \times \vec{R} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} i & k \\ 1 & 1 \end{vmatrix} = -i + k$$

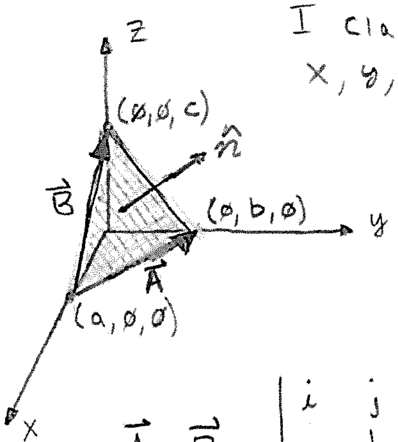
$$(\vec{P} \times \vec{Q}) \times \vec{R} = -j \times \vec{R} = \begin{vmatrix} i & j & k \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i \begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} = \vec{0}$$

✓

$$\vec{P} \times (\vec{Q} \times \vec{R}) = i \times (-i + k) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} j & k \\ 0 & 1 \end{vmatrix} = -j$$

CLEARLY $\vec{0} \neq -j$ because $0 \neq -1$ so by a counterexample the cross product is not associative.

$\boxed{1.4}$



I claim the plane intercepts the axis at a, b, c for x, y, and z respectively.

$\vec{A} = -ai + bj + 0k$ AND $\vec{B} = -ai + 0j + ck$

$\vec{A}$ AND $\vec{B}$ lie in the plane their cross product should give the perpendicular vector to the plane.

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = i\begin{vmatrix} b & 0 \\ 0 & c\end{vmatrix} - j\begin{vmatrix} -a & 0 \\ -a & c\end{vmatrix} + k\begin{vmatrix} -a & b \\ -a & 0\end{vmatrix} = (bc)i + (ac)j + (ab)k$$

let $\vec{C} = \vec{A}\times\vec{B}$ then $\frac{\vec{C}}{C}$ gives a vector of length 1 which is $\hat{n}$

$$C = \sqrt{(bc)^2 + (ac)^2 + (ab)^2}$$

then $$\hat{n} = \frac{bc}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}i + \frac{ac}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}j + \frac{ab}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}k$$

or more compactly $$\hat{n} = \left(\frac{1}{\sqrt{(bc)^2+(ac)^2+(ab)^2}}\right)\left((bc)i + (ac)j + (ab)k\right)$$

✓

1.6 PROVE $\left[A\times(B\times C)\right] + \left[B\times(C\times A)\right] + \left[C\times(A\times B)\right] = \vec{0}$

WHERE
$$A = ai + bj + ck$$
$$B = di + ej + fk$$
$$C = gi + hi + mk \quad \text{and} \quad a,b,c,d,e,f,g,h,m \in \mathbb{R}$$

Proof

$$\vec{B}\times\vec{C} = \begin{vmatrix} i & j & k \\ d & e & f \\ g & h & m \end{vmatrix} = i\begin{vmatrix} e & f \\ h & m\end{vmatrix} - j\begin{vmatrix} d & f \\ g & m\end{vmatrix} + k\begin{vmatrix} d & e \\ g & h\end{vmatrix} = (em-hf)i - (dm-fg)j + (dh-eg)k$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ a & b & c \\ em-hf & dm-fg & dh-eg \end{vmatrix} = i\begin{vmatrix} b & c \\ dm-fg & dh-eg\end{vmatrix} - j\begin{vmatrix} a & c \\ em-hf & dh-eg\end{vmatrix} + k\begin{vmatrix} a & b \\ em-hf & dm-fg\end{vmatrix}$$

$$\Rightarrow \vec{A}\times(\vec{B}\times\vec{C}) = \left[b(dh-eg) - c(dm-fg)\right]i - \left[a(dh-eg) - c(em-hf)\right]j + \left[a(dm-fg) - b(em-hf)\right]k$$
$$= \underbrace{(bdh - beg - cdm + cfg)i - (adh - aeg - cem + chf)j + (adm - afg - bem + bhf)k}_{①},$$

$$\vec{C}\times\vec{A} = \begin{vmatrix} i & j & k \\ g & h & m \\ a & b & c \end{vmatrix} = (hc-bm)i - (cg-am)j + (bg-ah)k$$

$$\vec{B}\times(\vec{C}\times\vec{A}) = \begin{vmatrix} i & j & k \\ d & e & f \\ hc-bm & cg-am & bg-ah \end{vmatrix} = \left[e(bg-ah) - f(cg-am)\right]i - \left[d(bg-ah) - f(hc-bm)\right]j + \left[d(cg-am) - e(hc-bm)\right]k$$
$$= \underbrace{(beg - aeh - cfg + afm)i - (bdg - adh - cfh + bfm)j + (cdg - adm - ceh + bem)k}_{②}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = (bf-ce)i - (af-cd)j + (ae-bd)k$$

$$C\times(\vec{A}\times\vec{B}) = \begin{vmatrix} i & j & k \\ g & h & m \\ bf-ce & af-cd & ae-bd \end{vmatrix} = \left[h(ae-bd) - m(af-cd)\right]i - \left[g(ae-bd) - m(bf-ce)\right]j + \left[g(af-cd) - h(bf-ce)\right]k$$
$$= \underbrace{(aeh - bdh - afm + cdm)i - (aeg - bdg - bfm + cem)j + (afg - cdg - bfh + ceh)k}_{③}$$

NOTE NOW I HAVE ALL 3 TRIPLE PRODUCTS IN TERMS OF THEIR VECTOR'S COMPONENTS.

$$\left[\vec{A}\times(\vec{B}\times\vec{C})\right] + \left[\vec{B}\times(\vec{C}\times\vec{A})\right] + \left[\vec{C}\times(\vec{A}\times\vec{B})\right] = \text{SOME NEW VECTOR } \vec{E} = E_x i + E_y j + E_z k$$

TO FIND $\vec{E}$'S COMPONENTS WE SUM THE LEFT SIDE IN EACH DIMENSION USING 1, 2 AND 3.

X DIMENSION YIELDS, $(bdf - beg - cdm + cfg) + (beg - aeh - cfg + afm) + (aeh - bdh - afm + cdm) = \emptyset = E_x$

Y DIMENSION YIELDS, $-(adh - aeg - cem + chf) - (bdg - adh - cfh + bfm) - (aeg - bdg - bfm + cem) = \emptyset = E_y$

Z DIMENSION YIELDS, $(adm - afg - bem + bhf) + (cdg - adm - ceh + bem) + (afg - cdg - bfh + ceh) = \emptyset = E_z$

IN EACH DIMENSION EVERY TERM IS SUMMED WITH ITS ADDITIVE INVERSE PRODUCING ZERO THUS $E_x = E_y = E_z = 0$ and $\vec{E} = \vec{0}$ AND SO THE PROOF IS COMPLETE.

1-6 $(\vec{A}\times\vec{B})\times\vec{C} = \vec{A}\times(\vec{B}\times\vec{C})$

$\vec{A}\times(\vec{B}\times\vec{C}) = -[C\times(\vec{A}\times\vec{B})]$ since $\vec{A}\times\vec{B} = -[\vec{A}\times\vec{B}]$

$(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = -(\vec{C}\cdot\vec{B})\vec{A} + (\vec{C}\cdot\vec{A})\vec{B}$ by 1.17

$(\vec{C}\cdot\vec{A})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = -(\vec{C}\cdot\vec{B})\vec{A} + (\vec{C}\cdot\vec{A})\vec{B}$ since $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

(*) $-(\vec{A}\cdot\vec{B})\vec{C} = -(\vec{C}\cdot\vec{B})\vec{A}$ subtracting $(\vec{C}\cdot\vec{A})\vec{B}$

if $\vec{A}=\vec{0}$ or $\vec{B}=\vec{0}$ or $\vec{C}=\vec{0}$ then (*) is true.

thus consider $\vec{A},\vec{B},\vec{C}\neq\vec{0}$

$\vec{A}\perp\vec{B} \iff \vec{C}\perp\vec{B}$ so $\vec{A},\vec{B}\ \vec{C}$ are $\perp$ to each other

$\vec{A}$ not $\perp$ $\vec{B}$ then $|\vec{A}\cdot\vec{B}|>0$ and $|\vec{C}\cdot\vec{B}|\neq 0$ so $|\vec{C}\cdot\vec{B}|>0$

and $(\vec{A}\cdot\vec{B})\vec{C} = (\vec{C}\cdot\vec{B})\vec{A}$ must hold

if $\vec{C}=m\vec{A} \Rightarrow (\vec{A}\cdot\vec{B})m\vec{A} = (m\vec{A}\cdot\vec{B})\vec{A}$

$\Rightarrow m(\vec{A}\cdot\vec{B})\vec{A} = m(\vec{A}\cdot\vec{B})\vec{A}$

but if $\vec{C}\neq m\vec{A}$ then (*) can't hold

① $\vec{A}$ OR $\vec{B}$ OR $\vec{C}$ IS THE ZERO VECTOR
② $\vec{A}$, $\vec{B}$ and $\vec{C}$ are all $\perp$ to each other
③ $\vec{A}$ is a scalar multiple of $\vec{C}$
AT LEAST ONE OF THE ABOVE MUST HOLD FOR THE TRIPLE CROSS PRODUCT TO BE ASSOCIATIVE

✓

PY 414, ELECTRODYNAMICS, HW #2, 1-8 AND 1-9, DR. CHUNG   JAMES COOK

$\boxed{1.8}$

20/20

(a) PROVE THAT $\bar{A}_x \bar{B}_x + \bar{A}_y \bar{B}_y = A_x B_x + A_y B_y$ UNDER

$R = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ AND $\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \end{pmatrix}$

matrix notation   equivalent plain algebra

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix} \iff \begin{matrix} A_x\cos\phi + A_y \sin\phi = \bar{A}_x \\ -A_x \sin\phi + A_y \cos\phi = \bar{A}_y \end{matrix}$$

$$\begin{pmatrix} \bar{B}_x \\ \bar{B}_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} B_x \\ B_y \end{pmatrix} \iff \begin{matrix} B_x\cos\phi + B_y \sin\phi = \bar{B}_x \\ -B_x \sin\phi + B_y \cos\phi = \bar{B}_y \end{matrix}$$

$$\bar{A}_x\bar{B}_x = (A_x\cos\phi + A_y\sin\phi)(B_x\cos\phi + B_y\sin\phi)$$
$$= A_xB_x\cos^2\phi + A_xB_y\cos\phi\sin\phi + A_yB_x\sin\phi\cos\phi + A_yB_y\sin^2\phi$$

$$\bar{A}_y\bar{B}_y = (-A_x\sin\phi + A_y\cos\phi)(-B_x\sin\phi + B_y\cos\phi)$$
$$= A_xB_x\sin^2\phi - A_xB_y\sin\phi\cos\phi - A_yB_x\cos\phi\sin\phi + A_yB_y\cos^2\phi$$

$$\bar{A}_x\bar{B}_x + \bar{A}_y\bar{B}_y = A_xB_x\cos^2\phi + A_xB_y\cos\phi\sin\phi + A_yB_x\sin\phi\cos\phi + A_yB_y\sin^2\phi$$
$$+ A_xB_x\sin^2\phi - A_xB_y\sin\phi\cos\phi - A_yB_x\cos\phi\sin\phi + A_yB_y\cos^2\phi$$
$$= A_xB_x(\cos^2\phi + \sin^2\phi) + A_yB_y(\sin^2\phi + \cos^2\phi)$$
$$= A_xB_x + A_yB_y$$

$\therefore$ the dot product is preserved under two dimensional rotation ✓

(b) what constraints must the elements (R_{ij}) of the 3-D rotation matrix satisfy to preserve length of transformed vectors $\vec{A}$? (Let $\vec{A}'$ be transformed vector)

A_i' is component of $\vec{A}'$, $\sqrt{\sum_{i=1}^{3}(A_i')^2} = \sqrt{\sum_{i=1}^{3}(A_i)^2}$, length preservation

where A_i' is related to A_i by $\{R_{ij}\}$ as follows

$$A_i' = \sum_{j=1}^{3} R_{ij}A_j = R_{i1}A_1 + R_{i2}A_2 + R_{i3}A_3$$

$$\sqrt{\sum_{i=1}^{3}(R_{i1}A_1 + R_{i2}A_2 + R_{i3}A_3)^2} = \sqrt{(R_{11}A_1 + R_{12}A_2 + R_{13}A_3)^2 + (R_{21}A_1 + R_{22}A_2 + R_{23}A_3)^2}$$

$$+ (R_{31}A_1 + R_{32}A_2 + R_{33}A_3)^2 = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

SQUARING BOTH SIDES YIELDS,

$$A_1^2 + A_2^2 + A_3^2 = (R_{11}A_1 + R_{12}A_2 + R_{13}A_3)^2 + (R_{21}A_1 + R_{22}A_2 + R_{23}A_3)^2 + (R_{31}A_1 + R_{32}A_2 + R_{33}A_3)^2$$

CARRYING OUT THE SQUARES we get $A_1^2 + A_2^2 + A_3^2 =$

$$(R_{11}A_1)(R_{11}A_1 + R_{12}A_2 + R_{13}A_3) + (R_{12}A_2)(R_{11}A_1 + R_{12}A_2 + R_{13}A_3) + (R_{13}A_3)(R_{11}A_1 + R_{12}A_2 + R_{13}A_3)$$

$$+ (R_{21}A_1)(R_{21}A_1 + R_{22}A_2 + R_{23}A_3) + R_{22}A_2(R_{21}A_1 + R_{22}A_2 + R_{23}A_3) + R_{23}A_3(R_{21}A_1 + R_{22}A_2 + R_{23}A_3)$$

$$+ (R_{31}A_1)(R_{31}A_1 + R_{32}A_2 + R_{33}A_3) + R_{32}A_2(R_{31}A_1 + R_{32}A_2 + R_{33}A_3) + R_{33}A_3(R_{31}A_1 + R_{32}A_2 + R_{33}A_3).$$

grouping by coefficients A_iA_j where $i,j \in \{1,2,3\}$ then $A_1^2 + A_2^2 + A_3^2 = A_1'^2 + A_2'^2 + A_3'^2$

$$A_1A_1(R_{11}R_{11} + R_{21}R_{21} + R_{31}R_{31})$$
$$+ A_1A_2(R_{11}R_{12} + R_{12}R_{11} + R_{21}R_{22} + R_{22}R_{21} + R_{31}R_{32} + R_{32}R_{31})$$
$$+ A_1A_3(R_{11}R_{13} + R_{13}R_{11} + R_{21}R_{23} + R_{23}R_{21} + R_{31}R_{33} + R_{33}R_{31})$$
$$+ A_2A_2(R_{12}R_{12} + R_{22}R_{22} + R_{32}R_{32})$$
$$+ A_2A_3(R_{12}R_{13} + R_{13}R_{12} + R_{22}R_{23} + R_{23}R_{22} + R_{32}R_{33} + R_{33}R_{32})$$
$$+ A_3A_3(R_{13}R_{13} + R_{23}R_{23} + R_{33}R_{33})$$
$$= A_1^2 + A_2^2 + A_3^2 + 0(A_1A_2 + A_2A_3 + A_{13})$$

Then we may equate coefficients since A_i is differentiable twice over.

from $A_{11} \Rightarrow R_{11}^2 + R_{21}^2 + R_{31}^2 = 1$
from $A_{22} \Rightarrow R_{12}^2 + R_{22}^2 + R_{32}^2 = 1$
from $A_{33} \Rightarrow R_{13}^2 + R_{23}^2 + R_{33}^2 = 1$

equivalently $\boxed{\sum_{i=1}^{3} (R_{ij})^2 = 1,\ j \in \{1,2,3\}}$

from $A_1A_2 \Rightarrow 2(R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32}) = 0$
from $A_1A_3 \Rightarrow 2(R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33}) = 0$
from A_2A_3 $2(R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33}) = 0$
can divide two out for Σ form,

equivalently $\boxed{\sum_{i=1}^{3} R_{ij}R_{ik} = 0 \quad j,k \in \{1,2,3\} \text{ and } j \neq k}$

of course $R_{ij}R_{ik} = R_{ik}R_{ij} \Rightarrow k$ and j interchangeable
for example. $R_{i2}R_{i3} = R_{i3}R_{i2}$

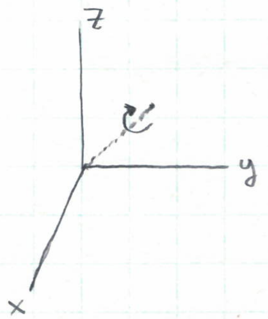
notation for this is δ_{ij}

10

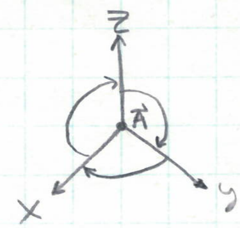
$$\sum_{k=1}^{3} R_{ki}R_{kj} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

1-9 What R describes $120°$ CW ROTATION ABOUT AXIS formed by $(0,0,0) \longrightarrow (1,1,1)$

Axis Along $\vec{A} = i + j + k$

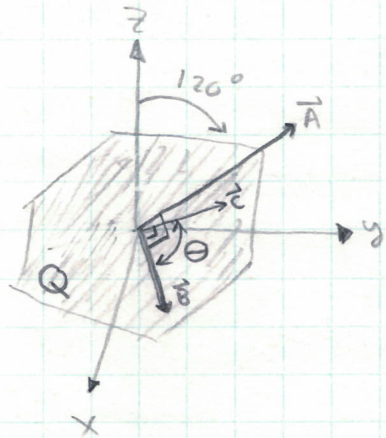


view down $\vec{A}$



$z \longrightarrow y$
$y \longrightarrow x$
$x \longrightarrow z$

so, $\bar{z} = y$
$\bar{y} = x$
$\bar{x} = z$



$\vec{A}$ is normal of Q plane
$\vec{A}$ projected onto Q is a point
x, y, and z axis projected onto Q form 3 lines $120°$ seperated.
when the coordinate system is rotated over the plane Q the angles observed projected on Q must stay the same so z goes to y, y goes to x and x goes to z. consequently,

$$i = k' \; , \; j = i' \; , \; k = j'$$

it is known that in general

$$R_{ij} = \lambda_{ij} = \hat{e}_i' \cdot \hat{e}_j \quad \text{where } \hat{e}_1 = i \; , \hat{e}_2 = j \; , \hat{e}_3 = k \; \ldots$$

since $k' \cdot i = 1$, $i' \cdot j = 1$ and $j' \cdot k = 1$

10

$\Rightarrow \lambda_{31} = 1$, $\lambda_{12} = 1$ and $\lambda_{23} = 1$

all other dot products are zero since if two unit vectors aren't identical they must be $\perp$, $(\lambda = 0)$

$$\boxed{R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} A_y \\ A_z \\ A_x \end{pmatrix}$$

which was the intial guess.

PY 414, ELECTRODYNAMICS, HW #3, DR. CHUNG — JAMES COOK

29/30

1.12

height of some hill is given as $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$ where y is north and x is east from origin of South Hadley

(a) Where is top of hill? Where $h(x,y)$ is maximized. OR WHERE $\nabla h(x,y) = \vec{0}$ (∇ is just 2 dimensional here)

South Hadley.

$$\frac{\nabla h(x,y)}{10} = \frac{\partial}{\partial x}(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)\,i + \frac{\partial}{\partial y}(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)\,j$$
$$= (2y - 6x - 18)\,i + (2x - 8y + 28)\,j$$

$\nabla h(x,y) = 0 \implies \dfrac{\nabla h(x,y)}{10} = 0 \implies 2y - 6x = 18$

$2x - 8y = -28$, $x = 4y - 14$

$2y - 6(4y - 14) = 18$

$2y - 24y + 84 = 18$

$-22y = -66 \implies \boxed{y = 3}$

$x = 4 \cdot 3 - 14 = \boxed{-2 = x}$

top of hill at 2 miles west and 3 miles north of South Hadley

(b)

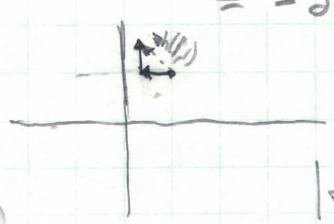
$$h(-2,3) = 10(2(-2)(3) - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12)$$
$$= 10(-12 - 12 - \cancel{36} + \cancel{36} + 84 + \cancel{12})$$
$$= 10(-12 + 84)$$
$$= 10(72) \Rightarrow \boxed{720 \text{ feet high}}$$

[note: $h(0,0) = 120 < 720$ thus $h(-2,3)$ is maximum because $\nabla h = 0$ only at one point and there exists another point where $h(x,y)$ is less than $h(-2,3)$.]

9

(c) $\nabla h(1,1) = 10(2 - 6 - 18)\,i + 10(2 - 8 + 28)\,j$
$= -220i + 220j \implies$ the hill is steepest in the $-i + j$ or NORTH-WESTERNLY Direction. It has a steepness of 311 ft/mile

what angle?



$$|\nabla h(1,1)| = \sqrt{(-220)^2 + (220)^2} = 311 \text{ ft/mile}$$

$\boxed{1.15}$ CALCULATE DIVERGENCE OF FOLLOWING VECTOR FUNCTIONS.

a) $$\nabla\cdot(x^2 i + 3xz^2 j - 2xzk) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(x^2 i + 3xz^2 j - 2xzk\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \underbrace{\frac{\partial}{\partial y}(3xz^2)}_{0} + \frac{\partial}{\partial z}(-2xz)$$

$$= 2x - 2x = \boxed{0}$$

b) $$\nabla\cdot\vec{V}_B = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(xyi + 2yzj + 3zxk\right)$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx) = \boxed{y + 2z + 3x}$$

c) $$\nabla\cdot\vec{V}_c = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(y^2 i + (2xy + z^2)j + 2yzk\right)$$

$$= \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= \boxed{2x + 2y}$$

10

$\boxed{1.17}$ Show that divergence transforms as a scalar under rotations ie. $\bar{\nabla}\cdot\bar{\vec{V}} = \nabla\cdot\vec{V}$

∇ and $\vec{V}$ transform by rotation matrix $R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ in the usual way,

$\bar{\nabla} = R\nabla$ and $\bar{\vec{V}} = R\vec{V}$ thus $\bar{\nabla}\cdot\bar{\vec{V}} = R\nabla \cdot R\vec{V}$

$$\bar{\nabla}\cdot\bar{\vec{V}} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} \cdot \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} R_{11}\nabla_1 + R_{12}\nabla_2 \\ R_{21}\nabla_1 + R_{22}\nabla_2 \end{pmatrix} \cdot \begin{pmatrix} R_{11}V_1 + R_{12}V_2 \\ R_{21}V_1 + R_{22}V_2 \end{pmatrix}$$

$$= (R_{11}\nabla_1 + R_{12}\nabla_2)(R_{11}V_1 + R_{12}V_2) + (R_{21}\nabla_1 + R_{22}\nabla_2)(R_{21}V_1 + R_{22}V_2)$$

$$= R_{11}^2\nabla_1V_1 + \cancel{R_{11}R_{12}\nabla_1V_2} + \cancel{R_{12}R_{11}\nabla_2V_1} + R_{12}^2\nabla_2V_2 + R_{21}^2\nabla_1V_1 + \cancel{R_{21}R_{22}\nabla_1V_2} + \cancel{R_{22}R_{21}\nabla_2V_1} + R_{22}^2\nabla_2V_2$$

$$= R_{11}^2\nabla_1V_1 + R_{12}^2\nabla_2V_2 + R_{21}^2\nabla_1V_1 + R_{22}^2\nabla_2V_2 \quad , \text{ terms cancel since } R_{12} = -R_{21}$$

$$= \nabla_1V_1(R_{11}^2 + R_{21}^2) + \nabla_2V_2(R_{12}^2 + R_{22}^2)$$

$$= \nabla_1V_1 + \nabla_2V_2 \quad \text{since } \cos^2\phi + (-\sin\phi)^2 = 1 \text{ and } \sin^2\phi + \cos^2\phi = 1$$

10

$$= \nabla\cdot\vec{V}$$

ELECTRODYNAMICS, HW#5, PY411, DR. CHUNG.

JAMES COOK

20/20

$\boxed{1-23}$

NOTATION:

$$\left(\frac{\partial F}{\partial x} = F_x\right) \quad \left(\frac{\partial A_1}{\partial y} = A_{1y}\right) \quad \left(\frac{\partial^2 H}{\partial x^2} = H_{xx}\right)$$

ADD VARIABLE OF differentiation to index to denote derivative.

$$\vec{A} = A_1 i + A_2 j + A_3 k$$

$$\nabla = \nabla_1 i + \nabla_2 j + \nabla_3 k \;,\quad \nabla_1 = \frac{\partial}{\partial x} \quad \nabla_2 = \frac{\partial}{\partial y} \;,\quad \nabla_3 = \frac{\partial}{\partial z} .$$

$$\nabla\left(\frac{f}{g}\right) = i\,\nabla_1\left(\frac{f}{g}\right) + j\,\nabla_2\left(\frac{f}{g}\right) + k\,\nabla_3\left(\frac{f}{g}\right) = i\left(\frac{g f_x - f g_x}{g^2}\right) + j\left(\frac{g f_y - f g_y}{g^2}\right) + k\left(\frac{g f_z - f g_z}{g^2}\right)$$

$$= \frac{g(f_x i + f_y j + f_z k) - f(g_x i + g_y j + g_z k)}{g^2} = \frac{g\nabla f - f\nabla g}{g^2}$$

$$\nabla\cdot\left(\frac{\vec{A}}{g}\right) = \nabla_1\left(\frac{A_1}{g}\right) + \nabla_2\left(\frac{A_2}{g}\right) + \nabla_3\left(\frac{A_3}{g}\right) = \frac{gA_{1x} - A_1 g_x}{g^2} + \frac{gA_{2y} - A_2 g_y}{g^2} + \frac{gA_{3z} - A_3 g_z}{g^2}$$

$$= \frac{g(A_{1x} + A_{2y} + A_{3z}) - (A_1 g_x + A_2 g_y + A_3 g_z)}{g^2} \;,\quad \left(\begin{array}{l} \nabla g = g_x i + g_y j + g_z k \\ \nabla\cdot A = A_{1x} + A_{2y} + A_{3z} \end{array}\right) \Rightarrow$$

$$= \frac{g(\nabla\cdot\vec{A}) - \vec{A}\cdot(\nabla g)}{g^2}$$

✓

1-23 Let $\nabla = \nabla_1 i + \nabla_2 j + \nabla_3 k$. and $\vec{A} = A_1 i + A_2 j + A_3 k$.
and $\frac{\partial A_1}{\partial x} = A_{1x}$ and $\frac{\partial A_3}{\partial y} = A_{3y}$ exc...

$$\nabla \times \left(\frac{\vec{A}}{g}\right) = \begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ \frac{A_1}{g} & \frac{A_2}{g} & \frac{A_3}{g} \end{vmatrix} = i \begin{vmatrix} \nabla_2 & \nabla_3 \\ \frac{A_2}{g} & \frac{A_3}{g} \end{vmatrix} - j \begin{vmatrix} \nabla_1 & \nabla_3 \\ \frac{A_1}{g} & \frac{A_3}{g} \end{vmatrix} + k \begin{vmatrix} \nabla_1 & \nabla_2 \\ \frac{A_1}{g} & \frac{A_2}{g} \end{vmatrix}$$

$$= i\left(\nabla_2\left(\frac{A_3}{g}\right) - \nabla_3\left(\frac{A_2}{g}\right)\right) - j\left(\nabla_1\left(\frac{A_3}{g}\right) - \nabla_3\left(\frac{A_1}{g}\right)\right) + k\left(\nabla_1\left(\frac{A_2}{g}\right) - \nabla_2\left(\frac{A_1}{g}\right)\right)$$

$$= i\left(\frac{gA_{3y} - A_3 g_y}{g^2} - \frac{gA_{2z} - A_2 g_z}{g^2}\right) - j\left(\frac{gA_{3x} - A_3 g_x}{g^2} - \frac{gA_{1z} - A_1 g_z}{g^2}\right)$$

$$+ k\left(\frac{gA_{2x} - A_2 g_x}{g^2} - \frac{gA_{1y} - A_1 g_y}{g^2}\right)$$

$$= \frac{g}{g^2}\left((A_{3y} - A_{2z})i - (A_{3x} - A_{1z})j + (A_{2x} - A_{1y})k\right)$$

$$+ \frac{1}{g^2}\left((-A_3 g_y + A_2 g_z)i - (-A_3 g_x + A_1 g_z)j + (-A_2 g_x + A_1 g_y)k\right)$$

$$= \frac{1}{g^2}\left[g\left(i\begin{vmatrix} \nabla_2 & \nabla_3 \\ A_2 & A_3 \end{vmatrix} - j\begin{vmatrix} \nabla_1 & \nabla_3 \\ A_1 & A_3 \end{vmatrix} + k\begin{vmatrix} \nabla_1 & \nabla_2 \\ A_1 & A_2 \end{vmatrix}\right) + \left(i\begin{vmatrix} A_2 & A_3 \\ g_y & g_z \end{vmatrix} - j\begin{vmatrix} A_1 & A_3 \\ g_x & g_z \end{vmatrix} + k\begin{vmatrix} A_1 & A_2 \\ g_x & g_y \end{vmatrix}\right)\right]$$

$$= \frac{1}{g^2}\left[g\begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ g_x & g_y & g_z \end{vmatrix}\right]$$

note that $g_x i + g_y j + g_z k = \nabla g$. thus

$$= \frac{1}{g^2}\left[g(\nabla \times \vec{A}) + A \times (\nabla g)\right] = \frac{g(\nabla \times \vec{A}) + A \times (\nabla g)}{g^2}$$ ✓

10

$\boxed{1-27}$ Prove $\nabla \times \nabla g = \vec{0}$ for all scalar functions g.

proof

Let $\nabla = \nabla_x i + \nabla_y j + \nabla_z k$ and $\frac{\partial g}{\partial x} = g_x$, $\frac{\partial g}{\partial y} = g_y$ and $\frac{\partial^2 g}{\partial z^2} = g_{zz}$

where $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ and g is some scalar function.

Also I make the observation $\nabla g = g_x i + g_y j + g_z k$.

thus $$\nabla \times \nabla g = \begin{vmatrix} i & j & k \\ \nabla_x & \nabla_y & \nabla_z \\ g_x & g_y & g_z \end{vmatrix}$$

$$= i\begin{vmatrix} \nabla_y & \nabla_z \\ g_y & g_z \end{vmatrix} - j\begin{vmatrix} \nabla_x & \nabla_z \\ g_x & g_z \end{vmatrix} + k\begin{vmatrix} \nabla_x & \nabla_y \\ g_x & g_y \end{vmatrix}$$

$$= i\left(\nabla_y g_z - \nabla_z g_y\right) - j\left(\nabla_x g_z - \nabla_z g_x\right) + k\left(\nabla_x g_y - \nabla_y g_x\right)$$

$$= i\left(\frac{\partial}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial}{\partial z}\frac{\partial g}{\partial y}\right) - j\left(\frac{\partial}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial}{\partial z}\frac{\partial g}{\partial x}\right) + k\left(\frac{\partial}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial}{\partial y}\frac{\partial g}{\partial x}\right)$$

then Clairaut's Theorem from Calculus gives us $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$ and so we may substitute the first term for the second in each pair,

$$= i\left(\frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 g}{\partial y \partial z}\right) - j\left(\frac{\partial^2 g}{\partial x \partial z} - \frac{\partial^2 g}{\partial x \partial z}\right) + k\left(\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial x \partial y}\right)$$

$$= 0i + 0j + 0k = \vec{0}$$

$\boxed{1.27}$ showing $\vec{0} = \nabla \times \nabla g$ true for $g = x^2 y^3 z^4$

$$\nabla g = \nabla_x (x^2 y^3 z^4) i + \nabla_y (x^2 y^3 z^4) j + \nabla_z (x^2 y^3 z^4) k$$

$$= (2x y^3 z^4) i + (3x^2 y^2 z^4) j + (4x^2 y^3 z^3) k$$

$$\nabla \times \nabla g = \begin{vmatrix} i & j & k \\ \nabla_x & \nabla_y & \nabla_z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

$$= i\left(\nabla_y (4x^2 y^3 z^3) - \nabla_z (3x^2 y^2 z^4)\right) - j\left(\nabla_x (4x^2 y^3 z^3) - \nabla_z (2x y^3 z^4)\right)$$

$$+ k\left(\nabla_x (3x^2 y^2 z^4) - \nabla_y (2x y^3 z^4)\right)$$

$$= i\,(12x^2y^2z^3 - 12x^2y^2z^3) - j\,(8xy^3z^3 - 8xy^3z^3)$$

$$+ k\,(6xy^2z^4 - 6xy^2z^4)$$

$$= i(0) + j(0) + k(0) = \vec{0}.$$

✓

10

ELECTRODYNAMICS, PY414, HWK #6, DR. CHUNG

JAMES COOK.

18/20

$\boxed{1-28}$ $\vec{v} = x^2 i + 2yz j + y^2 k$

a) Let P_A be $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

$$\int_{P_A} \vec{v}\cdot d\vec{l} = \int_{(0,0,0)}^{(1,0,0)} \vec{v}\cdot i\,dx + \int_{(1,0,0)}^{(1,1,0)} \vec{v}\cdot j\,dy + \int_{(1,1,0)}^{(1,1,1)} \vec{v}\cdot k\,dz = \frac{x^3}{3}\Big|_0^1 + \underset{(z=0)}{y^2(0)\Big|_0^1} + \underset{(y=1)}{(1)^2 z\Big|_0^1} = \frac{1}{3} + 0 + 1 = \boxed{1\tfrac{1}{3}}$$

b) Let P_B be $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

$$\int_{P_B} \vec{v}\cdot d\vec{l} = \int_{(0,0,0)}^{(0,0,1)} \vec{v}\cdot k\,dz + \int_{(0,0,1)}^{(0,1,1)} \vec{v}\cdot j\,dy + \int_{(0,1,1)}^{(1,1,1)} \vec{v}\cdot i\,dx = \underset{(y=0)}{(0)^2 z\Big|_0^1} + \underset{(z=1)}{y^2(1)\Big|_0^1} + \frac{x^3}{3}\Big|_0^1 = 0 + 1 + \frac{1}{3} = \boxed{1\tfrac{1}{3}}$$

c) NOTE THAT $x=y=z$ and $dx=dy=dz$ for the path of integration thus,

$$\int_{(0,0,0)}^{(1,1,1)} \vec{v}\cdot d\vec{l} = \int_{(0,0,0)}^{(1,1,1)} \vec{v}\cdot(i\,dx + j\,dy + k\,dz) = \int_{(0,0,0)}^{(1,1,1)} x^2 dx + \int_{(0,0,0)}^{(1,1,1)} 2yz\,dy + \int_{(0,0,0)}^{(1,1,1)} y^2 dz$$

next use $x=y=z$ for substitution.

$$= \int_{(0,0,0)}^{(1,1,1)} x^2 dx + \int_{(0,0,0)}^{(1,1,1)} 2y^2 + \int_{(0,0,0)}^{(1,1,1)} z^2 dz = \frac{x^3}{3}\Big|_0^1 + \frac{2}{3}y^3\Big|_0^1 + \frac{z^3}{3}\Big|_0^1 = \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \boxed{1\tfrac{1}{3}}$$

d) Fundamental Theorem for Gradients tells us that $\int_a^b \nabla T\cdot d\vec{l} = T(b) - T(a)$ if $\vec{v} = \nabla T$ where T is some scalar function then $\oint \vec{v}\cdot d\vec{l} = 0$. Previously we have shown $\nabla\times\nabla T = \vec{0}$

so if $\nabla\times\vec{v} = \vec{0}$ we may conclude that $\vec{v} = \nabla T$ and thus the integral that comes back to where it started has a value of zero. $\nabla\times\vec{v} = \begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ x^2 & 2yz & y^2 \end{vmatrix} =$

$$i\begin{vmatrix} \nabla_2 & \nabla_3 \\ 2yz & y^2 \end{vmatrix} - j\begin{vmatrix} \nabla_1 & \nabla_3 \\ x^2 & y^2 \end{vmatrix} + k\begin{vmatrix} \nabla_1 & \nabla_2 \\ x^2 & 2yz \end{vmatrix} = (2y - 2y)i - (0-0)j - (0-0) = \vec{0}$$

so $\oint v\cdot d\vec{l} = 0$.

0

$\boxed{1-30}$

$$\int_V T\, d\tau = \int_0^1 \int_0^{1-z} \int_0^{1-y} dx\, dy\, z^2 dz$$

$$= \int_0^1 \int_0^{1-z} (1-y)\, dy\, z^2 dz$$

$$= \int_0^1 \left(y - \frac{y^2}{2} \Big|_0^{1-z} \right) z^2 dz$$

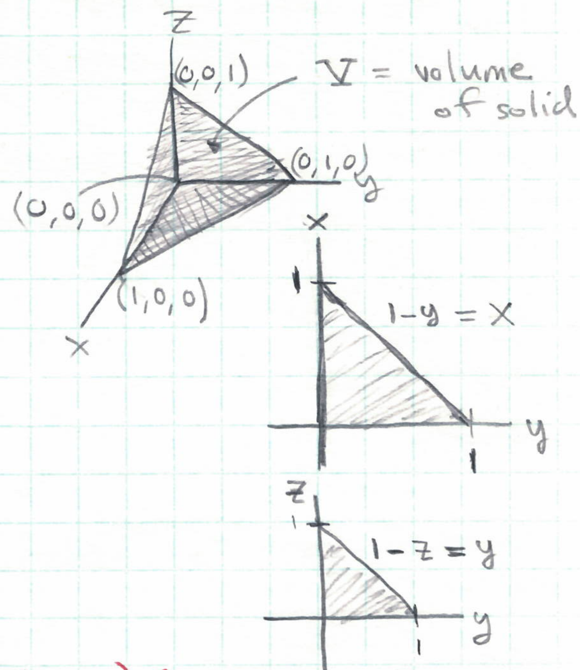
$$= \int_0^1 \left(1 - z - \frac{1}{2}\left(1 - 2z + z^2\right)\right) z^2 dz$$

$$= \int_0^1 \left(\frac{1}{2} z^2 - \frac{1}{2} z^4 \right) dz \quad I = \int z^2 \frac{(1-z)}{2}(1-z)\, dz$$

8

$$= \frac{z^3}{6} - \frac{z^5}{10} \Big|_0^1 = \frac{1}{6} - \frac{1}{10} = \frac{5}{30} - \frac{3}{30} = \frac{2}{30} = \boxed{\frac{1}{15}}$$

$$= \frac{1}{60}$$



PY 414, ELECTRODYNAMICS, HW # , DR. CHUNG JAMES COOK

$\boxed{1-38}$

Divergence Theorem : $\int_{\mathcal{V}} (\nabla \cdot \vec{V})\,d\tau = \oint_S \vec{V} \cdot d\vec{a}$ 20/20

(a) if $\vec{V} = r^2 \hat{r}$ check divergence theorem for $\vec{V}$ using sphere centered at origin of radius R.

$$\int_{\mathcal{V}} (\nabla \cdot r^2 \hat{r})\,d\tau = \int_{\mathcal{V}} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2)\,d\tau = \int_{\mathcal{V}} 4r\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \int_0^R 4r^3 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \int_0^R 4r^3 \int_0^{\pi} \sin\theta\, d\theta = 8\pi \int_0^R r^3 (-\cos\theta)\Big|_0^{\pi} = 8\pi \int_0^R r^3 (\cos 0 - \cos\pi)$$

$$= 16\pi \int_0^R r^3 = 16\pi\left(\frac{R^4}{4}\right) = 4\pi R^4$$

r constant over θ, ϕ, $r = R^4$.

$$\oint_S \vec{V} \cdot d\vec{a} = \oint_S r^2 \hat{r} \cdot (r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}) = \oint_S r^4 \sin\theta\, d\theta\, d\phi = R^4 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= R^4 \int_0^{\pi} \sin\theta\, d\theta\, (2\pi) = R^4 (\cos 0 - \cos\pi)\, 2\pi = 4\pi R^4$$

$$\therefore \int_{\mathcal{V}} (\nabla \cdot \vec{V})\,d\tau = \oint_S \vec{V} \cdot d\vec{a} = 4\pi R^4.$$

(b) if $\vec{V} = \frac{1}{r^2}\hat{r}$ does $\int_{\mathcal{V}} (\nabla \cdot \vec{V})\,d\tau = \oint_S \vec{V} \cdot d\vec{a}$ hold ?

$$\int_{\mathcal{V}} \left(\nabla \cdot \frac{1}{r^2}\hat{r}\right) d\tau = \int_{\mathcal{V}} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) d\tau = \int_{\mathcal{V}} \frac{1}{r^2}\frac{\partial}{\partial r}(1)\,d\tau = \int_{\mathcal{V}} 0\, d\tau = 0.$$

$$\oint_S \frac{1}{r^2}\hat{r} \cdot (r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}) = \oint_S \sin\theta\, d\theta\, d\phi = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi \int_0^{\pi} \sin\theta\, d\theta = 4\pi.$$

10 $\vec{V}$ is undefined in the region of the sphere at the origin. In more detailed statements of Gauss' Theorem it is noted that $\vec{V}$ must have continuous partial derivatives over the volume in question. Clearly $\frac{\partial}{\partial r}\frac{1}{r^2} = \frac{1}{r}$ which does not behave at $r=0$.

$\boxed{1-39}$ $\vec{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$

$$\int_v (\nabla\cdot\vec{v})\,d\tau = \int_v\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\right]d\tau$$

$$= \int_v \left(3\cos\theta + \frac{2\sin\theta\cos\theta}{\sin\theta} - \sin\phi\right)d\tau$$

$$= \int_v (5\cos\theta - \sin\phi)\, r^2\,dr\,\sin\theta\,d\theta\,d\phi$$

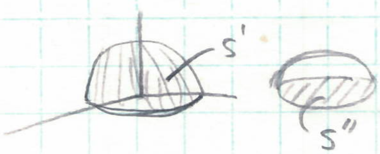
$$= \int_0^R 5r^2\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi - \int_0^R r^2\,dr\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{2\pi}\sin\phi\,d\phi$$

$$= 2\pi\int_0^R 5r^2\,dr\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right) - \int_0^R r^2\,dr\,(\cos 0 - \cos\tfrac{\pi}{2})\cancelto{0}{(\cos 0 - \cos 2\pi)}$$

$$= \pi\int_0^R 5r^2\,dr = \pi\frac{5}{3}R^3 = \frac{5\pi}{3}R^3$$

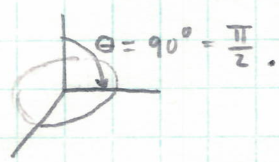
$$\int_{s'}\vec{v}\cdot d\vec{a}' = \int_{s'}\vec{v}\cdot(r^2\sin\theta\,d\theta\,d\phi)\hat{r} = \int_{s'} r\cos\theta\, r^2\sin\theta\,d\theta\,d\phi = \overset{\text{constant over surface.}}{R^3}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= R^3\, 2\pi\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}\right) = \pi R^3$$



$$\int_{s''}\vec{v}\cdot d\vec{a}'' = \int_{s''}(r\sin\theta)(r\,dr\,d\phi) = \int_0^R r^2\,dr\int_0^{2\pi}\overset{\text{constant in }\phi,\ \sin\theta = 1}{(\sin\theta)}\,d\phi = \frac{R^3}{3}(2\pi)$$

on disk at bottom



$$\oint_s \vec{v}\cdot d\vec{a} = \int_{s'}\vec{v}\cdot d\vec{a}' + \int_{s''}\vec{v}\cdot d\vec{a}'' \Rightarrow \boxed{\int_v(\nabla\cdot\vec{v})\,d\tau = \oint_s \vec{v}\cdot d\vec{a} = \frac{5\pi}{3}R^3}$$

where v = ½ sphere resting at origin on x-y plane.

10

$s'' + s' = s$.

ELECTRODYNAMICS, PY 414, DR. CHUNG — JAMES COOK

20/20

$\boxed{1-44}$

(a) $\int_{-2}^{2}(2x+3)\delta(3x)\,dx = \int_{-2}^{2}\frac{2x+3}{|3|}\delta(x)\,dx = \frac{3}{3} = 1.$ ✓

(b) $\int_{0}^{2}(x^3+3x+2)\delta(1-x)\,dx = (1^3+3+2) = 6$ ✓

(c) $\int_{-1}^{1} 9x^2\delta(3x+1)\,dx = \int_{-1}^{1}9x^2\delta(3(x+\frac{1}{3}))\,dx = \int_{-1}^{1}\frac{9x^2}{|3|}\delta(x+\frac{1}{3})\,dx = \frac{9(\frac{-1}{3})^2}{|3|} = \frac{1}{3}$ ✓

(d) $\int_{-\infty}^{a}\delta(x-b)\,dx = \begin{cases} 0 & \text{if } a<b \\ 1 & \text{if } a \geq b \end{cases}$ ✓ (argument of δ never zero) (argument of δ zero when evaluated at b.)

10

$\boxed{1-45}$

(a) if D_1 and D_2 involve delta functions and it can be shown $\int_{-\infty}^{\infty} f(x)\,D_1\,dx = \int_{-\infty}^{\infty} f(x)\,D_2\,dx$ then $D_1 = D_2$. note $f(x)=1$ for my problem. sol^{n}:

$$\int_{-\infty}^{\infty} x\frac{d}{dx}(\delta(x))\,dx = x\,\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(x)\,dx = 0 - 1$$

Let $u = x$ $v = \delta(x)$ which are used for integration
$du = dx$ $dv = \frac{d}{dx}(\delta(x))dx$ by parts above.

$$\int_{-\infty}^{\infty} -\delta(x)\,dx = -\int_{-\infty}^{\infty}\delta(x)\,dx = -1$$

$$\int_{-\infty}^{\infty}\left(x\frac{d}{dx}(\delta(x))\right)dx = \int_{-\infty}^{\infty}(-\delta(x))\,dx = -1 \quad \therefore \quad x\frac{d}{dx}(\delta(x)) = -\delta(x)$$ ✓

1-45

(b) $\Theta(x) \equiv \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$ $\frac{d\theta}{dx} = \delta(x) \iff d\theta = \delta(x)\,dx.$

choose $a \le b$

$$\int_a^b d\Theta = \Theta(b) - \Theta(a) = \begin{cases} a \le b \le 0 & \text{then } \Theta(b)-\Theta(a) = 0-0 = 0. \\ a \le 0 < b & \text{then } \Theta(b)-\Theta(a) = 1-0 = 1. \\ 0 < a \le b & \text{then } \Theta(b)-\Theta(a) = 1-1 = 0. \end{cases}$$

thus if zero is between a and b then $\int_a^b d\Theta = 1$
if not then the $\int_a^b d\Theta = 0$.

$$\int_a^b \delta(x)\,dx = \begin{cases} 1 & \text{if } 0 \in [a,b) \\ 0 & \text{if } 0 \notin [a,b) \end{cases}$$ because the $\delta(x)$ is zero for all $x \neq 0$.

zero is the only real number that allows a non-zero integral by def$^{\triangle}$

10

$$\therefore \int_a^b d\Theta = \int_a^b \delta(x)\,dx \iff d\Theta = \delta(x)\,dx \iff \frac{d\Theta}{dx} = \delta(x).$$

ELECTRODYNAMICS, PY 414, DR. CHUNG   JAMES COOK

18/20

$\boxed{1-46}$

(a) choose $\rho(\vec{r}) = q\delta(\vec{r}-\vec{r}') = q\delta(\vec{\imath})$

because $\delta(\vec{\imath}) = \delta(0)$ at the charge q

$\int_\tau q\,\delta(\vec{\imath})\,d\tau = q$ if τ contains $\vec{r}'$

so $\boxed{\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}')}$ ✓

(b) let $\rho(\vec{r}) = q\delta(\vec{r}-\vec{a}) - q\delta(\vec{r})$ note the arguments of the δ's are zero where the charge for the δ is.

$$\int_\tau q(\delta(\vec{r}-\vec{a}) - \delta(\vec{r}))\,d\tau = \begin{cases} q & \text{if just } \vec{a} \text{ in } \tau \\ -q & \text{if just the origin in } \tau \\ 0 & \text{if } \vec{0} \text{ and } \vec{a} \text{ in } \tau. \end{cases}$$

$$\boxed{\rho(\vec{r}) = q(\delta(\vec{r}-\vec{a}) - \delta(\vec{r}))}$$ ✓

(c) So we would like the $\rho(r)$ to behave such that,

$$\int_\tau \rho(\vec{r})\,d\tau = \begin{cases} 0 & \text{if } \vec{r} \text{ inside shell} \\ Q & \text{if } \vec{r} \text{ outside shell} \end{cases}$$

if τ is a shell of radius R (spherical)

$\rho(\vec{r}) = 0$ for $|\vec{r}| < R$
$\rho(\vec{r}) = $ "∞" for $|\vec{r}| = R$
$\rho(\vec{r}) = 0$ for $|\vec{r}| > R$

Let $\rho(\vec{r}) = \dfrac{Q\,\delta(|\vec{r}|-R)}{4\pi R^2}$, δ argument zero if τ contains $|\vec{r}|$ like vectors.

$$Q\int_\tau \frac{\delta(|\vec{r}|-R)}{4\pi R^2}\,d\tau = \begin{cases} 0 & \text{no singularity for } |\vec{r}| < R \\ Q & \text{if } \tau \text{ includes shell or the shell and more points outside the shell.} \end{cases}$$

So $\boxed{\rho(\vec{r}) = \dfrac{Q\,\delta(|\vec{r}|-R)}{4\pi R^2}}$ ✓

10

where $|\vec{r}|$ = length of $\vec{r}$, (from the origin).

$\boxed{1,48}$

$$\int_{\mathcal{V}} e^{-r}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)d\tau \qquad \mathcal{V} = \text{sphere centered at origin.}$$

FIRST METHOD:

$$\int_{\mathcal{V}} e^{-r}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)d\tau = \int_{\mathcal{V}} e^{-r}\left(4\pi\delta^3(\vec{r})\right)d\tau = e^0(4\pi) = \boxed{4\pi}.$$

SECOND METHOD:

$$\int_{\mathcal{V}} e^{-r}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)d\tau = \underbrace{-\int_{\mathcal{V}}\frac{\hat{r}}{r^2}\cdot\left(\nabla e^{-r}\right)d\tau}_{V} + \underbrace{\oint_S e^{-r}\left(\frac{\hat{r}}{r^2}\cdot d\vec{a}\right)}_{S}$$

all space

$$V = -\int_{\mathcal{V}}\frac{\hat{r}}{r^2}\cdot\left(\nabla e^{-r}\right)d\tau = -\int_{\mathcal{V}}\frac{1}{r^2}\left(-1e^{-r}\right)r^2\sin\theta\,dr\,d\theta\,d\phi = -\iint\sin\theta\,d\theta\,d\phi\int\frac{-e^{-r}r^2}{r^2}dr$$

$$= -4\pi\int_0^{R\,\infty} -e^{-r}dr = -4\pi\left[e^{-R} - e^0\right] = -4\pi\left(1 \cancel{- e^{-R}}\right).$$

$$S = \oint_S e^{-r}\left(\frac{\hat{r}}{r^2}\cdot r^2\sin\theta\,d\theta\,d\phi\,\hat{r}\right) = e^{-R}\oint_S \sin\theta\,d\theta\,d\phi = 4\pi e^{-R}. = 0$$

$r\to\infty$

8

THEN $$\int_{\mathcal{V}} e^{-r}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)d\tau = V + S = 4\pi \cancel{- 4\pi e^{-R}} \cancel{+ 4\pi e^{-R}} = \boxed{4\pi}.$$

1.49

(a) $\vec{F}_1 = x^2 k$ $\qquad \nabla = \nabla_1 i + \nabla_2 j + \nabla_3 k = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.

$\vec{F}_2 = xi + yj + zk$

20/20

$$\nabla \cdot \vec{F}_1 = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial x^2}{\partial z} = 0.$$ ✓

$$\nabla \cdot \vec{F}_2 = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$ ✓

$$\nabla \times \vec{F}_1 = \begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ 0 & 0 & x^2 \end{vmatrix} = i\cancel{\begin{vmatrix} \nabla_2 & \nabla_3 \\ 0 & x^2 \end{vmatrix}}^{\,0} - j\begin{vmatrix} \nabla_1 & \nabla_3 \\ 0 & x^2 \end{vmatrix} + k\cancel{\begin{vmatrix} \nabla_1 & \nabla_2 \\ 0 & 0 \end{vmatrix}}^{\,0} = -j\frac{\partial(x^2)}{\partial x} = -2xj.$$ ✓

$$\nabla \times \vec{F}_2 = \begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ x & y & z \end{vmatrix} = i\cancel{\begin{vmatrix} \nabla_2 & \nabla_3 \\ y & z \end{vmatrix}}^{\,0} - j\cancel{\begin{vmatrix} \nabla_1 & \nabla_3 \\ x & z \end{vmatrix}}^{\,0} + k\cancel{\begin{vmatrix} \nabla_1 & \nabla_2 \\ x & y \end{vmatrix}}^{\,0} = 0.$$ ✓

$\boxed{\vec{F}_2 \text{ may be written as the gradient of a scalar.}}$

$$\vec{F}_2 = -\nabla T = -\frac{\partial T}{\partial x} i + \left(-\frac{\partial T}{\partial y} j\right) + \left(-\frac{\partial T}{\partial z} k\right) = -xi - yj - zk.$$

choose $\boxed{T = -\left(\frac{x^2+y^2+z^2}{2}\right)}$. ✓ $\quad \frac{\partial T}{\partial x} = \frac{-2x}{2}, \quad \frac{\partial T}{\partial y} = \frac{-2y}{2}, \quad \frac{\partial T}{\partial z} = \frac{-2z}{2}.$

$\boxed{\vec{F}_1 \text{ may be written as the curl of a vector}}$

$$\vec{F}_1 = \nabla \times \vec{A} = i\begin{vmatrix} \nabla_2 & \nabla_3 \\ A_2 & A_3 \end{vmatrix} - j\begin{vmatrix} \nabla_1 & \nabla_3 \\ A_1 & A_3 \end{vmatrix} + k\begin{vmatrix} \nabla_1 & \nabla_2 \\ A_1 & A_2 \end{vmatrix}$$

$$= i(\nabla_2 A_3 - \nabla_3 A_2) - j(\nabla_1 A_3 - \nabla_3 A_1) + k(\nabla_1 A_2 - \nabla_2 A_1)$$

So,

$\nabla_2 A_3 - \nabla_3 A_2 = 0. \quad \nabla_1 A_3 - \nabla_3 A_1 = 0. \quad \nabla_1 A_2 - \nabla_2 A_1 = x^2$

OR,

$\frac{\partial}{\partial x}(A_2) - \frac{\partial}{\partial y}(A_1) = x^2 \quad$ Let $A_2 = x^3/3$, $A_1 = A_3 = 0$.

Then $i\cancel{(\nabla_2 A_3 - \nabla_3 A_2)}^{\,0} - j\cancel{(\nabla_1 A_3 - \nabla_3 A_1)}^{\,0} + k\left(\frac{\partial}{\partial x}\left(\frac{x^3}{3}\right) - \cancel{\nabla_2 A_1}^{\,0}\right) = \vec{F}_1$.

THUS I choose $\boxed{\vec{A} = \frac{x^3}{3} j}$ ✓

$\boxed{1.49}$

(b) $\vec{F}_3 = yz\,i + zx\,k + xy\,k$

$$\nabla \times \vec{B} = \begin{vmatrix} i & j & k \\ \nabla_1 & \nabla_2 & \nabla_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = i \begin{vmatrix} \nabla_2 & \nabla_3 \\ B_2 & B_3 \end{vmatrix} - j \begin{vmatrix} \nabla_1 & \nabla_3 \\ B_1 & B_3 \end{vmatrix} + k \begin{vmatrix} \nabla_1 & \nabla_2 \\ B_1 & B_2 \end{vmatrix}$$

Then we must insure $\vec{F}_3 = \nabla \times \vec{B}$ EQUATING COMPONENTS,

$$\nabla_2 B_3 - \nabla_3 B_2 = yz \implies B_3 = \frac{y^2 z}{2}$$
$$\nabla_1 B_3 - \nabla_3 B_1 = -zx \implies B_1 = \frac{z^2 x}{2}$$
$$\nabla_1 B_2 - \nabla_2 B_1 = xy \implies B_2 = \frac{x^2 y}{2}$$

The vector potential is $\boxed{\vec{B} = \frac{z^2 x}{2} i + \frac{x^2 y}{2} j + \frac{y^2 z}{2} k.}$ ✓

$\vec{F}_3 = -\nabla T = -\nabla_1 T i - \nabla_2 T j - \nabla_3 T k$

WE MUST FUFILL

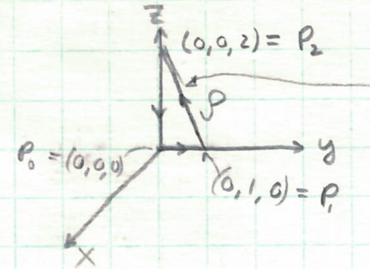
$$\left.\begin{aligned} -\nabla_1 T &= yz \\ -\nabla_2 T &= zx \\ -\nabla_3 T &= xy \end{aligned}\right\} \implies T = -xyz.$$

the scalar potential $\boxed{T = -xyz.}$ ✓

10

$\boxed{1.55}$

$\vec{V} = 6i + (yz^2)j + (3y+z)k$



$z = 2 - 2y$

$y = 1 - \frac{z}{2}$

$dy = -\frac{dz}{2}$

$$\int_S \vec{V}\cdot d\vec{l} = \int_{P_0}^{P_1} \vec{V}\cdot dy\,j + \int_{P_1}^{P_2} \vec{V}\cdot(dy\,j + dz\,k) + \int_{P_2}^{P_0} \vec{V}\cdot dz\,k$$

$$= \int_{P_0}^{P_1} yz^2 dy + \int_{P_1}^{P_2} yz^2 dy + (3y+z)dz + \int_{P_2}^{P_0} (3y+z)dz$$

$$= \int_{P_0}^{P_1} \emptyset\, dy + \int_{P_1}^{P_2} \left[\left(1-\frac{z}{2}\right)z^2\right]\left(-\frac{dz}{2}\right) + \left[3\left(1 - z/2\right) + z\right]dz + \int_{P_2}^{P_0} z\,dz$$

$$= \int_{(0,1,0)}^{(0,0,2)} \left(z^2 - \frac{z^3}{2}\right)\left(\frac{-1}{2}\right)dz + \left[3 - \frac{3}{2}z + z\right]dz + \frac{z^2}{2}\Big|_2^0$$

$$= \int_{(0,1,0)}^{(0,0,2)} \left(\frac{z^3}{4} - \frac{z^2}{2} - \frac{z}{2} + 3\right)dz - 2 = \frac{z^4}{16} - \frac{z^3}{6} - \frac{z^2}{4} + 3z\Big|_0^2 - 2$$

$$= 1 - 8/6 - 1 + 6 - 2 = 4 - 4/3 = 12/3 - 4/3 = \boxed{8/3}$$

$$\nabla\times\vec{V} = \begin{vmatrix} i & j & k \\ \nabla_x & \nabla_y & \nabla_z \\ 6 & yz^2 & 3y-z \end{vmatrix} = i(3 - 2yz) - j\cancelto{\emptyset}{\left(\nabla_x(3y-z) - \nabla_z(6)\right)} + k\cancelto{\emptyset}{\left(\nabla_x(yz^2) - \nabla_y(6)\right)}$$

$$\oint_S (\nabla\times\vec{V})\cdot(dydz\,i) = \oint_S (3 - 2yz)dydz = 3\oint_S dydz - 2\int_0^1 y\left(\int_0^{2-2y} z\,dz\right)dy$$

$$= 3(\text{AREA OF S}) - 2\int_0^1 \frac{y(2-2y)^2}{2}dy = 3\left(\tfrac{1}{2}(1\cdot 2)\right) - 2\int_0^1 4y - 8y^2 + 4y^3\,dy$$

$$= 3 - 2\left[\frac{4y^2}{2} - \frac{8y^3}{3} + \frac{4y^4}{4}\right|_0^1 = 3 - 2\left(2 - \frac{8}{3} + 1\right) = 3 - \frac{4}{3} = \frac{12}{3} - \frac{4}{3} = \boxed{\frac{8}{3}}$$

10