James Cook

20/20

3-3

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$$ as ϕ, θ are constant in $V(r)$.

$$= 2rV' + r^2 V''$$ where $V' = \frac{\partial V}{\partial r}$, $V'' = \frac{\partial^2 V}{\partial r^2}$

$$= 2V' + rV''$$

Guess $V = r^m$ and use method of Cauchy-Euler to solve DE.

$V' = m r^m$

$V'' = m(m-1) r^{m-1}$

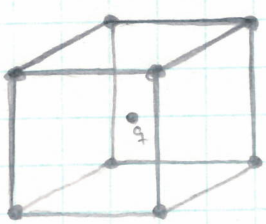
$$2V' + rV'' = 2mr^m + r(m(m-1))r^{m-1}$$
$$= r^m(2m + m^2 - m)$$ as $r^m \neq 0$ (just trivial sol^{n})

$$\nabla^2 V = 0 \implies m^2 + m = 0$$
$$m(m+1) = 0 \implies m = 0, -1$$

So our sol^{n} is

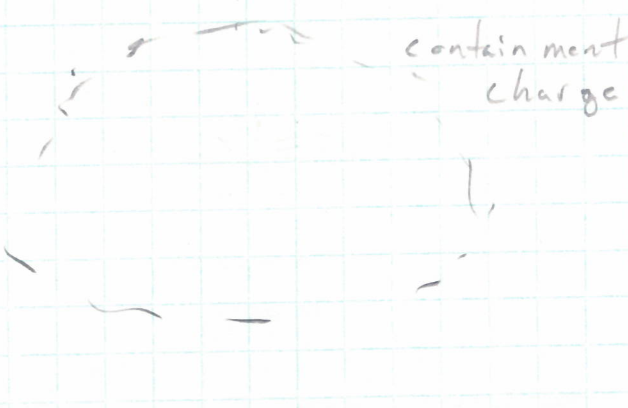
$$V(r) = c_1 r^0 + c_2 r^{-1} = \boxed{c_1 + c_2 \frac{1}{r} = V(r)}$$

3-2



Where is leak in electrostatic bottle

A CHARGED PARTICLE CANNOT BE HELD IN A STABLE EQUILIBRIUM (Potential well). As LAPLACES EQUATION DEMANDS THAT The potential has no max or min in a region void of the containment charge.



3-3

$\nabla^2(V(s))$ where $V(s)$ is some function in cylindrical coordinates with only s dependence

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 z}{\partial z^2}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)$$ as there is just s dependence ϕ, z constant.

$$= \frac{1}{s}\left(\frac{\partial V}{\partial s} + s\frac{\partial^2 V}{\partial s^2}\right)$$

$$\Rightarrow \frac{\partial V}{\partial s} + s\frac{\partial^2 V}{\partial s^2} = 0$$ LAPLACES EQUATION.

GUESS $V = s^m$ as this DE is solveable by the method of Cauchy-Euler.

$V' = m s^{m-1}$ $\quad V'' = (m-1)m s^{m-2}$

$\Rightarrow m s^{m-1} + s(m(m-1))s^{m-2} = 0$

$s^{m-1}(m + m^2 - m) = 0$ as $s^{m-1} = 0$ is just the trivial sol^{n}, $m^2 = 0 \Rightarrow m = \pm 0$. So we get the sol^{n}

$V(s) = c_1 s^0 + c_2 \ln s = \boxed{C_1 + C_2 \ln(s) = V(s)}$ $\checkmark$

and I know this is a valid sol^{n} because it checks for $\nabla^2 V = 0$.

$$V' + sV'' = \frac{C_2}{s} + s\left(C_2\left(\frac{-1}{s^2}\right)\right) = \frac{C_2}{s} - \frac{C_2}{s} = 0.$$

PY 414, DR. CHUNG, ELECTRODYNAMICS. JAMES COOK

3.4 PROVE THAT THE FIELD IS UNIQUELY DETERMINED WHEN THE CHARGE DENSITY ρ IS GIVEN AND EITHER V OR $\partial V/\partial n$ IS SPECIFIED ON EACH BOUNDARY SURFACE.

20/20

ρ IS KNOWN FOR $\mathbb{R}^3$, EXCEPT FOR SPECIAL BUBBLES WHERE WE KNOW EITHER THE BOUNDARY VOLTAGE OR THE $\frac{\partial V}{\partial n}$.

Pf/

SUPPOSE $\vec{E}_1$ AND $\vec{E}_2$ SATISFY THE CONDITIONS.
THEN $\nabla \cdot \vec{E}_1 = \frac{\rho}{\epsilon_0}$ AND $\nabla \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0}$

- NOTE THAT $\vec{E}_1 = \vec{E}_2$ FOR THE CASE WHERE V IS GIVEN ON THE BOUNDARYS AS $\vec{E}_1 = \nabla V_1$ and $\vec{E}_2 = \nabla V_2$ BUT BY THE Collary on pg. 118 we know $V_1 = V_2 \Rightarrow \vec{E}_1 = \vec{E}_2$
- Now consider we are just given $\frac{\partial V}{\partial n}$ @ the boundarys.

Again, $\nabla \cdot E_1 = \frac{\rho}{\epsilon_0}$, $\nabla \cdot E_2 = \frac{\rho}{\epsilon_0}$

$E_3 \equiv E_2 - E_1$ $\quad \nabla \cdot (E_3) = \nabla \cdot (E_2 - E_1) = (\nabla \cdot E_1) - (\nabla \cdot E_2) = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0.$

$$E_3 = E_2 - E_1 \qquad \nabla \cdot E_3 = \nabla(E_2 - E_1) = 0$$

$$\int (\nabla \cdot E_3)\, d\tau = \oint_s E_3 \cdot d\vec{a}$$

$$\int_v 0\, d\tau = \oint_s -\nabla V_3 \cdot da\,\hat{n} = \oint_s -\frac{\partial V_3}{\partial n}\, da$$

$$= \oint_s -\frac{\partial}{\partial n}(V_2 - V_1)\, da$$

$$\Rightarrow \oint_s -\frac{\partial V_2}{\partial n}\, da = \oint_s -\frac{\partial V_1}{\partial n}\, da$$ which holds if $\frac{\partial V}{\partial n}$ is given

$$\oint_s \vec{E}_1 \cdot d\vec{a} = \oint_s -\nabla V_1 \cdot da\,\hat{n} = \oint -\frac{\partial V_1}{\partial n}\, da = \oint \frac{-\partial V_2}{\partial n}\, da$$

$$\oint_s E_2 \cdot d\vec{a} = \oint_s -\nabla V_2 \cdot da\,\hat{n} = \oint_s \frac{-\partial V_2}{\partial n}\, da = \oint_s \frac{-\partial V_1}{\partial n}\, da$$

$$\Rightarrow \oint_s \vec{E}_1 \cdot d\vec{a} = \oint \vec{E}_2 \cdot d\vec{a}$$ AS LONG AS $\frac{\partial V}{\partial n}$ IS GIVEN so that we may CONNECT $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$.

As "S" was an arbitrary surface we may claim $\vec{E}_1 = \vec{E}_2$ and so the field is unique.

3.5

prob. 1.60 c) $\int_{\mathcal{V}} [T\nabla^2 u + (\nabla T)\cdot(\nabla u)]\,d\tau = \oint_S (T\nabla u)\cdot d\vec{a}$ Let $E_3 \equiv E_2 - E_1$

$T = U = V_3$ $\nabla^2 V_1 = -\rho/\epsilon_0$ $\nabla^2 V_2 = -\rho/\epsilon_0$ $\Rightarrow \nabla^2 V_3 = 0.$

$$\int_{\mathcal{V}} V_3 \nabla^2 V_3 + (\nabla V_3)\cdot(\nabla V_3)\,d\tau = \oint_S V_3 \nabla V_3 \cdot d\vec{a}$$ ✓

$$\int_{\mathcal{V}} V_3(\emptyset) + |\nabla V_3|^2\,d\tau = \oint_S V_3 \nabla V_3 \cdot d\vec{a}$$

$$\int_{\mathcal{V}} |\nabla V_3|^2\,d\tau = \int_{\mathcal{V}} |-\vec{E}_3|^2\,d\tau = \oint_S -V_3\vec{E}_3\cdot d\vec{a}$$

as V_3 is constant over S

$$-V_3\oint_S \vec{E}_3\cdot d\vec{a} = -V_3\oint(\vec{E}_2 - \vec{E}_1)\cdot d\vec{a} = -V_3\oint \vec{E}_2\cdot d\vec{a} + V_3\int \vec{E}_1\cdot d\vec{a}$$

$$= -V_3\left(\oint \vec{E}_2\cdot d\vec{a} - \oint \vec{E}_1\cdot d\vec{a}\right)$$

$$= -V_3\left(\frac{Q_{enc}}{\epsilon_0} - \frac{Q_{enc}}{\epsilon_0}\right)$$ by Gauss Law

$$= 0.$$

10

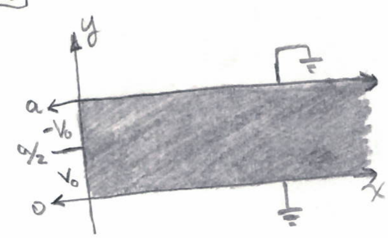
Then $\int_{\mathcal{V}} |-E_3|^2\,d\tau = \oint_S -V_3\vec{E}_3\cdot d\vec{a} = 0$ ✓.

$\Rightarrow |\vec{E}_3| = 0 \Rightarrow \vec{E}_2 - \vec{E}_1 = 0 \Rightarrow \vec{E}_1 = \vec{E}_2$ ✓

∴ the field is uniquely determined

PY 414, DR. CHANG JAMES COOK

3.12



Suppose $V(x,y) = w(x)\,z(y)$ then we know outside shaded region $\nabla^2 V = 0$.

18|20

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}(wz) + \frac{\partial^2}{\partial y^2}(wz) = z\frac{\partial^2 w}{\partial x^2} + w\frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{1}{w}\frac{\partial^2 w}{\partial x^2} = -\frac{1}{z}\frac{\partial^2 z}{\partial y^2} \Rightarrow \frac{1}{w}\frac{\partial^2 w}{\partial x^2} = k^2 \text{ and } \frac{1}{z}\frac{\partial^2 z}{\partial y^2} = -k^2$$

So $k \in \mathbb{R}$ and $\nabla^2 V = k^2 - k^2 = 0$.

$$\frac{\partial^2 w}{\partial x^2} = k^2 w \Rightarrow w = Ae^{kx} + Be^{-kx}$$

$$\frac{\partial^2 z}{\partial y^2} = -k^2 z \Rightarrow z = C\cos(ky) + D\sin(ky)$$

$V(x,y) =$ finite $\Rightarrow A = 0$ so we may rewrite $V(x,y)$ with new arbitrary constants c_3, c_2.

$$V(x,y) = e^{-kx}\left(c_2\cos(ky) + c_3\sin(ky)\right)$$

$$V(x,0) = e^{-kx}(c_2) = 0 \Rightarrow c_2 = 0,$$

$$V(x,a) = e^{-kx}(c_3\sin(ay)) = 0 \Rightarrow ka = n\pi, \quad n = 1,2,3,\dots$$

$$\Rightarrow k = \frac{n\pi}{a}$$

Consider that $V(x,y) = \sum_n c_n \sin\left(\frac{n\pi}{a}y\right)e^{-\frac{n\pi x}{a}}$, now choose C_n according to verticle bounds.

$$V(0,y) = \begin{Bmatrix} 0 \text{ if } y>a \text{ or } y<0 \\ -V_0 \text{ if } a/2 < y < a \\ V_0 \text{ if } a/2 > y > 0 \end{Bmatrix} = V_1(y) = \sum_n c_n \sin\left(\frac{n\pi}{a}y\right)$$

$$\int_0^a \left(\sin\frac{m\pi}{a}y\right)\left(\sum_n c_n \sin\frac{n\pi}{a}y\right)dy = \int_0^a \left(V_1(y)\sin\frac{m\pi}{a}y\right)dy$$

$\Downarrow m=n$

$$\int_0^a \left(\sin\frac{m\pi y}{a}\right)\left(c_n\sin\frac{m\pi y}{a}\right)dy = c_n\int_0^a \sin^2\left(\frac{m\pi y}{a}\right) = \frac{c_n}{2}(a) = \int_0^a \left(V_1(y)\sin\frac{m\pi}{a}y\right)dy$$

9

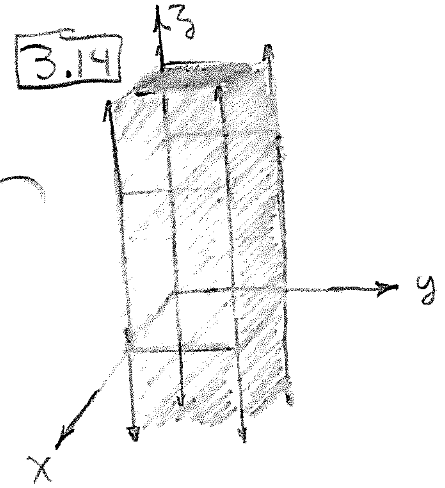
$$C_m = \frac{2}{a}\int_{a/2}^{a} -V_0\sin\frac{m\pi y}{a}\,dy + \frac{2}{a}\int_0^{a/2} V_0\sin\frac{m\pi y}{a}\,dy = \frac{2V_0}{a}\left[\frac{a}{m\pi}\left[\cos\frac{m\pi y}{a}\Big|_{a/2}^{a} - \cos\frac{m\pi y}{a}\Big|_0^{a/2}\right]\right]$$

$$= \frac{2V_0}{m\pi}\left[\cos(m\pi) - \cos\left(\frac{m\pi}{2}\right) - \cos\left(\frac{m\pi}{2}\right) + \cos(\emptyset)\right] = \frac{2V_0}{m\pi}\left[\cos(m\pi) - 2\cos\left(\frac{m\pi}{2}\right) + 1\right]$$

Simplify! n odd $\Rightarrow$ $n = 4m \Rightarrow 0$ $n = 4m+2$ $C_n = \frac{8V_0}{n\pi}$

Finally $\boxed{V(x,y) = \sum_n c_n \sin\left(\frac{n\pi y}{a}\right)e^{-\frac{n\pi x}{a}} \text{ where } C_n = \frac{2V_0}{n\pi}\left(1 + \cos(n\pi) - 2\cos\left(\frac{n\pi}{2}\right)\right)}$

3.14



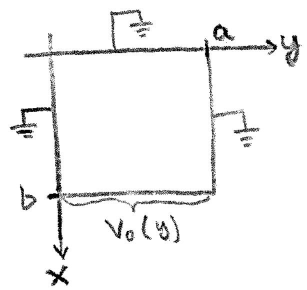
GEOMETRY DEMANDS No z dependence. So I suppose $V(x,y) = P(x)\,Q(y)$. Also denote

$$\frac{\partial^2 P}{\partial x^2} = P'' \quad \text{and} \quad \frac{\partial^2 Q}{\partial y^2} = Q''$$

$$\nabla^2 V = 0 = QP'' + PQ'' \Rightarrow -\frac{Q''}{Q} = \frac{P''}{P} = k^2$$

then $\frac{Q''}{Q} = -k^2 \Rightarrow Q(y) = f\cos ky + b\sin ky$

$\frac{P''}{P} = k^2 \Rightarrow P(x) = c\,e^{kx} + d\,e^{-kx}$



$$V(x,y) = (f\cos ky + b\sin ky)(ce^{kx} + de^{-kx})$$

$$V(0,y) = (f\cos ky + b\sin ky)(c+d) = 0 \Rightarrow c+d=0 \Rightarrow d=-c.$$

$$V(x,0) = (f\cos 0)(c(e^{kx} - e^{-kx})) = 0, \Rightarrow f = 0.$$

$$V(x,a) = (b\sin ka)(c(e^{kx} - e^{-kx})) = 0$$

$$\Rightarrow b\sin ka = 0 \Rightarrow ka = n\pi \;;\; k = \frac{n\pi}{a}$$

$$V(x,y) = \sum_n C_n \sin\left(\frac{n\pi y}{a}\right)\sinh kx$$

$$V(b,y) = \sum_n C_n \sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi b}{a}\right) = V_0(y)$$

$$\int_0^a \left(\sin\frac{m\pi y}{a}\right)\sum_n C_n \sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi b}{a}\right)dy = \int_0^a \sin\left(\frac{m\pi y}{a}\right)V_0(y)\,dy$$

$$\int_0^a C_m \sin^2\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{n\pi b}{a}\right)dy = \sinh\left(\frac{n\pi b}{a}\right)C_m\int_0^a \sin^2\left(\frac{m\pi y}{a}\right)dy$$

$$C_m = \frac{\int_0^a \sin\left(\frac{m\pi y}{a}\right)V_0(y)\,dy}{\sinh\left(\frac{n\pi b}{a}\right)\int_0^a \sin^2\left(\frac{m\pi y}{a}\right)} = \frac{\int_0^a V_0(y)\sin\left(\frac{m\pi y}{a}\right)dy}{\sinh\left(\frac{n\pi b}{a}\right)\left(\frac{a}{2}\right)}$$

$$\boxed{C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy \quad \text{in the below expression} \qquad V(x,y) = \sum C_n \sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)}$$

3.14

(b) $V_0(y) = V_0$ a plain old constant.

$$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{a\, 2V_0}{n\pi\, a\sinh\left(\frac{n\pi b}{a}\right)} \left\{ -\cos n\pi + \cos 0 \right\}$$

$$\boxed{\begin{aligned} C_n &= \frac{2V_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} (1 - \cos n\pi) \\ V(x,y) &= \sum_n C_n \sin\left(\frac{n\pi y}{a}\right) \sinh\left(\frac{n\pi x}{a}\right) \end{aligned}}$$

9

Simplify!

$$\frac{V_0 a}{n\pi}\left(1 - (-1)^n\right) = \begin{cases} 0 & \text{even} \\ \frac{2V_0 a}{n\pi} & \text{odd} \end{cases}$$

PY 414, DR. CHUNG. JAMES COOK

$\boxed{3.16}$ Derive $P_3(x)$ from the Rodrigues formula, and check that $P_3(\cos\theta)$ satisfies the angular equation (3.60) for $\ell = 3$. Check that P_3 and P_1 are orthogonal by explicit integration.

20/20

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \quad : \text{Rodrigues Formula}$$

$$P_3(x) = \frac{1}{2^3 3!}\left(\frac{d}{dx}\right)^3 (x^2-1)^3 \qquad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$= \frac{1}{48}\frac{d^3}{dx^3}\left(\binom{3}{0}(x^2)^0(-1)^3 + \binom{3}{1}x^2(-1)^2 + \binom{3}{2}x^4(-1)^1 + \binom{3}{3}x^6(-1)^0\right)$$

$$= \frac{1}{48}\frac{d^3}{dx^3}\left(-1 + 3x^2 - 3x^4 + x^6\right)$$

$$= \frac{1}{48}\frac{d^2}{dx^2}\left(6x - 12x^3 + 6x^5\right)$$

$$= \frac{1}{48}\frac{d}{dx}\left(6 - 36x^2 + 30x^4\right)$$

$$= \frac{1}{48}\left(-72x + 120x^3\right)$$

$$\boxed{P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)}$$

(I change $\Theta \to \gamma$ as I detest Θ)

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\gamma}{d\theta}\right) = -\ell(\ell+1)\sin\theta\,\gamma \qquad \text{we } \not{s}\ \gamma(\theta) = P_3(\cos\theta), \text{ cheking...}$$

$$\frac{d}{d\theta}\left\{\sin\theta \frac{d}{d\theta}\left(\frac{1}{2}\left(5(\cos\theta)^3 - 3(\cos\theta)\right)\right)\right\} = \frac{d}{d\theta}\left\{\sin\theta\left(\frac{15}{2}(\cos\theta)^2(-\sin\theta) + \frac{3}{2}\sin\theta\right)\right\} =$$

$$\frac{d}{d\theta}\left\{\frac{3}{2}\sin^2\theta - \frac{15}{2}\sin^2\theta\cos^2\theta\right\} = \frac{d}{d\theta}\left\{\sin^2\theta\left(\frac{3}{2} - \frac{15}{2}(1-\sin^2\theta)\right)\right\} =$$

$$\frac{d}{d\theta}\left\{\sin^2\theta\left(-\frac{12}{2} + \frac{15}{2}\sin^2\theta\right)\right\} = \frac{d}{d\theta}\left\{-\frac{12}{2}\sin^2\theta + \frac{15}{2}(\sin^2\theta)^2\right\} =$$

$$-\frac{12}{2}(2\sin\theta)(\cos\theta) + \frac{15}{2}(2\sin^2\theta)(2\sin\theta)(\cos\theta) = -\frac{12}{2}\sin 2\theta + \frac{15}{2}(2\sin^2\theta)\sin 2\theta =$$

$$-6\sin 2\theta + 15\sin^2\theta\sin 2\theta = (*)$$

$\boxed{3.16}$

now compute $-\ell(\ell+1)\sin\theta\, P_3(\cos\theta) = -3(3+1)\sin\theta\, \frac{1}{2}\left(5(\cos^3\theta) - 3\cos\theta\right)$

$-6\sin\theta\left(5\cos^3\theta - 3\cos\theta\right) = -6\sin\theta\cos\theta\left(5\cos^2\theta - 3\right) =$

$-6\sin\theta\cos\theta\left(5 - 5\sin^2\theta - 3\right) = -3\sin 2\theta\left(2 - 5\sin^2\theta\right) =$

$-6\sin 2\theta + 15(\sin^2\theta)(\sin 2\theta) = (**) = (*)$

note that

as $\frac{d}{d\theta}\left(\sin\theta \frac{d\gamma}{d\theta}\right) = (*)$ and $-\ell(\ell+1)\sin\theta\, P_3(\cos\theta) = (**)$

and $(*) = (**) \Rightarrow \frac{d}{d\theta}\left(\sin\theta \frac{d\gamma}{d\theta}\right) = -\ell(\ell+1)\sin\theta\, P_3(\cos\theta).$

So Eq. 3.60 is happy.

3.16 Check P_3 and P_1 are orthogonal.

$P_1 = x$

$P_3 = \frac{1}{2}(5x^3 - 3x)$

$$\int P_1 P_3 \, dx = \int \frac{x}{2}(5x^3 - 3x)\,dx = \int\left(\frac{5}{2}x^4 - \frac{3}{2}x^2\right)dx$$

$$= \frac{x^5}{2} - \frac{x^3}{2}\Big|_{-1}^{1} = \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

$P_1 = \cos\Theta$

$P_3 = \frac{1}{2}\left(5(\cos\Theta)^3 - 3(\cos\Theta)\right)$

$$\int P_1 P_3 \, dx = \int_0^{\pi} \cos\Theta\left(\frac{1}{2}\left(5(\cos\Theta)^3 - 3\cos\Theta\right)\right)\sin\Theta \, d\Theta$$

$$= \int \sin\Theta\cos\Theta\left(\frac{5}{2}\cos^3\Theta - \frac{3}{2}\cos\Theta\right)d\Theta$$

$$= \int\left(\frac{5}{2}\sin\Theta\cos^4\Theta - \frac{3}{2}\sin\Theta\cos^2\Theta\right)d\Theta$$

Let $u = \cos\Theta$

$du = -\sin\Theta\, d\Theta$

$d\Theta = \frac{du}{-\sin\Theta}$

$$= \int\left(-\frac{5}{2}u^4 + \frac{3}{2}u^2\right)du$$

18

$$= -\frac{u^5}{2} + \frac{u^3}{2} = \frac{-\cos^5\Theta}{2} + \frac{\cos^3\Theta}{2}\Big|_0^{\pi} = \frac{-(\cos\pi)^5}{2} + \frac{(\cos\pi)^3}{2} + \frac{1}{2} - \frac{1}{2}$$

$$= \frac{+1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 0.$$

and as $1 \neq 3$ we should expect that $\int P_1 P_3 \, dx = 0$ if they are indeed orthogonal, and they are. ✓

$\boxed{3.18}$

$V_0 = k\cos(3\theta)$: Find potential inside and outside sphere of radius R and find $\sigma(\theta)$. Assume no charge inside or out of sphere. So $\nabla^2 V = 0$ is valid for $\mathbb{R}^3$ except for the spherical shell.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(\left(A_l r^l + \left(\frac{B_l}{r^{l+1}}\right)\right)P_l(\cos\theta)\right)$$ from azimuthal symmetry this is the general sol$^{\underline{n}}$

$V(0,\theta) = \text{finite} \Rightarrow B_l = 0$ for all l.

$$V_0 = k\cos(3\theta) = V(R,\theta)$$
$$= k\cos((\theta+\theta)+\theta)$$
$$= k\left(\cos(\theta+\theta)\cos(\theta) - \sin(\theta+\theta)\sin(\theta)\right)$$
$$= k\left(\cos\theta(\cos\theta\cos\theta - \sin\theta\sin\theta) - \sin\theta(\sin\theta\cos\theta + \sin\theta\cos\theta)\right)$$
$$= k\left(\cos\theta(\cos^2\theta - \sin^2\theta) - 2\sin^2\theta\cos\theta\right)$$
$$= k\left(\cos\theta(\cos^2\theta - (1-\cos^2\theta)) - 2\cos\theta(1-\cos^2\theta)\right)$$

Let $x = \cos\theta$

$$= k\left(x(2x^2-1) - 2x(1-x^2)\right)$$
$$= k\left(2x^3 - x - 2x + 2x^3\right)$$
$$= k\left(4x^3 - 3x\right)$$
$$= \frac{k}{5}\left(20x^3 - 15x\right) = \frac{k}{5}\left((20x^3 - 12x) - (3x)\right)$$

remember that $\left\{ \begin{array}{l} P_1(x) = x \\ P_3(x) = \frac{1}{2}(5x^3 - 3x) \end{array} \right\}$

So $\frac{k}{5}\left(\frac{8}{2}(5x^3 - 3x) - 3(x)\right) = \frac{k}{5}\left(8P_3 - 3P_1\right) = \frac{8k}{5}P_3 - \frac{3k}{5}P_1$

So we can construct $k\cos 3\theta$ using P_1 and P_3

$$V(R,\theta) = k\cos 3\theta = \frac{8k}{5}P_3 - \frac{3k}{5}P_1 = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

then $\frac{8k}{5}P_3 - \frac{3k}{5}P_1 = A_0R^0P_0 + A_1R^1P_1 + A_2R^2P_2 + A_3R^3P_3 + \ldots$

comparing coefficients we may deduce

$$A_3 = \frac{8k}{5R^3} \quad \text{and} \quad A_1 = \frac{-3k}{5R}$$

3.18

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l P_l(\cos\theta) \right)$$ for $r < R$ and

$A_1 = \frac{-3K}{5R}$ and $A_3 = \frac{8K}{5R^3}$ but all other $A_i = 0$.

So

$$V_{inside}(r,\theta) = -\frac{3K}{5R} r P_1 + \frac{8K}{5R^3} r^3 P_3$$

$$\boxed{\text{if } r<R;\ V(r,\theta) = \frac{-3K}{5R} r(\cos\theta) + \frac{8K}{5R^3} r^3 \left(\frac{1}{2}\left[5\cos^3\theta - 3\cos\theta \right] \right)}$$

$$= \left(\frac{-K}{5R} r\cos\theta \right)\left(3 + \frac{4r^2}{R^2}\left[5\cos^2\theta - 3 \right] \right)$$

$$= \left(\frac{-K}{5R} r\cos\theta \right)\left(\frac{3R^2 + 20r^2\cos^2\theta - 3\frac{4r^2}{R^2}}{R^2} \right)$$

$$= \left(\frac{-K}{5R} r\cos\theta \right)\left(\frac{3(R^2 - 4r^2/R^2) + 20r^2\cos^2\theta}{R^2} \right)$$

⋮

I chose boxed version as its simple to take $\frac{\partial}{\partial r}$ for $\vec{E}$.

3-18

now fit Legendres polynomials to $V_{outside}$.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l / r^{l+1}\right) P_l(\cos\theta)$$

Now we know $A_l = 0$ for all l as r^l would blow up the potential function as $r \to \infty$.

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = k\cos(3\theta)$$

from previous work,

$$k\cos 3\theta = \frac{k}{5}\left(8\left(\tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)\right) - 3(\cos\theta)\right) = \frac{8k}{5}P_3 - \frac{3k}{5}P_1$$

So,

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l = \frac{B_0}{R^1}P_0 + \frac{B_1}{R^2}P_1 + \frac{B_2}{R^3}P_2 + \frac{B_3}{R^4}P_3 + \dots = \frac{8k}{5}P_3 - \frac{3k}{5}P_1$$

$$\Rightarrow B_l = \begin{cases} 0 & \text{if } l \neq 1,3 \\ \frac{8k}{5}R^4 & \text{if } l = 3 \\ -\frac{3k}{5}R^2 & \text{if } l = 1 \end{cases}$$

Then

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l = \boxed{-\frac{3kR^2}{5r^2}\cos\theta + \frac{4kR^4}{5r^4}(5\cos^3\theta - 3\cos\theta) \quad r > R}$$

for $r > R$

$\boxed{3-18}$

$\sigma(\theta) = ?$

$$E_{\perp}^{abo} - E_{\perp}^{bel} = \frac{\sigma}{\epsilon_0}\hat{r}$$

$$E_{\perp}^{abo} = -\frac{\partial}{\partial r}\left(V_{out}\right|_{r=R} = \frac{-\partial}{\partial r}\left(\frac{-3kR^2}{5r^2}\cos\theta + \frac{4kR^4}{5r^4}(5\cos^3\theta - 3\cos\theta)\right|_{r=R}$$

$$= -\left(\frac{6kR^2}{5r^3}\cos\theta - \frac{16kR^4}{5r^5}(\cos^3\theta - 3\cos\theta)\right|_{r=R}$$

$$= \frac{-6k}{5R}(\cos\theta) + \frac{16k}{5R}(\cos^3\theta - 3\cos\theta)$$

$$E_{\perp}^{bel} = \frac{-\partial}{\partial r}\left(V_{in}\right|_{r=R} = \frac{-\partial}{\partial r}\left(\frac{-3kr}{5R}(\cos\theta) + \frac{4k}{5R^3}r^3(5\cos^3\theta - 3\cos\theta)\right|_{r=R}$$

$$= \frac{6K}{5R}(\cos\theta) + \frac{12K}{5R^3}r^2(5\cos^3\theta - 3\cos\theta)\Big|_{r=R}$$

$$= \frac{6K}{5R}(\cos\theta) - \frac{12K}{5R}(5\cos^3\theta - 3\cos\theta)$$

Let $\beta = 5\cos^3\theta - 3\cos\theta$

$$E_{\perp}^{abo} - E_{\perp}^{bel} = -\frac{6k}{5R}(\cos\theta) + \frac{16k}{5R}(\beta) - \frac{6K}{5R}(\cos\theta) + \frac{12K}{5R}(\beta)$$

$$= \frac{K}{5R}(-12\cos\theta + 16\beta + 12\beta) = \frac{K}{5R}(-12\cos\theta + 28\beta)$$

$$= \frac{K}{5R}\left(-12\cos\theta + 28(5\cos^3\theta - 3\cos\theta)\right)$$

$$= \frac{K}{5R}(-12\cos\theta + 140\cos^3\theta - 84\cos\theta)$$

$$= \frac{K}{5R}(-93\cos\theta + 140\cos^3\theta) = \frac{\sigma}{\epsilon_0}$$

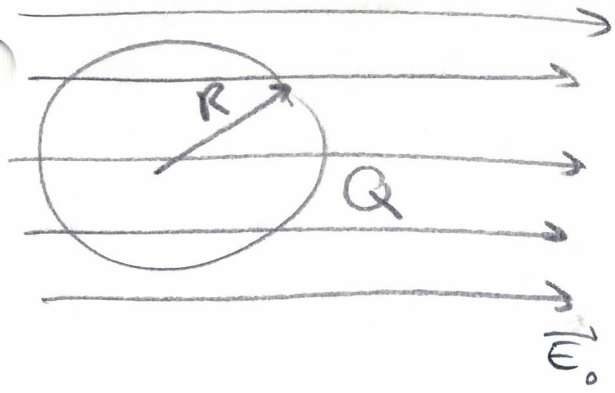
10 $\boxed{\sigma = \frac{\epsilon_0 K}{5R}(140\cos^3\theta - 93\cos\theta)}$ ✓

JAMES COOK

3-20 Find V outside charged metal sphere

Explain where V set to zero

20/20

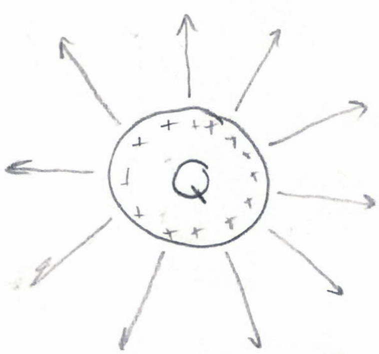


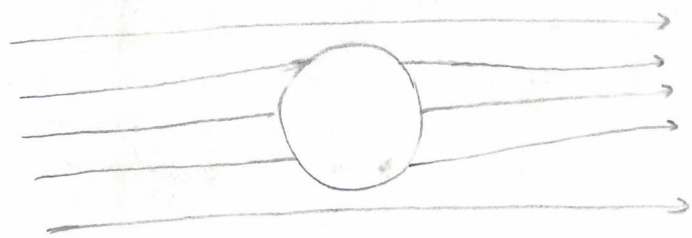
we set $V = 0$ at $r = R$

$$V_{sphere} = -\int_R^r \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + V_{sphere}$$

$$= -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)$$



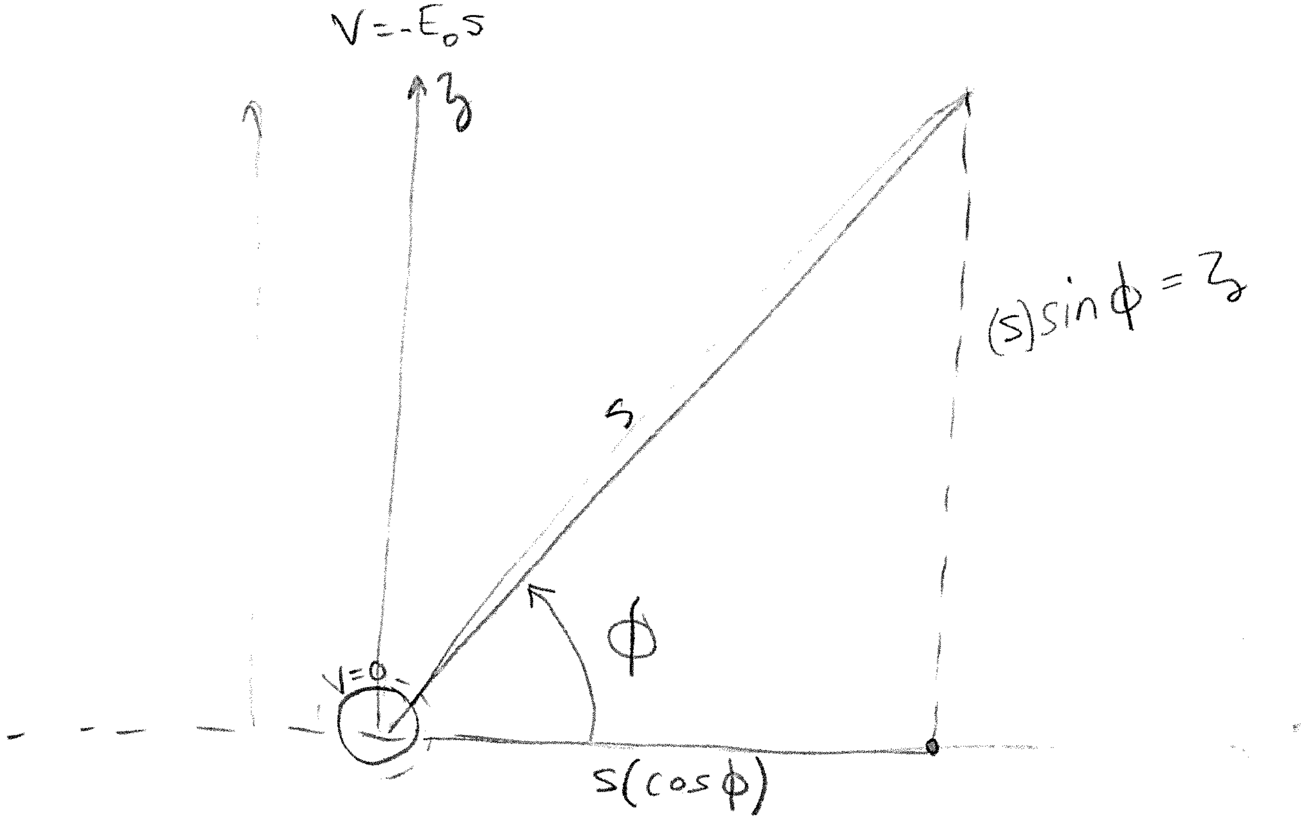


$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)}$$

10

$$-(E_0 r)\cos\theta = \frac{Q}{4\pi\epsilon_0 R}$$

3-24



$$E = E_0 \hat{z}$$

$$V = -E_0 z + \cancel{\text{const.}}^{\,0}$$

$$V = -E_0 (s) \sin\phi$$

$$s \gg R$$

$\boxed{3-24}$ from class,

$$V(s,\phi) = a_0 + b_0 \ln(s) + \sum_{n=1}^{\infty} \left\{ s^n (a_n \cos n\phi + b_n \sin n\phi) + s^{-n}(c_n \cos n\phi + d_n \sin n\phi) \right\}$$

from my picture I know that $V(\infty, \phi) = -E_0 s(\sin\phi)$ that is for $s >> R$ we would like the potential look like that of a uniform E field.

$$V(s,\phi) = -E_0 s(\sin\phi) = a_0 + b_0 \ln(s) + \sum_{n=1}^{\infty} \left\{ s^n (a_n \cos n\phi + b_n \sin n\phi) + s^{-n}(c_n \cos n\phi + d_n \sin n\phi) \right\}$$

$(s << R)$

equating coefficients $\Rightarrow$
$a_0 = 0$, set $V=0$ at $z=0$
$b_0 = 0$, sol^n can't match $V(s,\phi)$ $s<<R$
$a_i = 0$, want answer to look like $\sin\phi$
$b_i = 0$ $i \neq 1$ only want $\sin\phi$
$c_i = 0$, don't want $\cos\phi$
$d_i = 0$ $i \neq 1$ only want $\sin\phi$

$$V(s,\phi) = s\sin\phi\, b_1 + \tfrac{1}{s} d_1 \sin\phi$$

$$V(R,\phi) = R\sin\phi\, b_1 + \tfrac{1}{R} d_1 \sin\phi = 0$$

$$R b_1 = \frac{-1}{R} d_1 \Rightarrow d_1 = -R^2 b_1$$

$$V(s,\phi) = b_1 s \sin\phi - R^2 b_1 \tfrac{1}{s} \sin\phi = -E_0 s(\sin\phi) \Rightarrow b_1 = -E_0.$$

$$V(s,\phi) = -E_0 s \sin\phi + R^2 E_0 \tfrac{1}{s} \sin\phi$$

$$= E_0 \sin\phi \left[R^2/s - s \right]$$

$$E_{\perp}^{abo} - E_{\perp}^{bel} = \frac{\sigma}{\epsilon_0} \hat{n}$$

$$-\frac{\partial V}{\partial n}\bigg|_{s=R} = \frac{-\partial}{\partial s}\left(E_0 \sin\phi [R^2/s - s]\right)\bigg|_{s=R} = -E_0 \sin\phi \left[\frac{-R^2}{s^2} - 1 \right]\bigg|_{s=R}$$

10

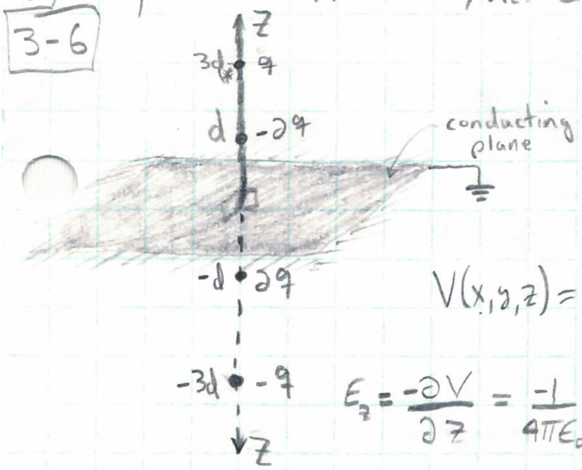
$$= -E_0 \sin\phi \left[-R^2/R^2 - 1 \right] = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \epsilon_0 E_0 \sin\phi \left[1 + R^2/R^2 \right]$$

$$\boxed{\sigma = 2\epsilon_0 E_0 \sin\phi}$$

PY 414, ELECTRODYNAMICS, Dr. CHUNG JAMES COOK

$\boxed{3-6}$



14/20

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{(x^2+y^2+(z-3d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+3d)^2)^{1/2}} + \frac{2q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{2q}{(x^2+y^2+(z+d)^2)^{1/2}}\right)$$

$$E_z = \frac{-\partial V}{\partial z} = \frac{-1}{4\pi\epsilon_0}\left\{\frac{q(z-3d)}{(x^2+y^2+(z-3d)^2)^{3/2}} - \frac{q(z+3d)}{(x^2+y^2+(z+3d)^2)^{3/2}} + \frac{2q(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{2q(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right\}$$

$$E_z^{(*)} = \frac{-\partial V}{\partial z} = \frac{-1}{4\pi\epsilon_0}\left(-\frac{q(z+3d)}{(z+3d)^3} + \frac{2q(z-d)}{(z-d)^3} - \frac{2q(z+d)}{(z+d)^3}\right)$$

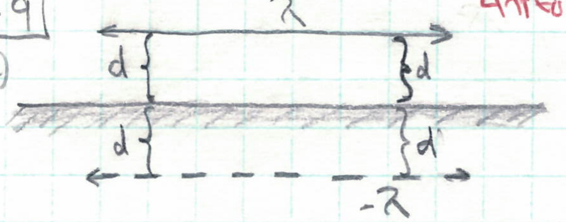
5

$$\boxed{F_z = qE_z = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{(z+3d)^2} - \frac{2q^2}{(z-d)^2} + \frac{2q^2}{(z+d)^2}\right), \quad F_x = F_y = 0 \text{ by symetry.}}$$

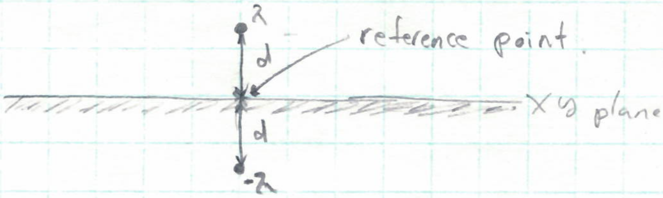
easier $\rightarrow$ $F = -\frac{q \cdot 2q}{4\pi\epsilon_0 (2d)^2} + 2qq/4\pi\epsilon_0(4d)^2 - q \cdot q/4\pi\epsilon_0(6d)^2 = \frac{-29}{72}\frac{q^2}{4\pi\epsilon_0 d^2}$

$\boxed{3-9}$

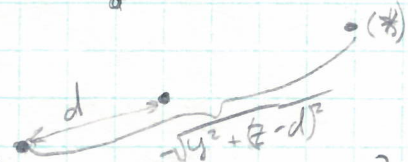
(a)



$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}$ which radiates radially out around wire if we could view wire looking along x-axis.



$$V_\lambda = \frac{-\lambda}{2\pi\epsilon_0}\int_d^r \frac{1}{r'}dr' = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{d}{r}\right) = V_\lambda \qquad r = \frac{d}{\sqrt{y^2+(z-d)^2}}$$



$$V_{-\lambda} = \frac{\lambda}{2\pi\epsilon_0}\int_d^r \frac{1}{r}dr = \frac{-\lambda}{2\pi\epsilon_0}\ln\frac{d}{r} \qquad r = \frac{d}{\sqrt{y^2+(z+d)^2}}$$

$$V_\lambda + V_{-\lambda} = \frac{\lambda}{2\pi\epsilon_0}\left\{\ln\left(\frac{d}{\sqrt{y^2+(z-d)^2}}\right) - \ln\left(\frac{d}{\sqrt{y^2+(z+d)^2}}\right)\right\}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left\{\ln\left(\frac{d}{\sqrt{y^2+(z-d)^2}}\right)\left(\frac{\sqrt{y^2+(z+d)^2}}{d}\right)\right\}$$

$$\boxed{V(x,y,z) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\sqrt{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}}\right)}$$ ✓

(b) at surface $z=0$, $\frac{-\partial V}{\partial z} = +E_z = \frac{-\lambda}{2\pi\epsilon_0}\left(\sqrt{\frac{y^2+(z-d)^2}{y^2+(z+d)^2}}\right)\frac{1}{2}\left(\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right)^{-1/2}\frac{d}{dz}\left(\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right)$

$$E_z = \frac{-\lambda}{2\pi\epsilon_0}\left(\frac{y^2+(z-d)^2}{y^2+(z+d)^2}\right)\frac{1}{2}\left(\frac{(y^2+(z-d)^2)(z+d)2 - 2(z-d)(y^2+(z+d)^2)}{(y^2+(z-d)^2)^2}\right)$$

10 9

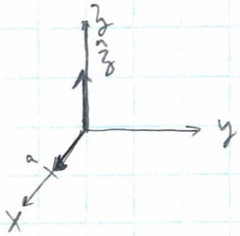
$$= \frac{-\lambda}{4\pi\epsilon_0}\left(\frac{y^2+d^2}{y^2+d^2}\right)\left(\frac{(y^2+d^2)(2d) - (-2d)(y^2+d^2)}{(y^2+d^2)^2}\right) = \frac{-\lambda}{4\pi\epsilon_0}\frac{4d(y^2+d^2)}{(y^2+d^2)^2}$$ as $z \to 0$ where $E_z = \frac{\sigma}{\epsilon_0}$

$$= \frac{-\lambda d}{\pi\epsilon_0}\frac{1}{y^2+d^2} = \frac{\sigma}{\epsilon_0} \implies \boxed{\sigma = \frac{-\lambda d}{\pi}\frac{1}{(y^2+d^2)}}$$

PY 414, Chapter Three #'s 31, 32 $\quad$ 14/20 $\quad$ JAMES COOK

$\boxed{3\text{-}31}$ a pure dipole is situated at origin pointing in $\hat{z}$

(a) $F(a,0,0) = q\,E(a,0,0)$ I have $E(r,\theta)$ so I would like to change from the cartesian to the coör in $E(r,\theta)$



$$\left.\begin{aligned} \hat{\theta} &= -\hat{z} \\ \hat{r} &= \hat{x} \\ \theta &= \pi/2 \\ r &= a \end{aligned}\right\}$$

$$\vec{E}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

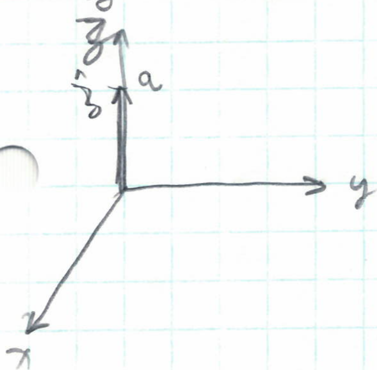
$$= \frac{p}{4\pi\epsilon_0 a^3}\left(2\cos\left(\tfrac{\pi}{2}\right)\hat{x} + \sin\tfrac{\pi}{2}\left(-\hat{z}\right)\right)$$

$$= \frac{-p}{4\pi\epsilon_0 a^3}\hat{z} = \vec{E}$$

$$\vec{F} = q\vec{E} = \boxed{-\frac{qp}{4\pi\epsilon_0 a^3}\hat{z} = \vec{F}}$$

(b) $F(0,0,a) = q\,E(0,0,a)$

again transform to desired coordinates for $E(r,\theta)$.



$$\left.\begin{aligned} \theta &= 0 \\ r &= a \\ \hat{\theta} &= \hat{x} \\ \hat{r} &= \hat{z} \end{aligned}\right\}$$

$$\vec{F} = q\vec{E} = \frac{qp}{4\pi\epsilon_0 a^3}\left(2\cos(0)(\hat{z}) + \sin(0)(\hat{x})\right)$$

$$\boxed{\vec{F} = \frac{2qp}{4\pi\epsilon_0 a^3}\hat{z}}$$

(c) $W = q\Delta V = q V_{ba} = q(V_b - V_a)$, $V(r,\theta) = \dfrac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \dfrac{p\cos\theta}{4\pi\epsilon_0 r^2}$

$$V_b = \frac{1}{4\pi\epsilon_0}\frac{p\cos 0}{a^2} = \frac{p}{4\pi\epsilon_0 a^2}$$

a) $\hat{x}\cdot\hat{z} = 0 \Rightarrow V_a = 0.$

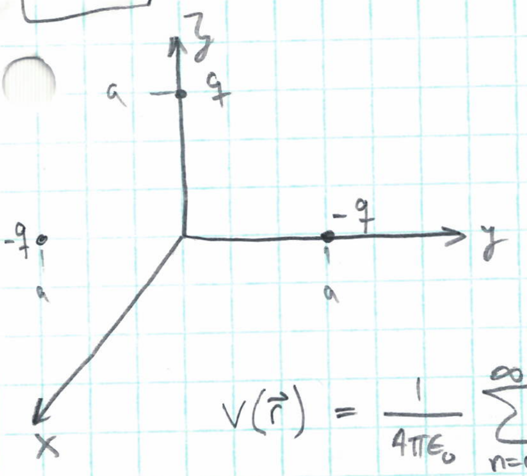
b) $\hat{z}\cdot\hat{z} = 0$

$$W = q(V_b - V_a) = \boxed{\frac{qp}{4\pi\epsilon_0 a^2} = W_{ba}}$$

9

$\boxed{3\text{-}32}$

find approximate electric field at points far from the origin, Express in Spherical coordinates keep just 2 lowest orders in multipole expansion.



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\vec{r}')\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{r}\int \rho(\vec{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta\,\rho(\vec{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')\,d\tau' + \dots\right\}$$

$\frac{1}{r}\int \rho(\vec{r}')\,d\tau' = \frac{-q}{r}$ since $r \gg a$ so $\int \rho(\vec{r}')\,d\tau'$ contains $-q - q + q = -q$.

$\frac{1}{r^2}\int r'\cos\theta\,\rho(\vec{r}')\,d\tau' = \frac{\hat{r}}{r^2}\cdot\int$

What is ρ? in this case

or

$$V = \sum \frac{q_i}{4\pi\epsilon_0 r} + \sum \frac{r \cdot p_i}{4\pi\epsilon_0 r^2}$$

5

$\cancel{10}$

$$= \frac{-q}{4\pi\epsilon_0 r} + \frac{q a \cos\theta}{4\pi\epsilon_0 r^2}$$

$$E_r = -\frac{\partial V}{\partial r} \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$