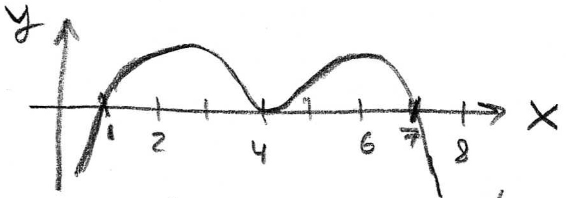
Homework 22 Calculus I

§4.3#7| Given the following graph of f'' state the x-coordinates for the inflection points of $y = f(x)$,



We find the concavity changes at $x = 1$ and $x = 7$.

§4.3#9| Find intervals of increase, decrease and concave up/down. Also find local maximums & minimums for $f(x) = 2x^3 + 3x^2 - 36x$.

$$f'(x) = 6x^2 + 6x - 36$$
$$= 6(x^2 + x - 6)$$
$$= 6(x+3)(x-2)$$

Critical numbers are $x = -3$ and $x = 2$.

+ + + + + | − − − − | + + + + + → $\frac{df}{dx}$

−3 2

Thus f is increasing on $(-\infty, -3)$ and $(2, \infty)$.
f is decreasing on $(-3, 2)$

By the 1st Derivative Test we find $f(-3) = -2(27) + 27 + 36(3) = 81$ is a local max. While $f(2) = 2(8) + 12 - 72 = -44$ is a local min.

$$f''(x) = 12x + 6$$
$$= 6(2x+1) = 0 \quad \text{when } x = -1/2$$

− − − − − − | + + + + + → f''

−1/2

We find f is concave down on $(-\infty, -1/2)$
f is concave up on $(-1/2, \infty)$

This means $x = -1/2$ has an inflection point.

§4.3 #11 | Same instructions as #9. Let $f(x) = x^4 - 2x^2 + 3$

$$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x+1)(x-1)$$

We find $f'(x) = 0$ for $x = 0, -1, 1$. The sign chart for df/dx is

$$\underset{}{- - -} \ \Big|_{-1} \ + + + + + \ \Big|_{0} \ - - - - \ \Big|_{1} \ + + + + \to f'$$

This shows that f is decreasing on $(-\infty, -1)$ and $(0,1)$ while f is increasing on $(-1, 0)$ and $(1, \infty)$.

$f(-1) = 1 - 2 + 3 = 2$: local min. by 1st Der. Test.
$f(0) = 3$: local max. by 1st Der. Test.
$f(1) = 1 - 2 + 3 = 2$: local min. by 1st Der. Test.

Next consider,

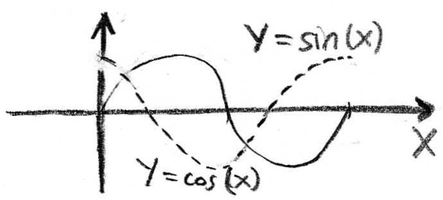
$$f''(x) = 12x^2 - 4$$

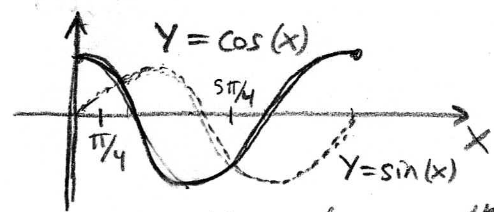
Thus $f''(x) = 0$ if $12x^2 = 4$ so $x^2 = 1/3 \to x = \pm 1/\sqrt{3}$.

$$+ + + + + \ \Big|_{-1/\sqrt{3}} \ - - - - - \ \Big|_{1/\sqrt{3}} \ + + + + \to f''$$

thus f is concave up on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$ and f is concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$. We have inflection points at $x = \pm 1/\sqrt{3}$.

§4.3 #13 | Again same as #9 or #11. Let $f(x) = \sin(x) + \cos(x)$, $0 \leq x \leq 2\pi$.
We calculate $f'(x) = \cos(x) - \sin(x)$ and $f''(x) = -\sin(x) - \cos(x)$.





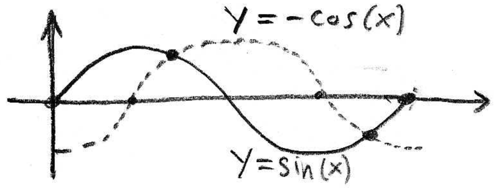
Notice $\cos(x) - \sin(x) = 0$ for $\sin(x) = \cos(x)$ this has sol$^{\underline{n}}$'s $x = \pi/4$ and $x = 5\pi/4$, you can see this from where the graphs of $y = \sin(x)$ and $y = \cos(x)$ intersect.

$$\Big[_{0} + + + + \ \Big|_{\pi/4} \ - - - - - - \ \Big|_{5\pi/4} \ + + + + + \Big]_{2\pi} \quad \frac{df}{dx} = \cos(x) - \sin(x)$$

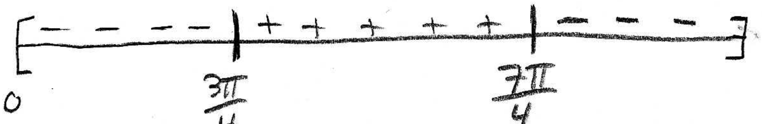
Thus f is inc. on $[0, \pi/4)$ and $(5\pi/4, 2\pi)$ and f dec. on $(\pi/4, 5\pi/4)$.
By 1st Der. Test $f(\pi/4) = 1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2}$ is local max.
while $f(5\pi/4) = 1/\sqrt{2} - 1/\sqrt{2} = 0$ is a local min.

§4.3#13) Consider $f''(x) = -\sin(x) - \cos(x)$. Where is this function zero on $[0, 2\pi]$? In other words, where does

$$-\sin(x) - \cos(x) = 0 \rightarrow \sin(x) = -\cos(x)$$



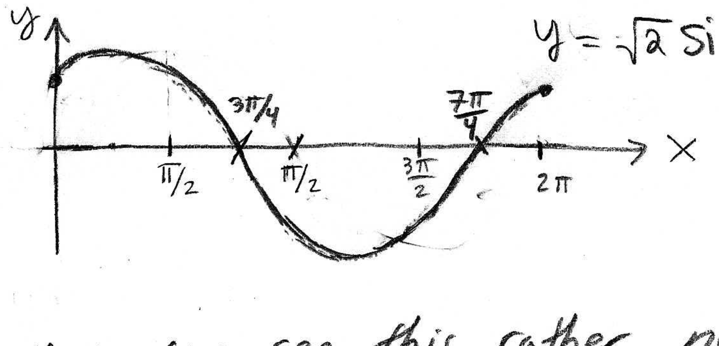
intersections at $x = 3\pi/4$ and $x = 7\pi/4$



$$\left(f''(\pi/4) = -1/\sqrt{2} - 1/\sqrt{2} < 0,\ f''(\pi) = 1,\ f''\left(\tfrac{15\pi}{8}\right) < 0 \right)$$

Thus $f(x) = \sin(x) + \cos(x)$ is concave down on $(0, 3\pi/4)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and f is concave up on $(3\pi/4, 7\pi/4)$. We find inflection points at $x = 3\pi/4$ and $x = 7\pi/4$.

Remark:
$$\begin{aligned} \sin(x) + \cos(x) &= \left(\frac{1}{\sqrt{2}}\sin(x) + \cos(x)\frac{1}{\sqrt{2}}\right)\sqrt{2} \\ &= \left(\cos(\pi/4)\sin(x) + \cos(x)\sin(\pi/4)\right)\sqrt{2} \\ &= \sqrt{2}\sin(x + \pi/4) \end{aligned}$$



just shift $y = \sin(x)$ $\frac{\pi}{4}$ to the left.

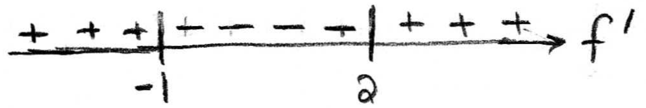
So if you can see this rather non-obvious trig-identity then the conclusions offered to us by the inc/dec and concavity tests can also be found from non-calculus arguments.

§4.3 #29 Let $f(x) = 2x^3 - 3x^2 - 12x$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

$f''(x) = 12x - 6$

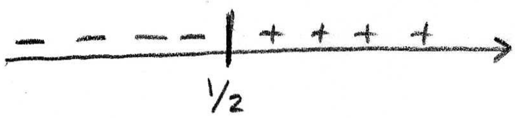
We find critical numbers $x = 2$ and $x = -1$.



f increases on $(-\infty, -1)$ and $(2, \infty)$.
f decreases on $(-1, 2)$.

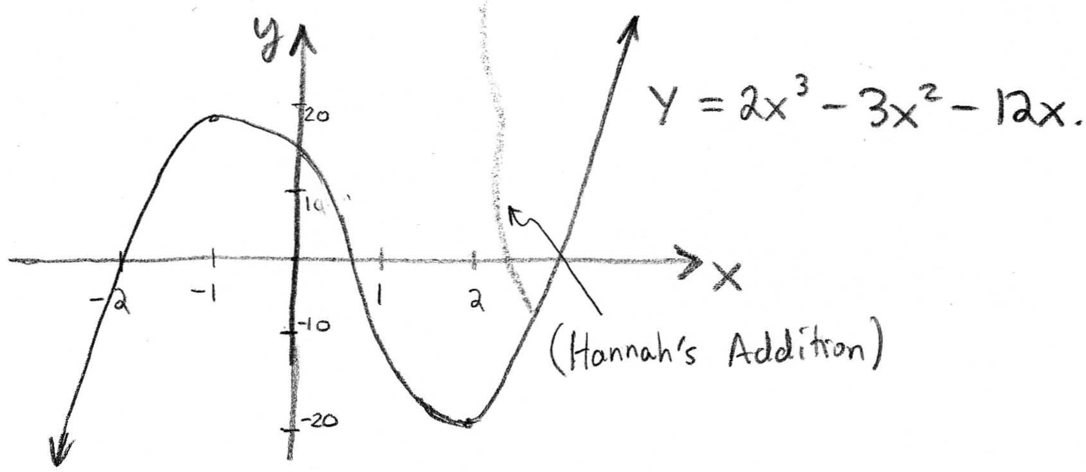
This means $f(-1) = -2-3+24 = 19$ is a local maximum and $f(2) = 16 - 12 - 24 = -20$ is a local minimum by 1st Derivative Test.

Notice $f''(x) = 0 = 12x - 6$ for $x = 1/2$ thus the sign chart is



f is concave down on $(-\infty, 1/2)$
f is concave up on $(1/2, \infty)$

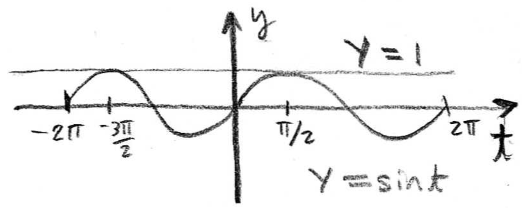
This means there is an inflection pt. at $x = 1/2$.

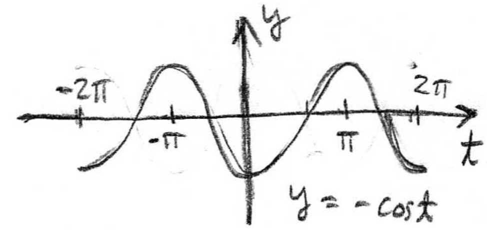


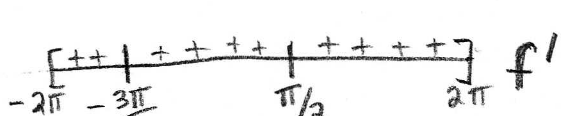
§4.3 #40 $f(t) = t + \cos t$ for $-2\pi \le t \le 2\pi$.

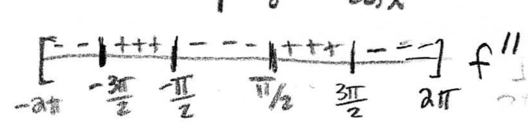
$f'(t) = 1 - \sin t$

$f''(t) = -\cos t$









Apparently f increases on $[-2\pi, -3\pi/2)$ and $(-3\pi/2, \pi/2)$ and also $(\pi/2, 2\pi)$. It decreases nowhere and there are no local extremes.

§4.3 #40 We found $f''(t) = -\cos t$ the sign chart followed from the graph

$$\begin{array}{ccccccccccc} [& - - - & | & + + + + & | & - - - & | & + + + + & | & - - - - &] \\ -2\pi & & -\frac{3\pi}{2} & & -\frac{\pi}{2} & & \frac{\pi}{2} & & \frac{3\pi}{2} & & 2\pi \end{array} \quad f''$$

We can read that f is concave up on $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ while f is concave down on $\left(-2\pi, -\frac{3\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$.
This means there are inflection points at $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

$$f(-2\pi) = -2\pi + \cos(-2\pi) = 1 - 2\pi \cong -5.28$$

$$f\left(-\frac{3\pi}{2}\right) = -\frac{3\pi}{2} + \cos\left(-\frac{3\pi}{2}\right) = -1.5\pi \cong -4.71$$

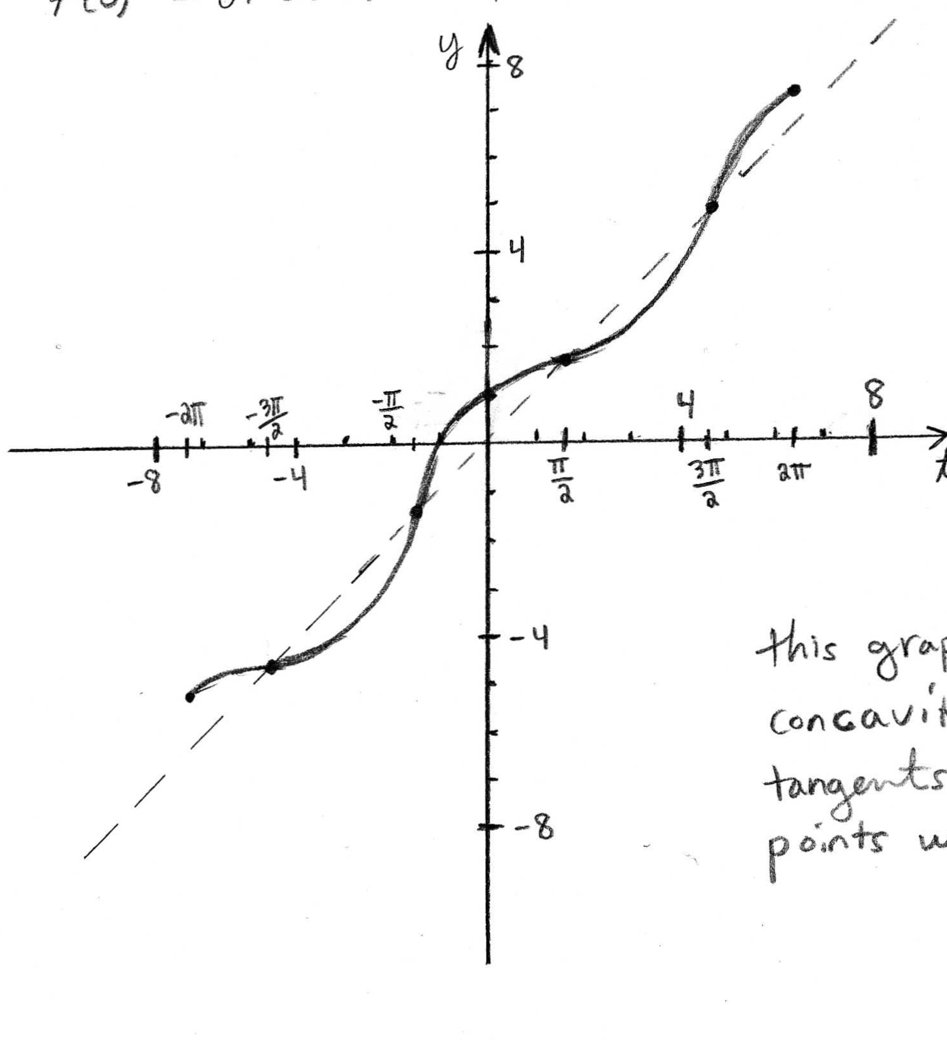
$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} \cong -1.57$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cong 1.57$$

$$f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} \cong 4.71$$

$$f(2\pi) = 2\pi + \cos(2\pi) = 1 + 2\pi = 7.28$$

Just gathering some values to make the graph better.
Notice $f(0) = 0 + \cos(0) = 1$.



this graph follows from concavity, horizontal tangents and the points we calculated.

§4.3 #53 Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

If $f(x) = ax^3 + bx^2 + cx + d$ then

$f'(x) = 3ax^2 + 2bx + c$ and,

$f''(x) = 6ax + 2b$

We know by Fermat's Th^{m} that

$$f'(-2) = 12a - 4b + c = 0 \quad \text{I}$$

$$f'(1) = 3a + 2b + c = 0 \quad \text{II}$$

Also we were given,

$$f(-2) = -8a + 4b - 2c + d = 3 \quad \text{III}$$

$$f(1) = a + b + c + d = 0 \quad \text{IV}$$

We have 4 eq^{n}'s and 4 unknowns. All that remains is algebra.

$$\text{I} - \text{II} : \quad 9a - 6b = 0$$

$$3a = 2b$$

$$\underline{a = \tfrac{2}{3}b} \quad (*)$$

$$\text{IV} - \text{III} : \quad 9a - 3b + 3c = -3$$

$$c = -1 + b - 3a$$

$$c = -1 + b - 3\left(\tfrac{2}{3}b\right) \quad : \text{using } (*)$$

$$\underline{c = -1 - b} \quad (**)$$

Take eq^{n} II and substitute (*) and (**)

$$3a + 2b + c = 2b + 2b - 1 - b = 0$$

$$\Rightarrow 3b = 1 \Rightarrow \boxed{b = 1/3}$$

Then by (*) we find $\boxed{a = 2/9}$ and by (**) $\boxed{c = -4/3}$

Finally use IV to solve for $d = -a - b - c = \frac{-2}{9} - \frac{3}{9} + \frac{12}{9} = \boxed{\frac{7}{9} = d}$

Thus

$$\boxed{f(x) = \frac{1}{9}\left(2x^3 + 3x^2 - 12x + 7\right)}$$