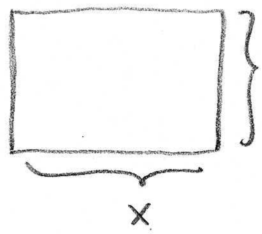
HOMEWORK 23, CALCULUS I

§4.7 #5 Find dimensions of rectangle of perimeter 100m and largest area.



It is given that $100 = 2x + 2y$
thus $y = 50 - x$ then the area A is

$$A = xy = x(50-x) = 50x - x^2$$

We use calculus to maximize A,

$\frac{dA}{dx} = 50 - 2x \longrightarrow$ critical # $x = 25$

$\frac{d^2A}{dx^2} = -2 < 0$ thus by 2nd derivative test $x = 25$ gives maximum area, $y = 50 - 25 = 25$ the case of a 25×25 square gives max. area of $A = 625$.

$\boxed{\text{§4.7\#19}}$ Find points on ellipse $4x^2 + y^2 = 4$ furthest from $(1,0)$.
The distance D from a point (x,y) on the ellipse to $(1,0)$ is

$$D = \sqrt{(x-1)^2 + y^2}$$

This gives $D = f(x) = \sqrt{(x-1)^2 + 4 - 4x^2}$ as $y^2 = 4 - 4x^2$
We can simplify more, $f(x) = \sqrt{x^2 - 2x + 1 + 5 - 4x^2}$ thus

$$D = \sqrt{6 - 2x - 3x^2}$$

$$\frac{dD}{dx} = \frac{1}{2\sqrt{6-2x-3x^2}}(-2-6x)$$

Critical #'s for D must have $-2 - 6x = 0 \rightarrow 6x = -2 \rightarrow x = -1/3$

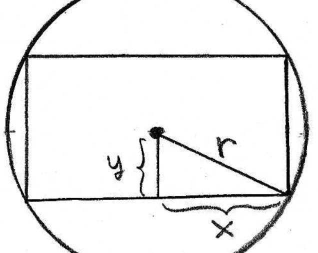
$$\underset{-1/3}{+ + + + + + \,|\, - - - - - -} \longrightarrow \frac{dD}{dx}$$

Thus $x = -1/3$ gives maximum distance D by 1st der. test.

$$y^2 = 4 - 4\left(\tfrac{-1}{3}\right)^2 = \frac{36-4}{9} = \frac{32}{9} \longrightarrow y = \pm\frac{4\sqrt{2}}{3}$$

We find the points furthest from $(1,0)$ on the given ellipse are $\boxed{(-1/3, 4\sqrt{2}/3) \text{ and } (-1/3, -4\sqrt{2}/3)}$

§4.7 # 21) Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.



To begin draw a picture and choose natural variables:

y = distance from center of rectangle to base of the rectangle.

x = distance of half of base of rectangle.

We know $x^2 + y^2 = r^2$

The area of the rectangle is $A = 4xy$. Notice that $y > 0$ since y is a distance $\Rightarrow$ $y = \sqrt{r^2 - x^2}$ and

$$A = 4x\sqrt{r^2 - x^2}$$

Now we can optimize $A = f(x)$. (it's a function of x alone)

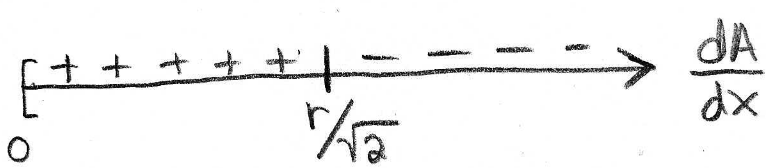
$$\frac{dA}{dx} = 4\sqrt{r^2-x^2} + 4x\frac{1}{\cancel{2}\sqrt{r^2-x^2}}(-\cancel{2}x)$$

$$= 4\left(\frac{(\sqrt{r^2-x^2})^2 - x^2}{\sqrt{r^2-x^2}}\right)$$

$$= 4\left(\frac{r^2 - 2x^2}{\sqrt{r^2-x^2}}\right) = 0$$ when $r^2 - 2x^2 = 0$

critical #'s $x = \pm r/\sqrt{2}$

x is a distance
only physically relevant critical # is $x = r/\sqrt{2}$



Thus $x = r/\sqrt{2}$ yields max. area

(note $x = 0$ is an endpoint if we allow rather silly rectangles, however $A = 0$ is not the max, the local max at $x = r/\sqrt{2}$ is the relevant point)

$$y = \sqrt{r^2 - (r/\sqrt{2})^2} = \sqrt{r^2 - r^2/2} = r/\sqrt{2}$$

The dimensions will be twice x and y

$$\boxed{\text{length} = 2x = \sqrt{2}\,r, \quad \text{width} = 2y = \sqrt{2}\,r}$$

And the area is $A = 2r^2$ which is less than πr^2, good.

§4.7 #23 After some deliberation we found the area as a function of z. Now optimize A.

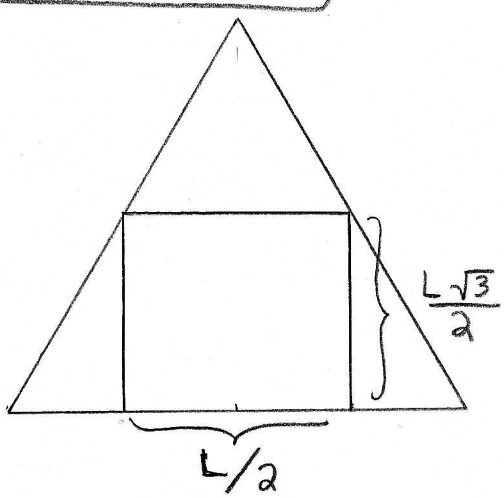
$$A = \sqrt{3}\,(Lz - 2z^2)$$

$$\frac{dA}{dz} = \sqrt{3}\,(L - 4z) = 0 \quad \text{for} \quad \underline{z = L/4}$$

critical #

$$\frac{d^2A}{dz^2} = -4\sqrt{3} < 0 \rightarrow z = L/4$$ yields max area by 2nd derivative test (no interesting end pts. here)

Thus $\boxed{\begin{array}{l} \text{length} = L - 2(L/4) = L/2 \\ \text{width} = \sqrt{3}\,L/2 \end{array}}$



Remark: the examples considered thus far we've had our optimal value be a local extrema. This need not be the case. Some word problems make the domain of the variable a closed interval. In such a case we also would need to check endpoints. (Like Example 5.4.6 in my notes)

§4.7 #43 A beehive is made of cells which are regular hexagonal prism. See the text for a picture. It turns out the surface area S is given by

$$S = 6sh - \frac{3}{2}s^2\cot(\theta) + (3s^2\sqrt{3}/2)\csc\theta$$

where s, b and h are constants.

(a.)
$$\frac{dS}{d\theta} = \frac{3}{2}s^2\csc^2\theta - \frac{3s^2\sqrt{3}}{2}\csc\theta\cot\theta$$
$$= \frac{3s^2}{2}\csc\theta\left[\csc\theta - \sqrt{3}\cot\theta\right]$$

(b.) We need to find critical numbers for S. Geometry suggests that $0 < \theta < \pi/2$. Recall $\csc\theta = \frac{1}{\sin\theta}$ and clearly $\csc\theta \neq 0$ for $\theta \in (0, \pi/2)$. So critical #'s must have

$$\csc\theta - \sqrt{3}\cot\theta = 0$$
$$\frac{1}{\sin\theta} = \sqrt{3}\frac{\cos\theta}{\sin\theta}$$
$$\cos\theta = \frac{1}{\sqrt{3}} \quad \therefore \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 70.53°$$

Use 2nd Derivative Test

$$\frac{d^2S}{d\theta^2} = -3s^2\csc\theta\csc\theta\cot\theta - \frac{3s^2\sqrt{3}}{2}\left(-\csc\theta\cot^2\theta - \csc^3\theta\right)$$
$$= 3s^2\left(-\csc^2\theta\cot\theta + \frac{\sqrt{3}}{2}\csc\theta\left(\cot^2\theta + \csc^2\theta\right)\right)$$
$$= 3s^2\left(-\frac{1}{\sin^2\theta}\frac{\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{2}\frac{1}{\sin\theta}\left(\frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta}\right)\right)$$

Notice $\cos\theta = 1/\sqrt{3} \Rightarrow \sin\theta = \sqrt{2/3}$ thus $\sin^2\theta = 2/3$, $\cos^2\theta = 1/3$

$$\left.\frac{d^2S}{d\theta^2}\right|_{\theta=\cos^{-1}(1/\sqrt{3})} = 3s^2\left(-\frac{3}{2}\frac{1/\sqrt{3}}{\sqrt{2/3}} + \frac{\sqrt{3}}{2}\sqrt{\frac{3}{2}}\left(\frac{1/3}{2/3} + \frac{1}{2/3}\right)\right)$$
$$= 3s^2\left(\frac{-3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}\left(\frac{1}{2} + \frac{3}{2}\right)\right)$$
$$= \frac{9s^2}{2\cdot\sqrt{2}} > 0 \quad \therefore \quad \boxed{S \text{ is minimized at } \theta \cong 70.53°}$$

§4.7 #43 Continued / $\cot(70.53°) \cong 1/\sqrt{2}$ and $\csc(70.53°) \cong \sqrt{3/2}$,

(c.) $S_{P_{min}} = 6sh - \frac{3}{2}s^2\cot(70.53°) + (3s^2\sqrt{3}/2)\csc(70.53°)$

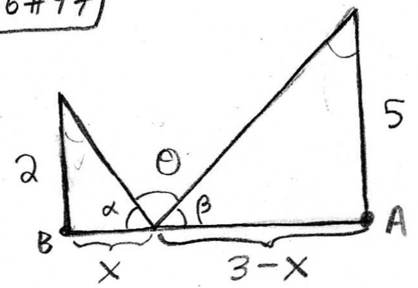
$$= 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + \left(\frac{3s^2\sqrt{3}}{2}\right)\sqrt{\frac{3}{2}}$$

$$= 6sh - \frac{3s^2}{2\sqrt{2}}(1-3)$$

$$= \boxed{6sh + 3s^2/\sqrt{2} = S_{min}}$$

Remark: I find the fact that beehives typically follow this model and actually minimize surface area S within about 2° (according to Stewart pg. 264) fascinating. How do the bees know to do this? Is it intinctual? How would bees obtain such instinct in an evolutionary storyline? I'm sure someone has a sufficiently convoluted explaination to see this as an evolutionary out come, But for me its seems logical to see the hand of God at work here. I think God has given bees this instinct. I can't fathom all the observed optimization in nature as the mere consequence of random mutation guided via natural selection. I think creation points to the Creator. Its not an absolute proof, its just one of many many examples.

§7.6#47



$$\alpha + \beta + \theta = \pi$$

$$\pi - \theta = \alpha + \beta$$

$$\cot(\alpha) = \frac{x}{2} \quad \text{and} \quad \cot(\beta) = \frac{3-x}{5}$$

Remark: this problem requires persistence.

Sol$^{\underline{n}}$ over

§7.6 #47

$$\cot^{-1}\left(\frac{x}{2}\right) + \cot^{-1}\left(\frac{3-x}{5}\right) + \theta = \pi$$

$$\frac{-1/2}{1+x^2/4} + \frac{1/5}{1+\left(\frac{3-x}{5}\right)^2} + \frac{d\theta}{dx} = 0$$

$$\frac{d\theta}{dx} = \frac{2}{4+x^2} - \frac{5}{25+(3-x)^2}$$

$$= \frac{2(25+(3-x)^2) - 5(4+x^2)}{(4+x^2)(25+(3-x)^2)}$$

$$= \frac{2(25+9-6x+x^2) - 20 - 5x^2}{(4+x^2)(x^2-6x+34)}$$

$$= \frac{68-12x+2x^2-20-5x^2}{(4+x^2)(x^2-6x+34)}$$

$$= \frac{48-12x-3x^2}{(4+x^2)(x^2-6x+34)} = 0$$

$$48 - 12x - 3x^2 = 0$$

$$\Rightarrow x^2 + 4x - 16 = 0$$

$$x = \frac{-4 \pm \sqrt{16+64}}{2} = -2 \pm \frac{\sqrt{80}}{2} = -2 \pm 2\sqrt{5}$$

We want $x > 0$ thus $x = 2\sqrt{5} - 2$ is the critical # of interest. The point is

$$3 - x = 3 - (2\sqrt{5} - 2) = \boxed{5 - 2\sqrt{5} \text{ away from A}}$$

Notice this maximizes θ because $48 - 12x - 3x^2$ changes from (+) to (−) around $x = -2 + 2\sqrt{5}$.