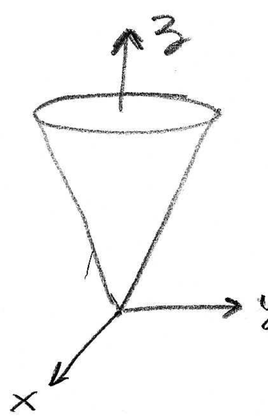
Homework 11, Calculus III



§16.8 #5

$\phi = \pi/3$ is a cone centered about the z-axis.

$$z = \rho\cos\phi = \rho\cos\left(\tfrac{\pi}{3}\right) = \frac{2\rho}{\sqrt{3}}.$$

$$\left.\begin{aligned} x &= \rho\cos\theta\sin\phi = \tfrac{1}{2}\rho\cos\theta \\ y &= \rho\sin\theta\sin\phi = \tfrac{1}{2}\rho\sin\theta \end{aligned}\right\} \text{ since } \sin\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}.$$

Thus $x^2 + y^2 = \frac{1}{4}\rho^2(\cos^2\theta + \sin^2\theta)$

$$4x^2 + 4y^2 = x^2 + y^2 + z^2$$

$$\boxed{z^2 = 3x^2 + 3y^2}$$

§16.8 #9

Write the equation $z^2 = x^2 + y^2$ in sphericals,

$$(\rho\cos\varphi)^2 = (\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2$$

$$\begin{aligned} \rho^2\cos^2\varphi &= \rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi \\ &= \rho^2(\cos^2\theta + \sin^2\theta)\sin^2\varphi \\ &= \rho^2\sin^2\varphi \end{aligned}$$

$$\Rightarrow \cos^2\varphi = \sin^2\varphi$$

$$\Rightarrow \tan^2\varphi = 1$$

$$\tan\varphi = \pm 1$$

$$\therefore \boxed{\varphi = \frac{\pi}{4} \text{ or } \frac{-\pi}{4}}$$

§16.8 # 21) Let B : (x, y, z) with $x^2+y^2+z^2 \le 25$. Recall $dV = \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$

$$\iiint_B (x^2+y^2+z^2)^2 \, dV = \int_0^5 \int_0^{\pi} \int_0^{2\pi} \rho^6 \sin\phi \, d\theta \, d\phi \, d\rho :$$

θ-integration gives $\theta \Big|_0^{2\pi}$ a.k.a 2π.

$$= \int_0^5 \int_0^{\pi} 2\pi \rho^6 \sin\phi \, d\phi \, d\rho$$

$$= \int_0^5 2\pi \rho^6 \left(-\cos\phi \Big|_0^{\pi}\right) d\rho$$

$$= \int_0^5 2\pi \rho^6 \left(-\cos\pi + \cos(0)\right) d\rho$$

$$= \frac{4\pi\rho^7}{7} \Bigg|_0^5$$

$$= \boxed{\frac{312{,}500\pi}{7}}$$

§16.8 # 23) E is $1 \le \rho \le 2$, and $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$

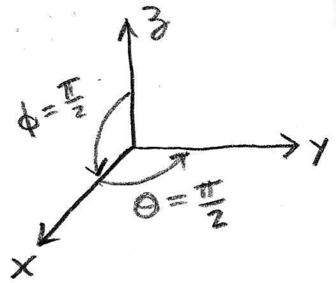
$$\iiint_E z \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 (\rho\cos\phi)(\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi \int_1^2 \rho^3 \, d\rho$$

$$= \left(\theta \Big|_0^{\pi/2}\right)\left(\frac{1}{2}\sin^2\phi \Big|_0^{\pi/2}\right)\left(\frac{1}{4}\rho^4 \Big|_1^2\right)$$

$$= \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{16-1}{4}\right)$$

$$= \boxed{\frac{15\pi}{16}}$$



§16.8 # 27) Find volume of part of ball $\rho \le a$ that lies between $\phi = \pi/6$ and $\phi = \pi/3$.

$$V = \iiint\limits_{\substack{\text{part}\\ \text{of}\\ \text{ball}}} dV = \int_0^a \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^a \rho^2 d\rho \int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\phi \, d\phi$$

why can we do this here?

$$= \left(\frac{1}{3}\rho^3 \Big|_0^a\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\cos\phi \Big|_{\pi/6}^{\pi/3}\right)$$

$$= \left(\frac{1}{3}a^3\right)(2\pi)\left(-\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{2\pi a^3}{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= \boxed{\frac{\pi a^3}{3}\left(\sqrt{3} - 1\right)}$$

Remark: You probably need to do more basic multiple integration problems. I doubt that the assigned problems are sufficient to build the requisite skill. Also I hope you understand why its ok to break up

$$\int_4^5\int_2^3\int_0^1 f(x)g(y)h(z)\,dx\,dy\,dz = \int_0^1 f(x)dx \int_2^3 g(y)dy \int_4^5 h(z)dz$$

we could not do that if instead

$$\int_4^5 \int_z^3 \underbrace{\int_{y+z}^1 f(x)g(y)h(z)\,dx\,dy\,dz}$$

variable bounds require certain orders of operations, in contrast constant bounds are easy.