In-class exercises 1-17-06

$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \int \frac{\sqrt{\tan^2\theta}\,\sec\theta\tan\theta\,d\theta}{\sec^3\theta}$$

$x = \sec\theta$
$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$
$dx = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta\,d\theta$$

$$= \int \sin^2\theta\,d\theta$$

$$= \int \frac{1}{2}\left(1-\cos(2\theta)\right)d\theta$$

$$= \frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + C$$

$$= \boxed{\frac{1}{2}\sec^{-1}(x) - \frac{1}{4}\sin\left(2\sec^{-1}(x)\right) + C}$$

$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{\sec^3\theta\sqrt{\tan^2\theta}}$$

$x = \sec\theta$
$x^2 - 1 = \tan^2\theta$
$dx = \sec\theta\tan\theta\,d\theta$

$$= \int 2\cos^2\theta\,d\theta$$

$$= \int \left(1+\cos(2\theta)\right)d\theta$$

$$= \theta + \frac{1}{2}\sin(2\theta) + C$$

$$= \boxed{\sec^{-1}(x) + \frac{1}{2}\sin\left(2\sec^{-1}(x)\right) + C}$$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9\cos^2\theta}}{9\sin^2\theta}\,3\cos\theta\,d\theta$$

$x = 3\sin\theta$
$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$
$dx = 3\cos\theta\,d\theta$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

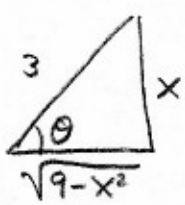
$$= \int \cot^2\theta\,d\theta$$

notice $\cot^2\theta + 1 = \csc^2\theta$

$$= \int \left(\csc^2\theta - 1\right)d\theta$$

$$= -\cot\theta - \theta + C$$

$$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$



$\sin\theta = \frac{x}{3}$

$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{9-x^2}}{x}$