EXAMPLES OF IMPROPER INTEGRALS

1.) $$\int_0^\infty e^{-3x}\,dx = \lim_{b\to\infty} \int_0^b e^{-3x}\,dx$$

$$= \lim_{b\to\infty} \left[\left(e^{-3x}/{-3}\right)\Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{3}e^{-3b} + \frac{1}{3}e^0 \right]$$

$$= \boxed{\frac{1}{3}}$$ (this integral converged)

2.) $$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \left[\int_1^b \frac{dx}{x} \right] = \lim_{b\to\infty} \left[\underbrace{\ln(b)}_{\infty} - \underbrace{\ln(1)}_{0} \right] = \boxed{\infty}$$

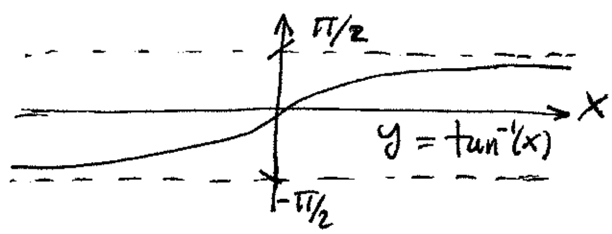
This integral <u>DIVERGED</u>.

3.) $$\int_{-\infty}^1 \frac{dx}{x^2+1} = \lim_{a\to-\infty} \left[\int_a^1 \frac{dx}{x^2+1} \right]$$

$$= \lim_{a\to-\infty} \left[\tan^{-1}(1) - \tan^{-1}(a) \right]$$

$$= \pi/4 - \left(-\frac{\pi}{2}\right)$$

$$= \boxed{3\pi/4}$$



4.) $$\int_{\pi/2}^\infty \cos(x)\,dx = \lim_{t\to\infty} \left[\int_{\pi/2}^t \cos(x)\,dx \right]$$

$$= \lim_{t\to\infty} \left[\sin(t) - \underbrace{\sin(\pi/2)}_{1} \right]$$

$$= \boxed{\text{d. n. e.}}$$ (due to oscillation at ∞)

(DIVERGENT)

5.) $$\int_0^\infty \frac{dx}{2^{x+3}} = \lim_{b\to\infty} \int_0^b \frac{dx}{2^{x+3}}$$

$$= \lim_{b\to\infty} \int_0^b 2^{-x-3}\, dx$$

$$= \lim_{b\to\infty} \left[\frac{-1}{\ln(2)} 2^{-x-3} \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{-1}{8\ln(2)} \underbrace{2^{-b}}_{\to 0} + \frac{1}{8\ln(2)} 2^{-0} \right]$$

$$= \boxed{\frac{1}{8\ln(2)}}$$ (this integral converged)

6.) For $x \geq 1$ notice $x^3 \geq x^2$ and so it follows that $\frac{1}{1+x^2} \geq \frac{1}{1+x^3} \geq 0$. Notice

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\tan^{-1}(b) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \underline{\underline{\frac{\pi}{4}}}$$

Hence, by comparison Th$^{\underline{m}}$,

$$\int_1^\infty \frac{dx}{1+x^3} \leq \int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}$$

it follows $\boxed{\int_1^\infty \frac{dx}{1+x^3} \text{ CONVERGES}}$

__Remark:__ convergence is easier than explicit calculation of a given improper integral.

7.) For $x \geq 0$ notice $e^{x^2} \geq 1$ thus $e^{x+x^2} \geq e^x$

It is simple to show $\int_0^\infty e^x\, dx = \lim_{b\to\infty}\left[e^b - 1\right] = \infty$.

Thus $\int_0^\infty e^{x+x^2}\, dx$ diverges to ∞ by comparison Th$^{\underline{m}}$.

8.) $$\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{a \to -2^+} \int_a^{14} (x+2)^{-1/4}\, dx$$

$$= \lim_{a \to -2^+} \left(\frac{4}{3}(x+2)^{3/4} \Big|_a^{14} \right)$$

$$= \lim_{a \to -2^+} \left(\frac{4}{3}(16)^{3/4} - \frac{4}{3}\underbrace{(a+2)^{3/4}}_{\to 0} \right)$$

$$= \boxed{\frac{32}{3}}$$

9.) $$\int_{-10}^{10} \frac{dx}{x-3} = \int_{-10}^{3} \frac{dx}{x-3} + \int_{3}^{10} \frac{dx}{x-3}$$: each of these is improper because $x-3 \to 0$ as $x \to 3$.

$$= \lim_{b \to 3^-} \left(\int_{-10}^{b} \frac{dx}{x-3} \right) + \lim_{a \to 3^+} \left(\int_a^{10} \frac{dx}{x-3} \right)$$

$$= \lim_{b \to 3^-} \left(\underbrace{\ln|b-3|}_{\to -\infty} - \ln|-13| \right) + \lim_{a \to 3^+} \left(\ln|10-3| - \underbrace{\ln|a-3|}_{\to -(-\infty)} \right)$$

$$= \boxed{\text{d.n.e}} \quad \text{(divergent integral)}$$

Remark: a careless student might ignore the division by zero and think ~~$\int_{-10}^{10} \frac{dx}{x-3} = \ln|x-3| \Big|_{-10}^{10} = \ln(7) - \ln(13)$~~ and they'd be wrong. This type of improper integral requires attention to the domain of expressions. In contrast $\int_a^\infty$ $\int_{-\infty}^b$ $\int_{-\infty}^\infty$ are easy to spot.

10.) $$\int_2^\infty \underbrace{y}_{u} \underbrace{e^{-3y}\,dy}_{dV} = -\frac{1}{3} y e^{-3y} \Big|_2^\infty + \frac{1}{3}\int_2^\infty e^{-3y}\,dy$$

$$= -\frac{1}{3} y e^{-3y} \Big|_2^\infty - \frac{1}{9} e^{-3y} \Big|_2^\infty$$

$$= \lim_{t\to\infty}\Big[\underbrace{-\frac{1}{3} t e^{-3t}}_{\text{indeterminant.}} + \frac{1}{3}(2)e^{-6}\Big] - \frac{1}{9}\lim_{t\to\infty}\big(\underset{0}{e^{-3t}} - e^{-6}\big)$$

$$= -\frac{1}{3}\lim_{t\to\infty}\left(t e^{-3t}\right) + \frac{7}{9}e^{-6}$$

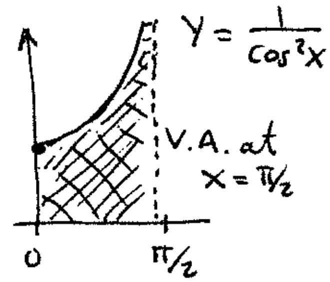
$$= -\frac{1}{3}\lim_{t\to\infty}\left(\frac{t}{e^{3t}}\right) + \frac{7}{9}e^{-6}$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{\mathcal{L}}{=}} -\frac{1}{3}\lim_{t\to\infty}\left(\frac{1}{3e^{3t}}\right)^{\to 0} + \frac{7}{9}e^{-6}$$

$$= \boxed{\frac{7}{9}e^{-6}}$$

11.) find area bounded by $0 \le x < \pi/2$, $0 \le y \le \sec^2 x$

(if it is finite, also sketch the region)



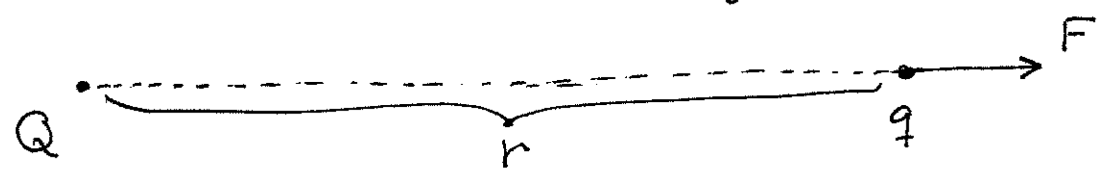
$$\int_0^{\pi/2} \sec^2(x)\,dx = \lim_{b\to} \int_0^b \sec^2 x\,dx$$

$$= \lim_{b\to \pi/2^-}\big(\tan(b) - \tan(0)\big)$$

$$= \boxed{\infty}$$ (the area of the indicated region is not finite)

12.) work done by F from $x=a$ to $x=b$ is given by $W = \int_a^b F(x)dx$ for force in the direction of the x - coordinate line (one - dimensional force / motion)

The force on q due to Q is given by Coulomb's Force Law $F = \dfrac{qQ}{4\pi\varepsilon_0 r^2}$, assuming $q, Q > 0$,



<u>QUESTION</u>: if q is initially <u>very</u> <u>very</u> far away from Q then it is moved radially from its initial position to a final position of $r = R$ then what is the work done by F ?

$$\int_\infty^R F(r)\,dr = -\int_R^\infty \frac{qQ\,dr}{4\pi\varepsilon_0 r^2}$$

$$= \frac{-qQ}{4\pi\varepsilon_0} \lim_{b\to\infty} \int_R^b \frac{dr}{r^2}$$

$$= \frac{-qQ}{4\pi\varepsilon_0} \left(\lim_{b\to\infty} \left[\cancelto{0}{\frac{-1}{b}} + \frac{1}{R} \right] \right.$$

$$= \boxed{\frac{-qQ}{4\pi\varepsilon_0 R}}$$

<u>Remark</u>: another force would be needed for this motion to occur, if $F_{net} \approx 0$ during the motion then the other force would be same magnitude as F but in opposite direction. If $F_{net} \neq 0$ then q would have some KE when it reached $r = R$... I leave the rest of this for your course in <u>PHYSICS</u>.