EXAMPLES OF TRIGONOMETRIC INTEGRALS

1.) $$\int \sin^2\theta \cos^3\theta \, d\theta = \int \sin^2\theta \underbrace{\cos^2\theta} \cos\theta \, d\theta$$

$\cos^2\theta = 1 - \sin^2\theta$ easy consequence of $\cos^2\theta + \sin^2\theta = 1$

$$= \int \underbrace{\sin^2\theta}_{u^2} \underbrace{[1 - \sin^2\theta]}_{[1 - u^2]} \underbrace{\cos\theta \, d\theta}_{du}$$

$$= \int (u^2 - u^4) \, du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta + C}$$

2.) $$\int \sin^2\theta \cos^2\theta \, d\theta = \int (\sin\theta\cos\theta)^2 \, d\theta$$

$\boxed{\underbrace{\sin 2\theta = 2\sin\theta\cos\theta}}$ trig. identity we should all know.

$$= \int \left[\frac{1}{2}\sin(2\theta)\right]^2 d\theta$$

$$= \frac{1}{4}\int \sin^2(2\theta) \, d\theta$$

$\boxed{\sin^2\beta = \frac{1}{2}(1 - \cos 2\beta)}$ $\beta = 2\theta$.

$$= \frac{1}{4}\int \frac{1}{2}(1 - \cos(4\theta)) \, d\theta$$

$$= \frac{1}{8}\int (1 - \cos 4\theta) \, d\theta$$

$$= \boxed{\frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C}$$

3.) $$\int \frac{\sin^2(1/t)}{t^2} \, dt = -\int \sin^2\theta \, d\theta$$

$\boxed{\theta = 1/t, \quad d\theta = -dt/t^2}$

$$= -\frac{1}{2}\int (1 - \cos(2\theta)) \, d\theta$$

$$= -\theta/2 + (1/4)\sin(2\theta) + C$$

$$= \boxed{-\frac{1}{2t} + \frac{1}{4}\sin\left(\frac{2}{t}\right) + C}$$

4.) $\int \cosh^3 x\,dx = \int \cosh^2 x \cosh x\,dx$

$\cosh^2 x - \sinh^2 x = 1$
$\cosh^2 x = 1 + \sinh^2 x$

$= \int (1+\sinh^2 x)\cosh x\,dx$

$u = \sinh x$
$du = \cosh x\,dx$

$= \int (1+u^2)\,du$

$= u + \frac{1}{3}u^3 + C$

$= \boxed{\sinh x + \frac{1}{3}\sinh^3 x + C}$

Remark: the above is a hyperbolic trig. integral but I include it here because the technique to solve this is nearly identical to the technique for the trig. case. FUN FACTS:

$\cosh\theta = \frac{1}{2}(e^{\theta}+e^{-\theta})$

$\sinh\theta = \frac{1}{2}(e^{\theta}-e^{-\theta})$

$\frac{d}{d\theta}\cosh\theta = \sinh\theta$

$\frac{d}{d\theta}\sinh\theta = \cosh\theta$

$\cosh^2\theta - \sinh^2\theta = 1$

$1 - \tanh^2\theta = \operatorname{sech}^2\theta$

$$\frac{d}{d\theta}(\tanh\theta) = \frac{d}{d\theta}\left(\frac{\sinh\theta}{\cosh\theta}\right) = \frac{(\sinh\theta)'\cosh\theta - \sinh\theta(\cosh\theta)'}{\cosh^2\theta} = \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta} = \operatorname{sech}^2\theta$$

Also, $\cosh x \sinh x = \frac{1}{2}\sinh(2x)$

$\left.\begin{aligned}\cos\theta &= \frac{1}{2}(e^{i\theta}+e^{-i\theta})\\ \sin\theta &= \frac{1}{2i}(e^{i\theta}-e^{-i\theta})\end{aligned}\right\} *$

the above follow from Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$

$\frac{d}{d\theta}(\sin\theta) = \cos\theta$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

Can use $*$ to derive trig. identities! For example,

$$\cos\theta\sin\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{4i}(e^{2i\theta} - e^{0} + e^{0} - e^{-2i\theta}) = \frac{1}{2}\left(\frac{1}{2i}\left[e^{2i\theta} - e^{-2i\theta}\right]\right) = \frac{1}{2}\sin(2\theta).$$

5.) a, b constants, $a \neq 0$

$$\int \tan^2(ax+b)\,dx = \int \tan^2\theta \, \frac{d\theta}{a} \qquad \boxed{\theta = ax+b, \ d\theta = a\,dx}$$

$$= \frac{1}{a}\int (\sec^2\theta - 1)\,d\theta$$

$$= \frac{1}{a}(\tan\theta - \theta) + C$$

$$= \frac{1}{a}(\tan(ax+b) - ax - b) + C$$

$$= \boxed{-x + \frac{1}{a}\tan(ax+b) + \tilde{C}}$$

6.) $$\int_0^{\pi/2} (2-\sin\theta)^2\,d\theta = \int_0^{\pi/2} (4 - 4\sin\theta + \sin^2\theta)\,d\theta$$

$$= (4\theta + 4\cos\theta)\Big|_0^{\pi/2} + \int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 2\pi - 4 + \frac{1}{2}\int_0^{\pi/2}(1 - \cos(2\theta))\,d\theta$$

$$= 2\pi - 4 + \frac{1}{2}\Big(\theta - \underbrace{\tfrac{1}{2}\sin(2\theta)}_{0}\Big)\Big|_0^{\pi/2}$$

$$= \boxed{3\pi - 4.}$$

7.) $$\int \sin(2\theta)\sin(6\theta)\,d\theta = \frac{1}{2}\int[\cos(4\theta) - \cos(8\theta)]\,d\theta \quad *$$

$$= \boxed{\frac{1}{8}\sin(4\theta) - \frac{1}{16}\sin(8\theta) + C}$$

* used $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$, and $\cos(-4\theta) = \cos(4\theta)$ since cosine is an <u>even</u> <u>function</u>.

8.) $$\int \sec^2 x \tan^3 x\,dx = \int \sec x \tan^2 x \sec x \tan x\,dx$$

$$= \int \sec x(\sec^2 x - 1)\sec x \tan x\,dx \qquad u = \sec x, \ du = \sec x \tan x\,dx$$

$$= \int (u^3 - u)\,du$$

$$= \frac{1}{4}u^4 - \frac{1}{2}u^2 + C = \boxed{\frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C}$$

9.)

$$\int \cos^4(e^t)\, e^t dt = \int \cos^4(\theta)\, d\theta$$

$$\boxed{\begin{aligned}\theta &= e^t \\ d\theta &= e^t dt\end{aligned}}$$

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$$= \int (\cos^2\theta)^2 d\theta$$

$$= \int \left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \frac{1}{4}\int \left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right] d\theta$$

$$= \frac{1}{4}\int \left[1 + 2\cos(2\theta) + \tfrac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos(2\theta) + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{3\theta}{8} + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C \quad !$$

oops... forgot ☆

$$= \boxed{\frac{3}{8}e^t + \frac{1}{4}\sin(2e^t) + \frac{1}{32}\sin(4e^t) + C}$$

10.) Hint:

$$\int \frac{dx}{\cos x - 1} = \int \frac{(\cos x + 1)\,dx}{(\cos x + 1)(\cos x - 1)}$$

$$= \int \frac{(\cos x + 1)\,dx}{\cos^2 x - 1}$$

$\sin^2 x + \cos^2 x = 1$

$$= \int \frac{(\cos x + 1)\,dx}{-\sin^2 x}$$