16.4. INTEGRATING FACTOR METHOD

The integrating factor method assumes that our starting point is a linear first order ODE which has been written in the so-called *standard form*

$$\boxed{\frac{dy}{dx} + Py = Q \quad \text{standard form}}$$

We assume that P, Q are continuous functions in the equation above. Notice that we cannot just separate variables and integrate. Let's see how the "integrating factor method" gets around the trouble. To start we need to define the integrating factor,

$$\boxed{\mu = exp\left(\int P(x)\,dx\right)}$$

Now observe that the integrating factor has an interesting derivative,

$$\begin{aligned}
\frac{d\mu}{dx} &= \frac{d}{dx}exp\left(\int P(x)\,dx\right) \\
&= exp\left(\int P(x)\,dx\right)\frac{d}{dx}\int P(x)\,dx \\
&= exp\left(\int P(x)\,dx\right)P(x) \\
&= \mu P
\end{aligned}$$

Now multiply the DEqn in standard form by the integrating factor and keep the identity above in mind,

$$\mu\frac{dy}{dx} + \mu P y = \mu Q \quad \Longrightarrow \quad \mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu Q$$

Notice that we can apply the product rule in reverse at this point,

$$\mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu Q \quad \Longrightarrow \quad \frac{d}{dx}(\mu y) = \mu Q \quad \Longrightarrow \quad d(\mu y) = \mu Q dx.$$

What this calculation shows is that the DEqn in standard form becomes separable if we change from y to μy, the term "separable" simply means we can separate and integrate. Integrating both sides we find,

$$\mu y = \int \mu Q\,dx \quad \Longrightarrow \quad \boxed{y = \frac{1}{\mu}\int \mu Q\,dx}$$

This formula gives a general solution for any first order ODE put in standard form. I don't want you to just use this formula. If you choose to use it then I require you to first prove it. The proof we just gave we will repeat again and again, each example follows the same pattern. The great advantage of mimicking the proof for each example is that there is a built-in redundancy to the calculation. If you just use the formula then there is no double check on your work. It's time for examples!

Example 16.4.1

$\boxed{E1}$ Find general sol^{n} to $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x)$. We know we can solve it once it's put into standard form, so make it so,

$$\frac{dy}{dx} - \underbrace{\left(\frac{2}{x}\right)}_{P(x)} y = \underbrace{x^2\cos(x)}_{Q(x)}$$

$$\mu = \exp\left(\int -\frac{2}{x}\,dx\right) = \exp\left(-2\ln|x|\right) = \exp\left(\ln\left(\frac{1}{|x|^2}\right)\right) = \frac{1}{x^2}$$

Multiplying by μ yields,

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = \cos(x)$$

Nice, $\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos(x) \Rightarrow \frac{y}{x^2} = \sin(x) + C$

$$\Rightarrow \boxed{y = x^2(C + \sin(x))}$$

Notice that if we calculated the integrating factor incorrectly then we would not have been able to make the reverse product rule work. This is the check and balance of the method, if you are paying attention you'd have to make a pair of errors simultaneously to get it wrong. Of course it's always possible to get stuck on an integration, but I hope that will not be your stumbling stone here.

Example 16.4.2

$\boxed{E2}$ $y\frac{dx}{dy} + 2x = 5y^3$ (this one is a bit weird, we'll need to think of x as the dependent variable and y as the independent variable)

$\boxed{\text{Standard Form}} \rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right)x = 5y^2$

$$\mu(y) = \exp\left(\int\left(\frac{2}{y}\right)dy\right) = \exp\left(2\ln|y|\right) = \exp\left(\ln|y|^2\right) = y^2$$

$$\underbrace{y^2\frac{dx}{dy} + 2yx}_{} = 5y^4 \quad \text{Multiplied by } \mu(y) = y^2.$$

$$\frac{d}{dy}\left(y^2x\right) = 5y^4 \Rightarrow y^2x = y^5 + C$$

$$\Rightarrow \boxed{x = y^3 + C/y^2}$$

Example 16.4.3

§2.3 #7 (long version) $\dfrac{dy}{dx} - y = e^{3x}$ is in standard form with $P(x) = -1$ and $Q(x) = e^{3x}$ H

Calculate $\mu(x) = \exp\left(\int -1\,dx\right) = \exp(-x)$. Then multiply the DE$_{\underline{n}}$ by $\mu(x)$,

$$\underbrace{e^{-x}\frac{dy}{dx} - e^{-x}y}_{} = e^{-x}e^{3x} = e^{2x}$$

product rule in reverse

$$\frac{d}{dx}\left(e^{-x}y\right) = e^{2x}$$

Integrate → use FTC $\underbrace{e^{-x}y = \frac{1}{2}e^{2x} + C}_{\star} \Rightarrow \boxed{y = \frac{1}{2}e^{3x} + Ce^{x}}$ (dividing by e^{-x})

Remark: Notice that we need not add a constant upon integrating $P(x)$ in $\mu(x) = exp(\int P(x)\,dx)$. If we did, it would cancel when we divide by $\mu(x)$ to solve for y. On the other hand in the final integration (marked by ★) we have no such expectation for that constant C to be cancelled. It is thus our custom to omit the integration constant when calculating the integrating factor.

Example 16.4.4

§2.3 #8 H 9

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1 \Rightarrow \overbrace{\frac{dy}{dx} - \frac{1}{x}y = 2x+1}^{\text{standard form.}}$$

$$\mu(x) = \exp\left(\int \frac{-1}{x}dx\right) = \exp\left(-\ln|x|\right) = \exp\left(\ln\left|\frac{1}{x}\right|\right) = \frac{1}{|x|}$$

Multiply by $\mu(x)$, assume $x > 0 \Rightarrow |x| = x$.

$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y}_{} = \frac{1}{x}(2x+1) = 2 + \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2 + \frac{1}{x}$$

integrating both sides, use for LHS that $\int \frac{d}{dx}(f(x))dx = f(x)$.

$$\frac{1}{x}y = \int\left(2 + \frac{1}{x}\right)dx = 2x + \ln|x| + C$$

$$\boxed{y = 2x^2 + x\ln(x) + Cx} \quad (x > 0)$$

Sol$^{\underline{n}}$ for $x < 0$ similar except $|x| = -x$.

Examples 16.4.5 through 16.4.8

H (10)

§2.3 #10 $\quad y + \frac{x}{2}\frac{dy}{dx} = \frac{1}{2x^3}$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^4} \quad \Rightarrow \mu(x) = \exp\left(\int \frac{2}{x}dx\right) = e^{2\ln|x|} = |x|^2 = x^2$$

$$\underbrace{x^2\frac{dy}{dx} + 2xy}_{} = \frac{x^2}{x^4}$$

$$\frac{d}{dx}\left(x^2 y\right) = \frac{1}{x^2} \quad \Rightarrow \quad x^2 y = \int \frac{dx}{x^2} = \frac{-1}{x} + C \quad \therefore \boxed{y = \frac{-1}{x^3} + \frac{C}{x^2}}$$

§2.3 #11 $\quad (t+y+1)dt - dy = 0 \quad \Rightarrow \quad \frac{dy}{dt} = t+y+1 \quad \Rightarrow \quad \frac{dy}{dt} - y = t+1$

(standard form)

$$\mu = \exp\left(\int -dt\right) = \exp(-t)$$

$$\underbrace{e^{-t}\frac{dy}{dt} - e^{-t}y}_{} = e^{-t}(t+1)$$

$$\int \frac{d}{dt}\left(e^{-t}y\right) = \int\left(te^{-t} + e^{-t}\right)dt$$

$$= \int \underset{u}{t}\,\underbrace{e^{-t}dt}_{dv} + \int e^{-t}dt$$

$$= -te^{-t} + \int e^{-t}dt - e^{-t}$$

$$= -te^{-t} - e^{-t} - e^{-t} + C \quad \Rightarrow e^{-t}y = -e^{-t}(2+t) + C$$

$$\Rightarrow \boxed{y = -2 - t + Ce^{t}}$$

§2.3 #12 $\quad \frac{dy}{dx} + 4y = x^2 e^{-4x} \quad \Rightarrow \mu(x) = \exp\left(\int 4dx\right) = e^{4x}.$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = x^2 \quad \Rightarrow \frac{d}{dx}\left(e^{4x}y\right) = x^2$$

$$\Rightarrow \quad e^{4x}y = \int x^2 dx = \frac{x^3}{3} + C$$

$$\Rightarrow \quad \boxed{y = e^{-4x}\left(\tfrac{1}{3}x^3 + C\right)}$$

§2.3 #18 $\quad \frac{dy}{dx} + 4y = e^{-x} \quad \Rightarrow \mu = \exp\left(\int 4dx\right) = e^{4x}$

$$e^{4x}\frac{dy}{dx} + 4e^{4x}y = e^{4x}e^{-x} \quad \Rightarrow \quad \frac{d}{dx}\left(e^{4x}y\right) = e^{3x} \quad \Rightarrow e^{4x}y = \frac{1}{3}e^{3x} + C$$

$$y(0) = 4/3 \quad \Rightarrow \quad e^0\frac{4}{3} = \frac{1}{3}e^0 + C \quad \Rightarrow \quad \underline{C = 1}$$

$$\boxed{y = \frac{1}{3}e^{-x} + e^{-4x}}$$

Example 16.4.9 and 16.4.10

§2.3 #22 $\sin(x)\frac{dy}{dx} + y\cos(x) = x\sin(x)$

H (III)

$$\frac{dy}{dx} + \frac{\cos(x)}{\sin(x)}y = x$$

$$\mu = \exp\left(\int \frac{\cos(x)dx}{\sin(x)}\right) = \exp\left(\int \frac{du}{u}\right) = \exp(\ln|\sin(x)|) = |\sin(x)|$$

$u = \sin(x)$, $du = \cos(x)dx$

$$|\sin(x)|\frac{dy}{dx} + \frac{|\sin(x)|}{\sin(x)}\cos(x)y = |\sin(x)|x$$

$$\sin(x)\frac{dy}{dx} + \cos(x)y = x\sin(x)$$

$$\frac{d}{dx}(\sin(x)y) = x\sin x \Rightarrow y\sin(x) = \cancel{\frac{x^2}{2} + c}$$

$$\int \underbrace{x}_{u}\underbrace{\sin x\,dx}_{dv} = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

$$\cancel{y(\pi/2) = 2 \Rightarrow 2\sin(\pi/2) = \frac{\pi^2}{8} + c \quad \therefore \quad c = 2 - \pi^2/8}$$

Thus $\cancel{\boxed{y = \frac{1}{\sin(x)}\left(\frac{x^2}{2} + 2 - \frac{\pi^2}{8}\right)}}$

$$\therefore \boxed{y = \frac{C + \sin x - x\cos x}{\sin x}}$$

$y(\pi/2) = 2$

$2 = \frac{C+1}{1}$

$\therefore C = 1$

§2.3 #29 $\frac{dy}{dx} = \frac{1}{e^{4y} + 2x} \longrightarrow \frac{dx}{dy} = e^{4y} + 2x$

Then $\frac{dx}{dy} - 2x = e^{4y}$ thus $\mu(y) = \exp\left(\int -2dy\right) = e^{-2y}$.

$$\underbrace{e^{-2y}\frac{dx}{dy} - 2e^{-2y}x}= e^{2y}$$

$$\frac{d}{dy}\left(e^{-2y}x\right) = e^{2y} \rightarrow e^{-2y}x = \frac{1}{2}e^{2y} + C$$

$$\rightarrow \boxed{x = \frac{1}{2}e^{4y} + ce^{2y}}$$

Summary

1. Put linear first order ODEn into standard form, identify the "P" and "Q".
2. Calculate the integrating factor $\mu = \int P\,dx$.
3. Multiply the standard form DEqn by the integrating factor.
4. Group terms and apply the product rule in reverse.
5. Integrate both sides, don't forget to add the constant.
6. Solve for y.
7. Apply initial condition if you have one.

You could do 7.) after 5.) instead, for certain problems that is very labor-saving.

Well, it was fun, but there may be a question gnawing away as you follow the recipe. Why? Why even think to invent this integrating factor? And who thought to multiply by it and so forth? I'm not certain the history of the method, it would be interesting to research for fun. There are ways of deriving the integrating factor, but the idea stems from some fairly sophisticated geometry. We'd need to understand what a "symmetry" of a differential equation