EXAMPLES OF INTEGRATION BY PARTS

1.) $$\int \underbrace{x}_{u} \underbrace{\cos(5x)\,dx}_{dV} = uv - \int v\,du$$

$u = x,\ du = dx$
$dV = \cos(5x)dx$
$V = \frac{1}{5}\sin(5x)$

$$= \frac{1}{5}x\sin(5x) - \int \frac{1}{5}\sin(5x)\,dx$$

$$= \boxed{\frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C}$$

Remark: to be ~~immediately~~ successful with IBP we need to select dv such that $V = \int dV$ is a calculation which is known.

2.) $$\int \underbrace{\sin^{-1}(3x)}_{u}\,\underbrace{dx}_{dV} = uv - \int V\,du$$

$u = \sin^{-1}(3x)$	$dV = dx$
$du = \frac{3dx}{\sqrt{1-9x^2}}$	$V = x$

$$= x\sin^{-1}(3x) - \int \frac{x \cdot 3\,dx}{\sqrt{1-9x^2}}$$

$W = 1 - 9x^2$
$dW = -18x\,dx$

$3x\,dx = \frac{-dW}{6}$

$$= x\sin^{-1}(3x) - \int \frac{-dW/6}{\sqrt{W}}$$

$$= x\sin^{-1}(3x) + \frac{1}{6}\int \frac{dW}{\sqrt{W}}$$

$$= x\sin^{-1}(3x) + \frac{1}{3}\sqrt{W} + C$$

$$= \boxed{x\sin^{-1}(3x) + \frac{1}{3}\sqrt{1-9x^2} + C}.$$

3.) $$\int \underbrace{\ln(x)}_{u}\,\underbrace{dx}_{dV} = uv - \int v\,du$$

$u = \ln x$ $\quad$ $dV = dx$
$du = dx/x$ $\quad$ $V = x$

$$= x\ln(x) - \int \frac{x\,\cancel{dx}}{x}$$

$$= \boxed{x\ln x - x + C}$$

Remark: we assume $x > 0$ throughout (3.)

4.) "I"

$$\underbrace{\int e^{-x}}_{u_1}\,\underbrace{\cos(3x)\,dx}_{dV_1} = \overbrace{\frac{1}{3}e^{-x}\sin(3x)}^{u_1V_1} - \int \overbrace{\frac{1}{3}\sin(3x)\left(-e^{-x}dx\right)}^{V_1\quad du_1}$$

$$= \frac{1}{3}e^{-x}\sin(3x) + \frac{1}{3}\int \underbrace{e^{-x}}_{u_2}\,\underbrace{\sin(3x)\,dx}_{dV_2}$$

$$= \frac{1}{3}e^{-x}\sin(3x) + \frac{1}{3}\left[e^{-x}\left(\frac{-1}{3}\cos 3x\right) + \frac{1}{3}\int \cos 3x\left(-e^{-x}dx\right)\right]$$

$$= \frac{1}{3}e^{-x}\sin(3x) - \frac{1}{9}e^{-3x}\cos 3x - \frac{1}{9}\int e^{-x}\cos 3x\,dx$$

Hence $\quad I = \frac{1}{3}e^{-x}\sin 3x - \frac{1}{9}e^{-3x}\cos 3x - \frac{1}{9}I$

which implies $\quad \frac{10}{9}I = \frac{1}{3}e^{-x}\sin 3x - \frac{1}{9}e^{-3x}\cos 3x$

thus $\quad I = \frac{9}{10}\left(\frac{1}{3}e^{-x}\sin 3x - \frac{1}{9}e^{-3x}\cos 3x\right) + C$

That is, $\quad \boxed{\int e^{-x}\cos 3x\,dx = \left(\frac{3}{10}\sin(3x) - \frac{1}{10}\cos(3x)\right)e^{-x} + C}$

5.)

$$\int \underbrace{x^2}_{u_1}\,\underbrace{\sin(x)\,dx}_{dV_1} = -x^2\cos x + \int \cos(x)\,(2x\,dx)$$

$u_1 = x^2$
$du_1 = 2x\,dx$
$dV_1 = \sin x\,dx$
$V_1 = -\cos x$

$$= -x^2\cos x + 2\int \underbrace{x}_{u_2}\,\underbrace{\cos(x)\,dx}_{dV_2}$$

$$= -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right)$$

$$= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$$

6.) $$\int_0^2 \underbrace{y}_{u}\,\underbrace{\sinh y\,dy}_{dv} = y\cosh y\Big|_0^2 - \int_0^2 \cosh y\,dy$$

$$= 2\cosh(2) - \sinh(y)\Big|_0^2$$

$$= 2\cosh(2) - \sinh(2) + \cancel{\sinh(0)}^{\,0}$$

$$= \boxed{2\cosh(2) - \sinh(2)}$$

7.) Derive $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \left(\frac{n-1}{n}\right)\int \cos^{n-2}(x)\,dx$

$$\cancel{\int \cos^n(x)dx = \int (\cos^{n-2}x)(1-\sin^2 x)dx} \quad \left(\text{since } \cos^2 x = 1-\sin^2 x\right)$$

$$\cancel{= \int \cos^{n-2}(x)dx - \int \sin^2 x \cos^{n-2} x\,dx}$$

NOPE, TRY AGAIN!

$$I = \int \cos^n(x)dx = \int \underbrace{\cos^{n-1}(x)}_{u}\ \underbrace{\cos(x)\,dx}_{dv}$$

$$= \sin(x)\cos^{n-1}(x) - \int \sin(x)\cdot(n-1)\cos^{n-2}(x)\,(-\sin x\,dx)$$

$$= \sin x\cos^{n-1}(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)\,dx$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)\int (1-\cos^2 x)\cos^{n-2}(x)\,dx$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)dx \underbrace{- (n-1)\int \cos^n(x)dx}_{-(n-1)\,I}$$

Solving for I yields,

$$\boxed{\int \cos^n(x)\,dx = \frac{1}{n}\sin(x)\cos^{n-1}(x) + \left(\frac{n-1}{n}\right)\int \cos^{n-2} x\,dx}$$

<u>Remark</u>: similar recursive formulas for sine, tangent and other functions can be derived. I recommend using a more direct approach for $\int \sin^2 x\,dx$ or $\int \sin^3 x\,dx$ etc. (see the pdf on TRIG. INTEGRALS)

8.) $\int \underbrace{t}_{u} \underbrace{\csc^2 t\, dt}_{dv} = -t\cot(t) + \int \cot(t)\,dt \leftarrow \left\{ \begin{array}{l} dv = \csc^2 t\, dt \\ v = -\cot(t) \\ u = t,\ du = dt \end{array} \right.$

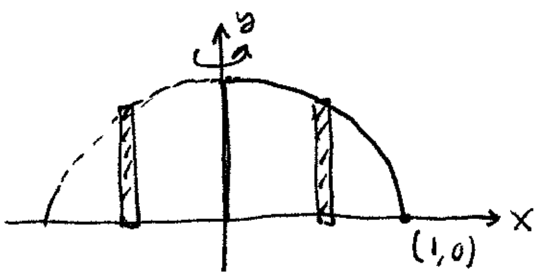
$= -t\cot(t) + \int \frac{\cos(t)\,dt}{\sin t}$ — $\left\{ \begin{array}{l} w = \sin t \\ dw = \cos t\, dt \end{array} \right.$

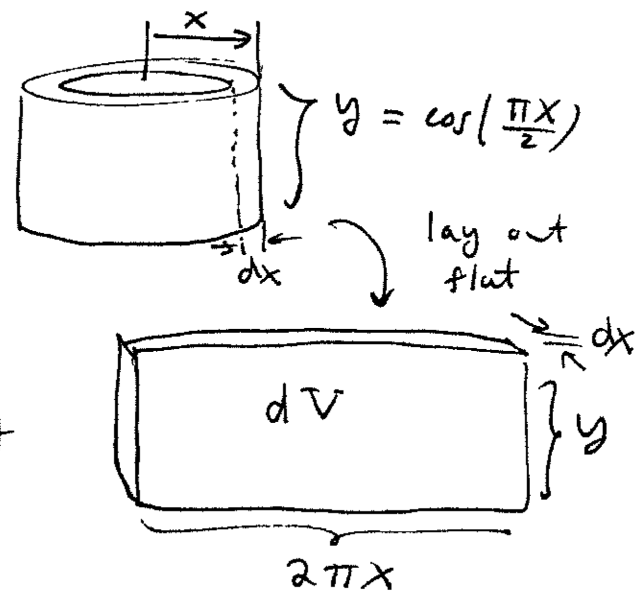
$= -t\cot(t) + \int \frac{dw}{w}$

$= -t\cot(t) + \ln|w| + c$

$= \boxed{-t\cot(t) + \ln|\sin t| + c}$

9.) use cylindrical shells to find volume bounded by rotating bounded area $y = \cos(\pi x/2)$, $y = 0$ for $0 \le x \le 1$ about the y-axis





$dV = (2\pi x)\left(\cos\left(\frac{\pi x}{2}\right)\right)dx$

for $0 \le x \le 1$ we can construct such a cylindrical shell. To add up all the little dV's we integrate,

$V = \int_0^1 \underbrace{2\pi x}_{u} \underbrace{\cos\left(\frac{\pi x}{2}\right)dx}_{dv} = 2\pi x\left(\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right)\Big|_0^1 - \int_0^1 \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)2\pi\,dx$

$= 2\pi\left(\frac{2}{\pi}\sin\left(\frac{\pi}{2}\right)\right) - 4\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx$

$= (2\pi)\left(\frac{2}{\pi}\right) - 4\left(\frac{-2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right)\Big|_0^1 = 4 + \frac{8}{\pi}\left(\overset{0}{\cancel{\cos\frac{\pi}{2}}} - \overset{1}{\cancel{\cos(0)}}\right)$

$= \boxed{4 - 8/\pi}$

10.) Calculate the average of $f(x) = x\sec^2 x$ on the interval $[0, \pi/4]$

$$f_{avg} = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} f(x)\,dx \quad (*)$$

I'll calculate $\int f(x)dx$ to begin then return to $*$

$$\int \underbrace{x}_{u}\, \underbrace{\sec^2 x\,dx}_{dv} = x\tan(x) - \int \tan x\,dx$$

$$= x\tan x - \int \frac{\sin x\,dx}{\cos x}$$

$w = \cos x$
$dw = -\sin x\,dx$

$$= x\tan x + \int \frac{dw}{w}$$

$$= x\tan x + \ln|\cos x| + C.$$

Thus,

$$f_{avg} = \frac{4}{\pi}\int_0^{\pi/4} x\sec^2 x\,dx$$

$$= \frac{4}{\pi}\Big[\, x\tan x + \ln|\cos x|\,\Big]\Big|_0^{\pi/4}$$

$$= \frac{4}{\pi}\Big[\,\frac{\pi}{4}\cancelto{1}{\tan\left(\frac{\pi}{4}\right)} + \ln|\cos \pi/4|\,\Big]$$

$\tan(0) = 0$
$\cos(0) = 1$
$\ln(1) = 0.$

$$= \boxed{1 + \frac{4}{\pi}\ln\left(\frac{1}{\sqrt{2}}\right)}$$

11.) Suppose $f(0) = g(0) = 0$ and f'', g'' are continuous
Show $\int_0^a f(x)\, g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x) g(x)\,dx$

$$\int_0^a \underbrace{f(x)}_{u}\, \underbrace{g''(x)\,dx}_{dv} = f(x)g'(x)\Big|_0^a - \int_0^a g'(x) f'(x)\,dx \quad *$$

where we have noted $f(x) = u$ implies $du = f'(x)\,dx$ whereas $\int dv = \int g''(x)\,dx \Rightarrow v = g'(x)$. Then $*$ follows from IBP. Continuing,

$$\int_0^a f(x)\,g''(x)\,dx = f(a)g'(a) - \cancelto{0}{f(0)}g'(0) - \int_0^a \underbrace{f'(x)}_{u_2}\, \underbrace{g'(x)\,dx}_{dv_2}$$

$$= f(a)g'(a) - \left[f'(x)\,g(x)\Big|_0^a - \int_0^a g(x) f''(x)\,dx \right]$$

$$= f(a)g'(a) - \left[f'(a)g(a) - f'(0)g(0) - \int_0^a f''(x)g(x)\,dx \right]$$

$$= f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx.$$

12.) HYPERBOLIC FUNCTIONS: Recall, $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$
hence $\cosh^2 x - \sinh^2 x = 1$ etc.

$$\int \underbrace{\cosh^{-1}(x)}_{u}\, \underbrace{dx}_{dv} = x\cosh^{-1}(x) - \int x \frac{dx}{\sqrt{x^2-1}}$$

$$= x\cosh^{-1}(x) - \int \frac{x\,dx}{\sqrt{x^2-1}} \; : \; w = x^2 - 1$$

$$= x\cosh^{-1}(x) - \frac{1}{2}\int \frac{dw}{\sqrt{w}}$$

$$= x\cosh^{-1}(x) - \sqrt{w} + C$$

$$= \boxed{x\cosh^{-1}(x) - \sqrt{x^2-1} + C}$$

$y = \cosh^{-1}(x)$

$\cosh y = x$

$\sinh y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sinh y}$

$\frac{d}{dx}\left[\cosh^{-1}(x)\right] = \frac{1}{\sqrt{x^2-1}}$