Multivariate Integration

To begin we will study double & triple integrals over boxlike or rectangular regions, these are as easy as integrations from calc. I & II. Then we find how to integrate over type I & II regions in the xy-plane. and general volumes in xyz-space, this is not as easy but once you understand the importance of graphing and set-up it becomes clear. After exhausting topics in Cartesian Coordinates we'll study the Jacobian, this will give us a derivation of how to integrate in sphericals or cylindricals or polars or whatever system of coordinates you might invent. Then we apply our Jacobian theory to do integrals in polars, cylindricals and sphericals. We scatter select applications throughout our discussion.

Definitions: integrals are defined as the limit of a weighted sum over f,

$$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad : \quad \Delta x = \frac{b-a}{n}$$

$$\iint_R f(x,y)\,dA \equiv \lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{k} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$$

$$R = [a,b]\times[c,d] \quad \text{and} \quad \Delta x = \frac{b-a}{n} \quad \text{while} \quad \Delta y = \frac{d-c}{k}$$

$$\iiint_B f(x,y,z)\,dV \equiv \lim_{m\to\infty}\lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{k} f(x_i^*, y_j^*, z_l^*)\,\Delta x\,\Delta y\,\Delta z$$

$$B = [a,b]\times[c,d]\times[p,q] \quad \text{and} \quad \Delta x = \frac{b-a}{n},\ \Delta y = \frac{d-c}{n},\ \Delta z = \frac{q-p}{k}$$

We note that in Cartesian coordinates $dA = dx\,dy$ = infinitesimal area element in xy-plane. $dV = dx\,dy\,dz$ = infinitesimal volume element. As in calc I and II. the sample points are chosen randomly, but it doesn't matter in the limit. In practice the limit is rarely seen, instead the F.T.C. or evaluation rule and here the Fubini Th$^{\text{m}}$ will keep us from ever using the def$^{\text{n}}$ directly (THANKFULLY!)

Several Properties of the integral follow directly from the properties of the limit itself: Let R be a rectangular region

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$$

$$f(x,y) \geq g(x,y) \quad \forall (x,y)\in R \Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

Likewise for $f(x,y,z)$ and $g(x,y,z)$ over a boxlike region. We assume that f, g are continuous most everywhere. Meaning we can integrate $f(x,y)$ if it has a finite number of curve discontinuities, or $f(x,y,z)$ if it has a finite number of planar discontinuities. We just chop the integral into a finite # of regions on which f is continuous.

FUBINI'S THM (WEAK FORM): known to Cauchy for continuous f in early 19th century. Let $R = [a,b]\times[c,d]$ and let f be a mostly continuous fnct. of (x,y)

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

where the expressions on the RHS are "iterated integrals" which you work inside out, treating the outside variable as a constant to begin.

$\boxed{E87}$ Let $R=[0,\pi]\times[0,2]$ that is $0\leq x\leq\pi$ and $0\leq y\leq 2$. Integrate $f(x,y)=\sin(x)+y$ over R.

$$\iint_R f(x,y)\,dA = \int_0^\pi\left(\int_0^2 [\sin(x)+y]\,dy\right)dx$$: parentheses added to emphasize the order of operations here.

$$= \int_0^\pi \left[y\sin(x)+\tfrac{1}{2}y^2\right]\Big|_0^2\,dx$$: note $\sin(x)$ is regarded as a constant in the dy integration.

$$= \int_0^\pi [2\sin(x)+2]\,dx$$

$$= -2\cos(x)\Big|_0^\pi + 2x\Big|_0^\pi$$

$$= -2\cos\pi + 2\cos(0) + 2\pi$$

$$= \boxed{4+2\pi}$$

Exercise: compute $\int_0^2\int_0^\pi (\sin(x)+y)\,dx\,dy$ you should get same answer.

E88 Let $R = \{(x,y) \mid 0 \le x \le \pi/2,\ 0 \le y \le 2\}$.

$$\iint_R y\cos(xy)\,dA = \int_0^2 \left(\int_0^{\pi/2} y\cos(xy)\,dx\right)dy \quad : \int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

$$= \int_0^2 \left(\frac{y}{y}\sin(xy)\Big|_0^{\pi/2}\right)dy$$

$$= \int_0^2 \left(\sin\left(\frac{\pi y}{2}\right) - \sin(0)\right)dy$$

$$= \frac{-2}{\pi}\cos\left(\frac{\pi y}{2}\right)\Big|_0^2 = \frac{-2}{\pi}\left(\cos(\pi) - \cos(0)\right) = \boxed{\frac{4}{\pi}}$$

Remark: the order of $\int$ here is easier than if we reversed to $dy\,dx$

GEOMETRY: the double integral of $f(x,y)$ over R is the volume of the solid bounded by $z = f(x,y)$, $z = 0$ and $(x,y) \in R$. The Th$^{\underline{m}}$ of Fubini can be seen as merely saying you can slice up the volume along x or y crossections. Infinitesimally

$$dV = \underbrace{(z_{TOP} - z_{bottom})}_{\text{height of the box}}\underbrace{dx\,dy}_{\text{area of the box}}$$

So if $z_{bottom} = 0$ and $z_{TOP} \ge 0$ then we get the volume, however as in $\int f(x)\,dx$ we count volume below the xy-plane as negative so the integral calculates the "signed" volume. I'm not even going to pretend I can draw these things ... I'll let Maple do the artistry. I do hope we can all see these things in our "minds eye" in the end.

E89 Let $B = \{(x,y,z) \mid 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c\}$

$$\iiint dV = \int_0^c \int_0^b \int_0^a dx\,dy\,dz$$

$$= \int_0^c \int_0^b x\Big|_0^a\,dy\,dz$$

$$= \int_0^c \left(\int_0^b a\,dy\right)dz$$

$$= \int_0^c ab\,dz$$

$$= \boxed{abc = V}$$

- If we integrate 1 over B we find the volume of B. Likewise if we had integrated 1 over a rectangle $R \subset \mathbb{R}^2$ we would have found the area.

E90 Let $B = [0,1]\times[0,2]\times[0,3]$. Let $\rho = \frac{dm}{dV} = xyz$. Consider,

$$\iiint_B xyz\, dV = \int_0^3\int_0^2\int_0^1 xyz\, dx\, dy\, dz$$

$$= \int_0^3 z\,dz \int_0^2 y\,dy \int_0^1 x\,dx$$

: only allowed if our functions factors into functions of x, y, z
$f(x,y,z) = f_1(x) f_2(y) f_3(z)$

$$= \frac{1}{2}z^2\Big|_0^3 \; \frac{1}{2}y^2\Big|_0^2 \; \frac{1}{2}x^2\Big|_0^1$$

$$= \frac{1}{8}(3)^2(2)^2$$

$$= \boxed{27/2}$$

What is the meaning of such an integration? Well if ρ = mass density $= dm/dV$ then $\rho\, dV = dm$ thus

$$m = \int_B dm = \iiint_B \rho\, dV = \boxed{27/2}$$ = mass of object B with density $\rho = xyz$.

Or you could interpret it as $\rho = dq/dV$ = charge/volume so $q = \int \rho\, dV = 27/2$ = charge of object B. I'm sure you could imagine other densities.

E91 Another interpretation of $\iint_R f(x,y)\, dA$ is that $f(x,y)$ represents an area density. So say $f(x,y) = \sigma(x,y)$

$$\sigma(x,y) = \frac{dq}{dA} \Rightarrow q = \iint_R \sigma(x,y)\, dA$$ = charge on the planar region R

$$\sigma(x,y) = \frac{dm}{dA} \Rightarrow m = \iint_R \sigma(x,y)\, dA$$ = mass of the rectangle R.

E92 Another interpretation of $\int_a^b f(x)dx$ is that $f(x)$ represents a linear density. So say $f(x) = \lambda(x)$ and,

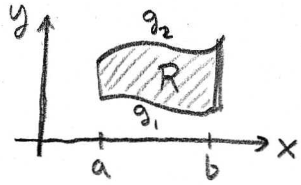
$$\lambda(x) = \frac{dq}{dx} \Rightarrow q = \int_a^b \lambda(x)\,dx \;, \quad \lambda = \frac{dm}{dx} \Rightarrow m = \int_a^b \lambda(x)dx$$

<u>Remark</u>: linear density is more exciting once we know about line-integrals (which are actually along curves generally). Note "density" is multifaceted.

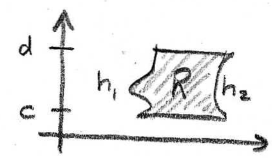
Double Integrals over General Regions:

Given an arbitrary connected region in the xy-plane there are two primary descriptions of the region, say R (not necessarily a rectangle any more). Your text classifies them as,

TYPE I: $\begin{cases} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{cases}$



TYPE II: $\begin{cases} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{cases}$



Of course, you can imagine regions which don't conveniently fit either TYPE. And on the other hand a rectangle is both TYPES at once, $g_1(x) = c$, $g_2(x) = d$ to get TYPE I, $h_1(y) = a$, $h_2(y) = b$ to get TYPE II.

Th^{m} (Fubini, strong version): Suppose f is mostly continuous.
Given $R_I = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ a TYPE I region,

$$\iint_{R_I} f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

Given $R_{II} = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$ a TYPE II region,

$$\iint_{R_{II}} f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

E93 Let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$.

$$\iint_R e^{x^2} dA = \int_0^1 \int_0^x e^{x^2} dy\, dx = \int_0^1 \left(e^{x^2} y\Big|_0^x\right) dx = \int_0^1 x e^{x^2} dx$$

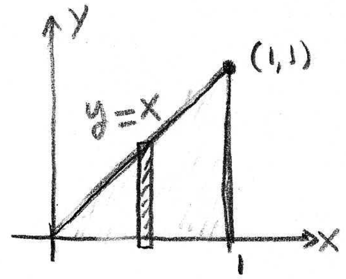
$$= \frac{1}{2} e^{x^2}\Big|_0^1$$

$$= \frac{1}{2}(e^1 - e^0)$$

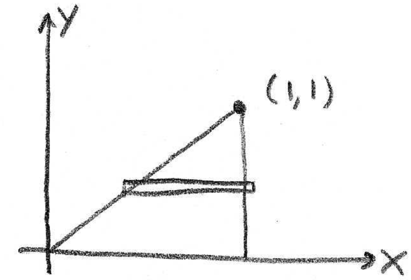
$$= \boxed{\frac{1}{2}(e-1)}$$

Remark: we could just as well describe R as a TYPE II region. However, then we'd be faced with $\int e^{x^2} dx$. This is not an elementary integral.

$\boxed{E94}$ Using R from $\boxed{E93}$ calc. $\iint_R e^{y^2}\,dA$. Since treating R as TYPE I leads us to $\int e^{y^2}dy$ we need to make dx come first, so convert R to a TYPE II region. A picture helps,



$y_{TOP} = x$
$y_{bottom} = 0$



$x_{LEFT} = y$
$x_{RIGHT} = 1$

$$\underbrace{\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x\}}_{\text{TYPE I description}} = R = \underbrace{\{(x,y)\mid 0\le y\le 1,\ y\le x\le 1\}}_{\text{TYPE II description}}$$

$$\iint_R e^{y^2}dA = \int_0^1\int_y^1 e^{y^2}\,dx\,dy \quad :$$

$$= \int_0^1 (1-y)e^{y^2}dy \quad :$$

the integral of the constant e^{y^2} is the product of the integration region length $(1-y)$ and the constant.

$$= \int_0^1 e^{y^2}dy - \int_0^1 ye^{y^2}dy$$

$$= \int_0^1 e^{y^2}dy - \tfrac{1}{2}(e-1) \quad : \text{ curses, I had hoped for better.}$$

$$= 1.463 - \tfrac{1}{2}(e-1) \quad : \int_0^1 e^{y^2}dy \text{ req's numerical sol}^{\underline{n}}.$$

__Remark:__ not all integrals result in pretty sums & products, if we just make up some example on a hunch then it can get ugly. Incidentally while __indefinite__ integrals of e^{x^2} are not known in terms of elementary functions, there are __improper__ integrals of e^{-x^2} which do come out quite nicely. See §16.4 #36, we need a few toys to make it easier.

E95 Another application of double integrals is finding the area of a region. For example, $S = \{(x,y) \mid 0 \le x \le R,\ 0 \le y \le \sqrt{R^2-x^2}\}$

$$A(R) = \iint_S dA = \int_0^R \int_0^{\sqrt{R^2-x^2}} dy\,dx$$

$$= \int_0^R \sqrt{R^2-x^2}\,dx$$: use trig-substitution $x = R\cos\theta$ so $dx = -R\sin\theta\,d\theta$ and $\sqrt{R^2-x^2} = \sqrt{R^2\sin^2\theta} = R\sin\theta$ and the bounds change to $\pi/2 \to 0$

$$= \int_{\pi/2}^0 -R^2\sin^2\theta\,d\theta$$

$$= R^2\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$: $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$= \frac{R^2}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{R^2}{2}\left(\frac{\pi}{2}\right) = \boxed{\pi R^2/4.}$$: since S is a quarter-circle this makes good sense.

Remark: this will be much easier in polar coordinates, see E106 on 344.

E96 We may also define the average of a function over R as

$$\boxed{f_{avg}^R = \frac{1}{A(R)}\iint_R f(x,y)\,dA}$$

Consider $f(x,y) = xy$. If $R = [0,1]\times[0,1]$ and S = quarter circle with $R=1$ from E95, do you think $f_{avg}^R > f_{avg}^S$ or vice-versa?

$$\iint_R xy\,dA = \int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 x\,dx \int_0^1 y\,dy = \frac{1}{2}\frac{1}{2} = \frac{1}{4}.$$

$$\iint_S xy\,dA = \int_0^1\int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \int_0^1\left(\frac{1}{2}xy^2\Big|_0^{\sqrt{1-x^2}}\right)dx = \int_0^1 \frac{1}{2}(x-x^3) = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{8}$$

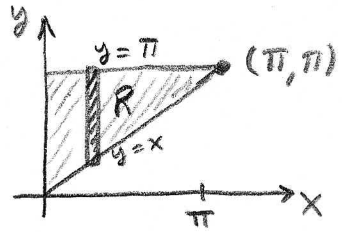
Thus

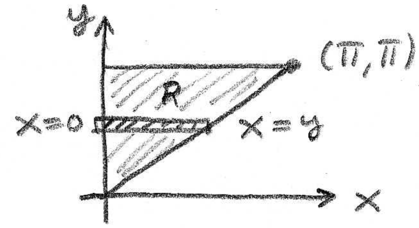
$$f_{avg}^R = \frac{1/4}{A(R)} = \frac{1/4}{1} = \frac{1}{4} \quad \text{whereas} \quad f_{avg}^S = \frac{1/8}{A(S)} = \frac{1/8}{\pi/4} = \frac{1}{2\pi}$$

The average of xy is larger on the unit-square since $\frac{1}{4} > \frac{1}{2\pi}$.

Remark: Notice $\iint_S xy\,dA$ was considerably easier than $\iint_S dA$.

E97 Calculate $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$. Notice we need to reverse the order of integration to do a TYPE II integration (has $dx\,dy$ instead). Our given integral suggests $0 \le x \le \pi$ and $x \le y \le \pi$,





$\underline{0 \le y \le \pi,\quad 0 \le x \le y}$

$$\iint_R \frac{\sin y}{y}\, dA = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \sin y\, dy = -\cos y\Big|_0^{\pi} = \boxed{2}.$$

E98 Calculate,

$$\int_0^{3/2}\int_0^{9-4x^2} 16x\, dy\, dx = \int_0^{3/2} 16x(9-4x^2)\, dx$$

: $16x$ is a constant w.r.t dy integration, simply multiply by int. reg. length.

$$= \int_0^{3/2} (144x - 64x^3)\, dx$$

$$= 72x^2\Big|_0^{3/2} - 16x^4\Big|_0^{3/2}$$

$$= 72(3/2)^2 - 16(3/2)^4$$

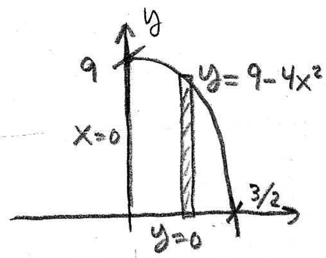
$$= 162 - 81 = \boxed{81}$$

Lets reverse the order of integration for fun. Note $9-4x^2 = 0 \Rightarrow x^2 = \frac{9}{4}$

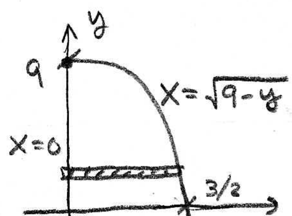
$y = 9-4x^2$ is a parabola with x-intercepts $x = \pm 3/2$ and y-intercept 9. Solve for x and keep positive root,

$$x^2 = \frac{1}{4}(9-y)$$

$$x = \frac{1}{2}\sqrt{9-y}$$



TYPE I: find y bounds in terms of x



TYPE II: find x bounds in terms of y.

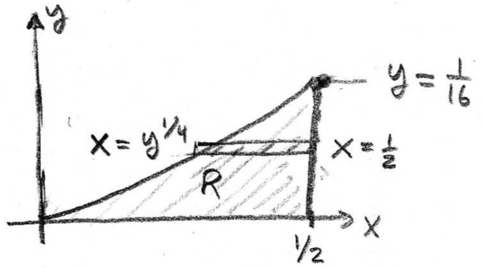
$$\iint_R 16x\, dA = \int_0^{9}\int_0^{\frac{1}{2}\sqrt{9-y}} 16x\, dx\, dy = \int_0^9 \left(8x^2\Big|_0^{\frac{1}{2}\sqrt{9-y}}\right) dy = \int_0^9 2(9-y)\, dy$$

$$= (18y - y^2\Big|_0^9$$

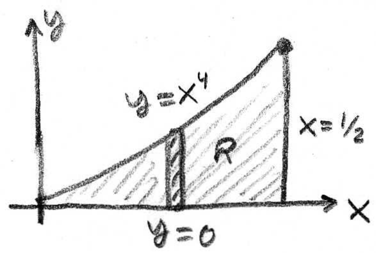
$$= 18(9) - 81$$

$$= \boxed{81}$$

E99 Calculate $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy$. Seems changing bounds may be helpful here. To begin $0 \le y \le 1/16$ and $y^{1/4} \le x \le \frac{1}{2}$ which is type II, lets graph to guide our conversion to TYPE I,



$\frac{1}{2} = y^{1/4} \Rightarrow y = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.
thus $x = 1/2$ and $x = y^{1/4}$ intersect at the point $(1/2, 1/16)$ as graphed.



$0 \le x \le 1/2$
$0 \le y \le \underline{x^4}$
well isn't that convenient.

$$\iint_R \cos(16\pi x^5)\,dA = \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)\,dy\,dx$$: notice $\cos(16\pi x^5)$ is constant in the dy-integration.

$$= \int_0^{1/2} x^4 \cos(16\pi x^5)\,dx$$: let $u = 16\pi x^5$,

$$= \frac{1}{80\pi}\sin(16\pi x^5)\Big|_0^{1/2}$$

$$= \frac{1}{80\pi}\left(\sin\left(\frac{16\pi}{32}\right) - \sin(0)\right)$$

$$= \boxed{\frac{1}{80\pi}}$$

Remark: these arguments should be familiar from Calc. II, see p. 132 – 139d. Our methods for finding area were more specialized, now we add $f(x,y)$ into the integration but the essential idea of viewing the graph as TYPE I or TYPE II was there as well. I'd say type II regions needed horizontal slicing whereas type I were vertically sliced. We can see the formula on 132 as type I

$$A = \int_a^b \int_{g(x)}^{f(x)} dy\,dx = \int_a^b (f(x) - g(x))\,dx = \int_a^b (y_{Top} - y_{bottom})\,dx$$

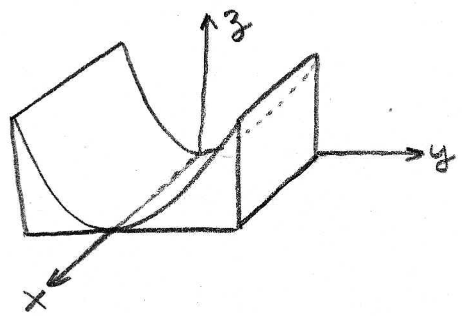
or for TYPE II : $A = \int_c^d \int_{x_L(y)}^{x_R(y)} dx\,dy = \int_c^d (x_R - x_L)\,dy$

these formulas are special cases of our double integrals.

Triple Integrals over General Bounded Regions in $\mathbb{R}^3$

Rather than explicitly stating the 3-d Fubini Th$^{\underline{m}}$ I will simply illustrate with a few examples. Usually we can bound z in terms of x & y then we can bound y in terms of x or vice-versa, that gives two orders of integration. Then other problems allow x to be bound in terms of y & z or possibly y in terms of x & z, in total there are 6 ways to write a particular integral over a volume. In §12.7 # 31 I explicitly show 6 ways to write a particular integral. I don't give general advice on how to rewrite and switch bounds, its a subtle buisness and I would advocate double checking with Mathematica. Generalities aside, lets do a few typical problems.

E100 Let us find the volume of the region between $z = y^2$ and the xy-plane bounded by $x=0$, $x=1$, $y=1$ and $y=-1$. Notice $dV = dx\,dy\,dz$ so integrating dV gives volume V,



$$0 \le z \le y^2$$
$$0 \le x \le 1$$
$$-1 \le y \le 1$$

- we must integrate w.r.t. z either first or second.

$$V = \int_{-1}^{1}\int_{0}^{1}\int_{0}^{y^2} dz\,dx\,dy \quad : \text{ we work inside out as usual.}$$

$$= \int_{-1}^{1}\int_{0}^{1} y^2\,dx\,dy \quad : \text{ back to 2-d integrals.}$$

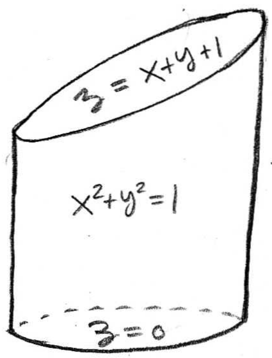
$$= \int_{-1}^{1} y^2\,dy \quad : \text{ back to 1-d integral.}$$

$$= \frac{1}{3}y^3\Big|_{-1}^{1}$$

$$= \frac{1}{3}\left(1-(-1)^3\right)$$

$$= \boxed{2/3}$$

E101 Consider the cylinder $x^2+y^2=1$, let $z=0$ bound it from below and let $z=x+y+1$ bound it above. Call this solid B. A sketch of B reveals the inequalities to the right of it



$$0 \le z \le x+y+1$$

$$0 \le x^2+y^2 \le 1 \longrightarrow -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad -1 \le x \le 1$$

Calculate then,

$$\iiint_B x\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x+y+1} x\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+xy+x)\,dy\,dx$$

$$= \int_{-1}^{1}\left[(x^2+x)y + \tfrac{1}{2}xy^2\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\Big(\underbrace{2x^2\sqrt{1-x^2}}_{\text{I need to think about this one.}} + \overbrace{\cancel{2x\sqrt{1-x^2}}}^{\text{zero, odd function over even}}\Big)dx$$

trig substitution.

$$\boxed{\begin{array}{ll} x=\sin\theta & dx=\cos\theta\,d\theta \\ 1-x^2=\cos^2\theta & \sqrt{1-x^2}=\cos\theta \end{array}}$$

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\cos\theta\cos\theta\,d\theta$$

$$= \int(\sin^2\theta-\sin^4\theta)\,d\theta$$

$$= \int\left[\tfrac{1}{2}(1-\cos(2\theta)) - \tfrac{1}{4}(1-2\cos(2\theta)+\cos^2(2\theta))\right]d\theta$$

$$= \int\left(\tfrac{1}{2}-\tfrac{1}{2}\cos(2\theta)-\tfrac{1}{4}+\tfrac{1}{2}\cos(2\theta)-\tfrac{1}{4}\cos^2(2\theta)\right)d\theta$$

$$= \int\left[\tfrac{1}{4}-\tfrac{1}{8}(1-\cos(4\theta))\right]d\theta$$

$$= \theta/8 + \sin(4\theta)/32$$

Change bounds on x from $-1=x \longrightarrow 1=x \implies -\frac{\pi}{2}=\theta \longrightarrow \frac{\pi}{2}=\theta$.

$$\iiint_B x\,dV = \left[\frac{\theta}{8}+\frac{1}{32}\sin(4\theta)\right]_{-\pi/2}^{\pi/2} = \boxed{\frac{\pi}{4}}$$

$\boxed{E102}$ Let B be bounded by coordinate planes and the plane passing through $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$. We find the eq$^{\underline{n}}$ of this plane to begin, note $\vec{V}$ & $\vec{W}$ are on the plane

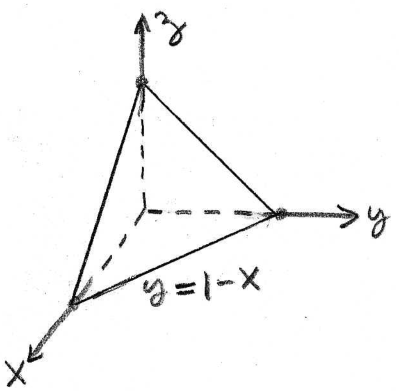
$$\vec{V} = (0,0,1)-(0,1,0) = \langle 0,-1,1\rangle$$
$$\vec{W} = (0,0,1)-(1,0,0) = \langle -1,0,1\rangle$$

$$\vec{V}\times\vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1,-1,-1\rangle = \langle a,b,c\rangle \text{ the normal.}$$

Choose $r_o = (0,0,1)$ to base the plane eq$^{\underline{n}}$,

$$-x-y-(z-1) = 0 \quad \Rightarrow \quad \underline{z = 1-x-y}$$

Lets plot it, note $z = 1-x-y$ intersects $z=0$ on the line $y = 1-x$ in the xy-plane.



$$\left.\begin{aligned} 0 &\le z \le 1-x-y \\ 0 &\le y \le 1-x \\ 0 &\le x \le 1 \end{aligned}\right\}$$ this is a useful description of B.

Lets find the volume of B,

$$V = \iiint_B dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx$$
$$= \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$
$$= \int_0^1 \Big[(1-x)y - \tfrac{1}{2}y^2 \Big|_0^{1-x} \Big]dx$$
$$= \int_0^1 \left[(1-x)^2 - \tfrac{1}{2}(1-x)^2 \right]dx$$
$$= \int_0^1 \tfrac{1}{2}(1-2x+x^2)\,dx$$
$$= \tfrac{1}{2}\left(1 - \tfrac{2}{2} + \tfrac{1}{3}\right) = \boxed{1/6}$$

Remark: these problems are pretty much easy once you figure out how to describe the solid.

$\boxed{E103}$ find the average value of $f(x,y,z) = x$ on the solid region from $\boxed{E102}$. The average is defined to be

$$\boxed{f_{avg}^{B} \equiv \frac{1}{\text{vol}(B)} \iiint_B f(x,y,z)\, dV}$$

we just found $\text{vol}(B) = 1/6$, lets focus on the $\iiint_B f\, dV$,

$$\iiint_B f\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz\, dy\, dx$$

$$= \int_0^1 x \int_0^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_0^1 x \left[(1-x)y - \tfrac{1}{2}y^2 \Big|_0^{1-x} \right] dx$$

$$= \int_0^1 \tfrac{1}{2} x (1-x)^2\, dx$$

$$= \int_0^1 \tfrac{1}{2}(x - 2x^2 + x^3)\, dx$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right)$$

$$= \tfrac{1}{2}\left(\tfrac{3}{4} - \tfrac{2}{3}\right)$$

$$= \frac{1}{24} \quad \Rightarrow \quad f_{avg}^{B} = \frac{1/24}{1/6} = \boxed{\frac{1}{4}}$$

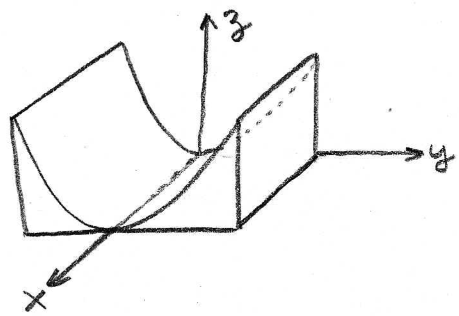
that seems pretty reasonable from the picture in $\boxed{E102}$.

<u>Remark</u>: We have studied how to integrate in Cartesian Coordinates in some detail. It turns out that this is quite limiting. To do many interesting problems with better efficiency it pays to employ cylindrical or spherical coordinates. Before getting to those special choices we consider a general coordinate change briefly and in the process derive what we later use for the cylindrical & spherical coordinates.

TRIPLE INTEGRALS OVER GENERAL BOUNDED REGIONS IN $\mathbb{R}^3$

Rather than explicitly stating the 3-d Fubini Th$^{\underline{m}}$ I will simply illustrate with a few examples. Usually we can bound z in terms of x & y then we can bound y in terms of x or vice-versa, that gives two orders of integration. Then other problems allow x to be bound in terms of y & z or possibly y in terms of x & z, in total there are 6 ways to write a particular integral over a volume. In §12.7 # 31 I explicitly show 6 ways to write a particular integral. I don't give general advice on how to rewrite and switch bounds, its a subtle buisness and I would advocate double checking with Mathematica. Generalities aside, lets do a few typical problems.

$\boxed{\text{E100}}$ Let us find the volume of the region between $z = y^2$ and the xy-plane bounded by $x=0$, $x=1$, $y=1$ and $y=-1$. Notice $dV = dx\,dy\,dz$ so integrating dV gives volume V,



$$0 \le z \le y^2$$
$$0 \le x \le 1$$
$$-1 \le y \le 1$$

- we must integrate w.r.t. z either first or second.

$$V = \int_{-1}^{1}\int_0^1\int_0^{y^2} dz\,dx\,dy \quad : \text{ we work inside out as usual.}$$

$$= \int_{-1}^{1}\int_0^1 y^2\,dx\,dy \quad : \text{ back to 2-d integrals.}$$

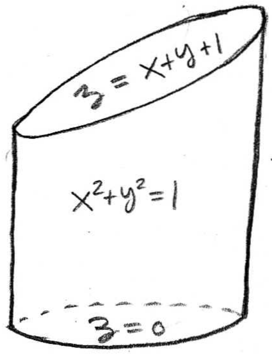
$$= \int_{-1}^{1} y^2\,dy \quad : \text{ back to 1-d integral.}$$

$$= \frac{1}{3}y^3\Big|_{-1}^{1}$$

$$= \frac{1}{3}\left(1-(-1)^3\right)$$

$$= \boxed{2/3}$$

E101 Consider the cylinder $x^2+y^2=1$, let $z=0$ bound it from below and let $z=x+y+1$ bound it above, Call this solid B. A sketch of B reveals the inequalities to the right of it



$$0 \le z \le x+y+1$$

$$0 \le x^2+y^2 \le 1 \quad \to \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \quad -1 \le x \le 1$$

Calculate then,

$$\iiint_B x\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x+y+1} x\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+xy+x)\,dy\,dx$$

$$= \int_{-1}^{1} \Big[(x^2+x)y + \tfrac{1}{2}xy^2\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= \int_{-1}^{1}\Big(\underbrace{2x^2\sqrt{1-x^2}}_{\text{I need to think about this one.}} + \overbrace{\cancel{2x\sqrt{1-x^2}}}^{\text{zero, odd function over even}}\Big)dx$$

trig substitution.

$$\boxed{\begin{array}{ll} x=\sin\theta & dx=\cos\theta\,d\theta \\ 1-x^2=\cos^2\theta & \sqrt{1-x^2}=\cos\theta \end{array}}$$

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\cos\theta\cos\theta\,d\theta$$

$$= \int(\sin^2\theta - \sin^4\theta)\,d\theta$$

$$= \int\Big[\tfrac{1}{2}(1-\cos(2\theta)) - \tfrac{1}{4}(1-2\cos(2\theta)+\cos^2(2\theta))\Big]d\theta$$

$$= \int\Big(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\theta) - \tfrac{1}{4} + \tfrac{1}{2}\cos(2\theta) - \tfrac{1}{4}\cos^2(2\theta)\Big)\,d\theta$$

$$= \int\Big[\tfrac{1}{4} - \tfrac{1}{8}(1-\cos(4\theta))\Big]d\theta$$

$$= \theta/8 + \sin(4\theta)/32$$

Change bounds on x from $-1=x \to 1=x \Rightarrow -\frac{\pi}{2}=\theta \to \frac{\pi}{2}=\theta$.

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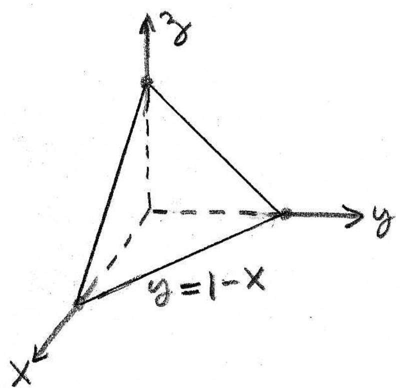

$$\left.\begin{aligned} 0 &\leq z \leq 1-x-y \\ 0 &\leq y \leq 1-x \\ 0 &\leq x \leq 1 \end{aligned}\right\} \text{ this is a useful description of } B.$$

Lets find the volume of B,

$$V = \iiint_B dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx$$
$$= \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$
$$= \int_0^1 \left[(1-x)y - \tfrac{1}{2}y^2 \Big|_0^{1-x} \right] dx$$
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Remark: these problems are pretty much easy once you figure out how to describe the solid.

E103 Find the average value of $f(x,y,z) = x$ on the solid region from E102. The average is defined to be

$$\boxed{f^B_{avg} \equiv \frac{1}{\text{vol}(B)} \iiint_B f(x,y,z)\,dV}$$

we just found vol$(B) = 1/6$, lets focus on the $\iiint_B f\,dV$,

$$\iiint_B f\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx$$

$$= \int_0^1 x \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 x\left[(1-x)y - \tfrac{1}{2}y^2\Big|_0^{1-x}\right]dx$$

$$= \int_0^1 \tfrac{1}{2}x(1-x)^2\,dx$$

$$= \int_0^1 \tfrac{1}{2}(x - 2x^2 + x^3)\,dx$$

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$$= \frac{1}{24} \quad \Rightarrow \quad f^B_{avg} = \frac{1/24}{1/6} = \boxed{\frac{1}{4}}$$

that seems pretty reasonable from the picture in E102.

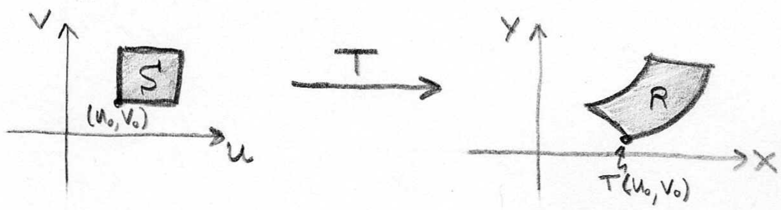
<u>Remark</u>: We have studied how to integrate in Cartesian Coordinates in some detail. It turns out that this is quite limiting. To do many interesting problems with better efficiency it pays to employ cylindrical or spherical coordinates. Before getting to those special choices we consider a general coordinate change briefly and in the process derive what we later use for the cylindrical & spherical coordinates.

Change of Variables in Multivariate Integration

I'll sketch the proof that leads to the Th^{m}'s on the next page. You can see Colley §5.5 for a more complete argument or Stewart. As usual the correct proof is most likely to be found in advanced calculus. Also we stick to two dimensions until the Th^{m}'s.

Coordinate Change

Usually we assign the Cartesian Coordinates (x,y) to the plane $\mathbb{R}^2$. However we can change the coordinates to (u,v). Its helpful to consider two planes, the (x,y)-plane and the (u,v)-plane, the coordinate change map T takes (u,v) to $T(u,v) = (x(u,v), y(u,v))$



Here $T : S \subset \mathbb{R}^2 \longrightarrow R \subset \mathbb{R}^2$. We insist that T be invertible, except possibly on the boundaries, this means the eq^{n}'s relating x, y and u, v can be solved for either x, y or u, v. From our discussion of differentiability we learned T can be approximated by the function,

$$\boxed{h(u,v) = T(u_o, v_o) + DT(u_o, v_o)\begin{bmatrix} u - u_o \\ v - v_o \end{bmatrix}}$$

where $DT(u_o, v_o)$ is the 2×2 Jacobian Matrix. If we consider a little parallelogram at (u_o, v_o) then transports it to $T(u_o, v_o)$ and $DT(u_o, v_o)$ distorts it into a modified parallelogram. Moreover the following proposition tells us how the areas of the parallelograms are related.

> Prop: (5.1 of Colley): Let $T(u,v) = A\begin{bmatrix} u \\ v \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A \neq 0$, then if D^* is a parallelogram then $T(D^*) = D$ is also a parallelogram and $\boxed{(\text{Area of } D) = |\det A| \cdot (\text{Area of } D^*)}$

While T itself is not usually linear, its best linear approx h is and close to some particular point (u_o, v_o) it's matrix is $DT(u_o, v_o)$. Thus for a tiny rectangle $(\Delta u \Delta v)$ we find

$$\boxed{\Delta u \, \Delta v = \det(DT(u_o, v_o)) \, \Delta x \, \Delta y}$$

then in the limit these become differentials, and also as we sum over S and $T(S) = R$ we find the theorems that follow,

How to change coordinates on a double integral

I sketched the general idea, now let me give the practical formulas so we can apply these to specific problems.

Def^{n}/ The Jacobian of the transformation $T(u,v) = (x(u,v), y(u,v))$ is,

$$\frac{\partial(x,y)}{\partial(u,v)} \equiv \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u y_v - x_v y_u$$

The "Jacobian" is the determinant of what I called the "Jacobian matrix",

E104 Consider polar coordinates: $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Lets calculate the JACOBIAN, note $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

Th^{m} (Changing variables in double integrals): Suppose $T : S \to R$ is a differentiable mapping that is mostly invertible (except possibly on the boundary) from TYPE I OR II region S to TYPE I OR II region R and suppose that f is a continuous function whose domain includes R,

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the $|\cdot|$ on the Jacobean are absolute value bars.

E105 Lets apply this Th^{m} to POLAR COORDINATES, Suppose f is continuous etc...

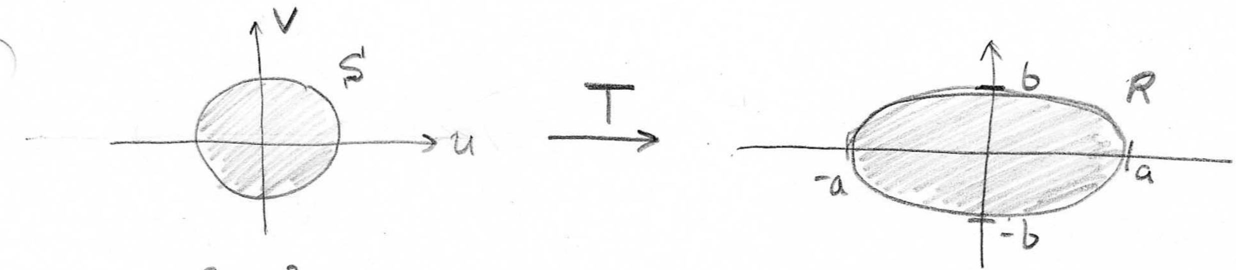
$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

E106 Using **E105** calculate the area of a circle of radius A, call it R

$$\iint_R dx\,dy = \iint_S r\,dr\,d\theta = \int_0^{2\pi}\int_0^{A} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}A^2\,d\theta = \frac{1}{2}A^2 \cdot 2\pi = \boxed{\pi A^2}$$

Remark: this is considerably easier than the direct Cartesian calculation of area, although the same geometry makes both sol^{n}'s work. Notice "mostly invertible" is a needed qualifier since the angle θ doubles up on $\theta = 0$ and 2π given (x,y) along $\theta = 0$ should we say it corresponds to $\theta = 0$ or $\theta = 2\pi$? Fortunately a curve or two will not change double integral's result.

E107 Consider the ellipse $x^2/a^2 + y^2/b^2 = 1$ where $a, b > 0$.
Find the Area, use a change of coordinates that makes it a circle



$u^2 + v^2 \leq 1$ $\qquad\qquad$ $x^2/a^2 + y^2/b^2 \leq 1$

we want to find T that morphs the circle in the uv-plane into our ellipse, a bit of reflection reveals

$$\begin{aligned} x &= au \\ y &= bv \end{aligned} \qquad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = u^2 + v^2, \text{ see it works} \right)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = ab - 0 = ab$$

Now calculating the area is easy given **E106**

$$\text{Area}(R) = \iint_R dx\,dy = \iint_S |ab|\,du\,dv = ab\iint_S du\,dv = \boxed{\pi ab = \text{Area}(R)}$$

in the last step I have used that the area of the unit circle in the uv-plane has area π.

<u>Remark</u>: we can do an analogous change of variables to obtain the volume of an ellipsoid from the volume of a sphere.

- Next we consider an example where it is not obvious how the coordinate change map's domain should be defined. The form of $T(u,v)$ will be suggested by the integrand rather than R as in **E107**. Please understand E107 is a novelty, on the other hand **E108** is more typical. The main use of the coordinate change Th^{m}'s is certainly the standard non-Cartesian systems (Polars, Cylindricals, Sphericals).

E108 Evaluate the integral by performing an appropriate coordinate change,

$$I = \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

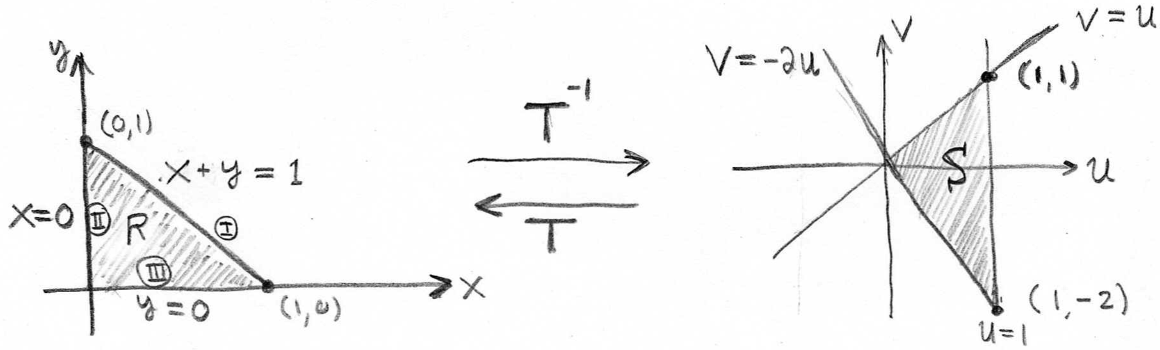
This suggests we choose $u = x+y$ and $v = y-2x$. Solving for x, y yields $x = \frac{u}{3} - \frac{v}{3}$ and $y = \frac{2u}{3} + \frac{v}{3}$. Thus

$$T(u,v) = \left(\tfrac{1}{3}(u-v),\ \tfrac{1}{3}(2u+v)\right)$$

if $T: S \to R$ then what is S in this case? We are interested in R that is indicated by the integral I, namely

$$R: \left\{ \begin{array}{c} 0 \leq y \leq 1-x \\ 0 \leq x \leq 1 \end{array} \right\}$$

WHAT IS S HERE?



To figure out the boundaries in the uv-triangle we are guided by the knowledge that for a simple linear T as we have here triangles go to triangles, vertices to vertices.

I: $x + y = \frac{1}{3}u - \frac{1}{3}v + \frac{2}{3}u + \frac{1}{3}v = u = 1$

II: $0 = x = \frac{1}{3}u - \frac{1}{3}v \Rightarrow u = v$

III.: $0 = y = \frac{2}{3}u + \frac{1}{3}v \Rightarrow v = -2u$

And we may explicitly describe S now,

$$S: \left\{ \begin{array}{c} -2u \leq v \leq u \\ 0 \leq u \leq 1 \end{array} \right\}$$

Notice then, for $x = \frac{1}{3}(u-v)$, $y = \frac{1}{3}(2u+v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) - \left(\tfrac{-1}{3}\right)\left(\tfrac{2}{3}\right) = \tfrac{3}{9} = \tfrac{1}{3}.$$

E108 Continued Apply what we've learned.

$$I = \int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

$$= \int_0^1 \int_{-2u}^{u} \sqrt{u}\; v^2 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\,du$$

$$= \int_0^1 \int_{-2u}^{u} \tfrac{1}{3} v^2 \sqrt{u}\,dv\,du$$

$$= \int_0^1 \frac{\sqrt{u}}{9}\, v^3 \Big|_{-2u}^{u}\,du$$

$$= \int_0^1 \frac{\sqrt{u}}{9}\left(u^3 - (-2u)^3\right) du$$

$$= \int_0^1 u^{3+1/2}\,du$$

$$= \frac{2}{9} u^{9/2}\Big|_0^1$$

$$= \boxed{2/9}$$

- note, this example borrowed from Thomas' Calculus 10^{th} Ed. pg. 1040.
- We move on to triple integrals.

Def^n/ Let $T(u,v,w) = (x(u,v,w),\, y(u,v,w),\, z(u,v,w))$ be a differentiable function from $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, the JACOBIAN of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

E109 Cylindrical Coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & y_r & z_r \\ x_\theta & y_\theta & z_\theta \\ x_z & y_z & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (r\cos^2\theta + r\sin^2\theta)\,1 = \boxed{r}$$

here $r^2 = x^2 + y^2$.

E110 Spherical Coordinates:

$$x = \rho\cos\theta\sin\varphi \qquad 0 \le \theta \le 2\pi$$
$$y = \rho\sin\theta\sin\varphi \qquad 0 \le \varphi \le \pi$$
$$z = \rho\cos\varphi \qquad \rho^2 = x^2+y^2+z^2 = r^2+z^2.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\theta & y_\theta & z_\theta \\ x_\varphi & y_\varphi & z_\varphi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\rho\sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi & -\rho\sin\varphi \end{vmatrix}$$

$$= \cos\theta\sin\varphi\,(\rho\cos\theta\sin\varphi)(-\rho\sin\varphi) - \sin\theta\sin\varphi\,[\rho^2\sin\theta\sin^2\varphi] +$$
$$+ \cos\varphi\,[-\rho\sin^2\theta\sin\varphi\cos\varphi - \rho\cos^2\theta\sin\varphi\cos\varphi]$$
$$= -\rho^2(\cos^2\theta\sin^3\varphi + \sin^2\theta\sin^3\varphi + \cos^2\varphi\sin\varphi)$$
$$= -\rho^2(\sin^3\varphi + \sin\varphi\cos^2\varphi)$$
$$= -\rho^2\sin\varphi\,(\sin^2\varphi+\cos^2\varphi)$$
$$= \boxed{-\rho^2\sin\varphi}$$

(notice that $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin\varphi$ thanks to determinants skew symmetry)

E111 Volume of a sphere of radius A, call it S

$$\text{vol}(S) = \iiint_S dV$$
$$= \int_0^\pi\int_0^{2\pi}\int_0^A \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

using the triple integral coordinate change Thm which is stated on next page,

$$= \left(\int_0^\pi \sin\varphi\, d\varphi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^A \rho^2\, d\rho\right)$$
$$= 2\cdot 2\pi\cdot\frac{1}{3}A^3$$
$$= \boxed{\frac{4}{3}\pi A^3}$$

Th^m (Coordinate Change in Triple Integrals): Suppose that we have a differentiable, mostly invertible $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ and a function f which is continuous on S, where $T(S) = R$.

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

- notice that $f(T(u,v,w))$ is simply notation for saying that f is to be written in terms of u,v,w as indicated by the formulas for $x(u,v,w)$, $y(u,v,w)$, $z(u,v,w)$.

E112 Sphericals.

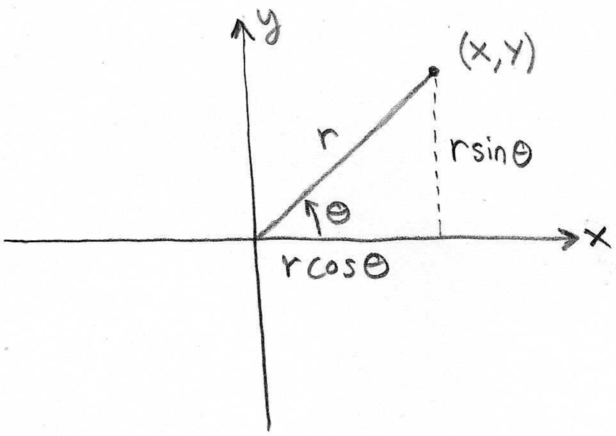
$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) \cdot \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

we'll discuss how to see S from R in more detail later on.

E113 Cylindricals.

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

- We pause now to go back and cover §9.6 on the geometry of spherical and cylindrical coordinates. To begin we recall the set-up for POLAR COORDINATES,



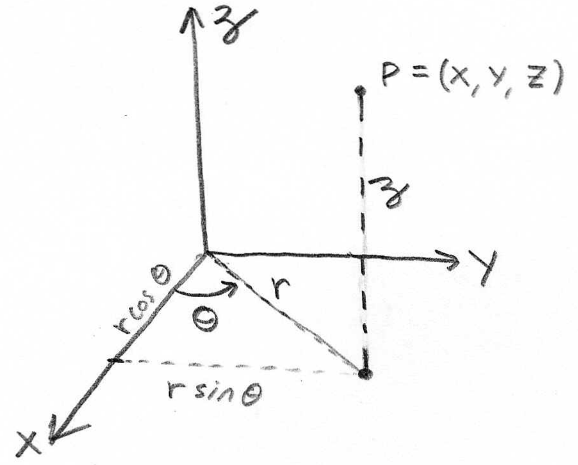
Useful Facts.

$x = r\cos\theta$ $y = r\sin\theta$	$\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$
$r^2 = x^2 + y^2$ $\tan\theta = y/x$	
$0 \leq \theta \leq 2\pi$ $0 \leq r$	

Notice that the sign of x & y is automatically encoded by the sign of $\cos\theta$ & $\sin\theta$. Also notice that θ is ambiguous at the origin and θ is double valued along $y=0$, $x>0$.

CYLINDRICAL COORDINATES:

Given a point $P \in \mathbb{R}^3$ we can describe the point P by the Polar Coordinates of $\pi_{xy}(P)$ its projection into the xy-plane and the z-coordinate of P. As usual we relate them to the standard CARTESIAN COORDINATES,



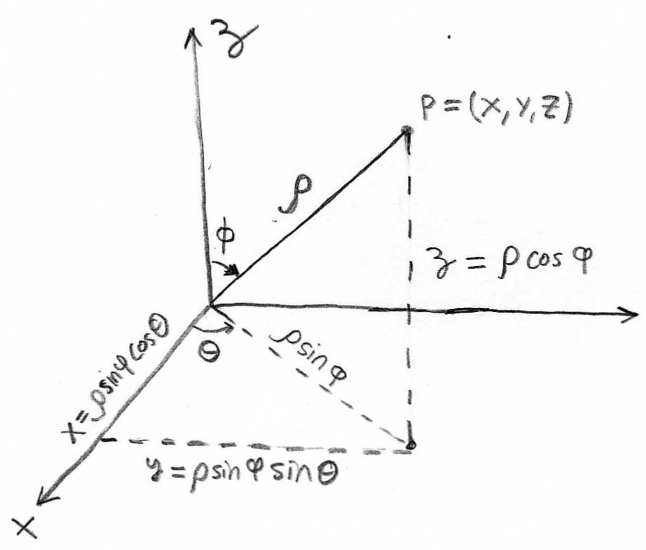
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

here the cylindrical coordinates (r, θ, z) are req^{d} to satisfy

$$r \geq 0, \quad 0 \leq \theta \leq 2\pi$$
$$r^2 = x^2 + y^2$$

SPHERICAL COORDINATES

Given a point $P \in \mathbb{R}^3$ we describe the location of P by its distance from the origin "rho" ρ and the polar angle θ plus the azimuthial angle $\phi = \varphi$ (I use both, sorry).



$$\boxed{\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}}$$

where you can easily prove

$$\rho^2 = x^2 + y^2 + z^2$$

and we require (define)

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$$

OTHER CONVENTIONS, BEWARE: (see 383 for more) $\quad \rho \geq 0$

Remark: I'm not particularly enamored with these math-conventions. To my taste the physics conventions of switching $\theta \longleftrightarrow \varphi$ are nice,

$$x = r\sin\theta\cos\varphi \qquad x = s\cos\varphi$$
$$y = r\sin\theta\sin\varphi \qquad y = s\sin\varphi$$
$$z = r\cos\theta \qquad z = z$$

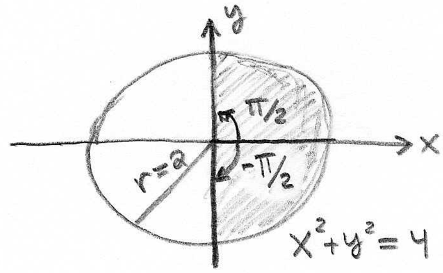
here $r^2 = x^2 + y^2 + z^2 = s^2 + z^2$

I will use math conventions unless otherwise explicitly stated.

Double Integrals using Polars (§16.4)

We now have all the theory we need to justify these calculations.

E114 Let $R = \{(x,y) \mid x^2+y^2 \leq 4,\ x \geq 0\}$. Convert this region to Polars and integrate $f(x,y) = \sqrt{4-x^2-y^2}$.



$$S = \{(r,\theta) \mid 0 \leq r \leq 2,\ \underbrace{-\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}}\}$$

ok, so technically we ought to instead use $\theta \in [0,\pi/2] \cup [\frac{3\pi}{2}, 2\pi]$. but $[-\pi/2, \pi/2]$ gives the same results. I think you'll find most folks are not super picky about the domain of θ.

$$\iint_R \sqrt{4-x^2-y^2}\, dA = \int_0^2 \int_{-\pi/2}^{\pi/2} \sqrt{4-r^2\cos^2\theta - r^2\sin^2\theta}\, \underline{r\,d\theta\,dr}$$

$\boxed{dA = r\,d\theta\,dr}$

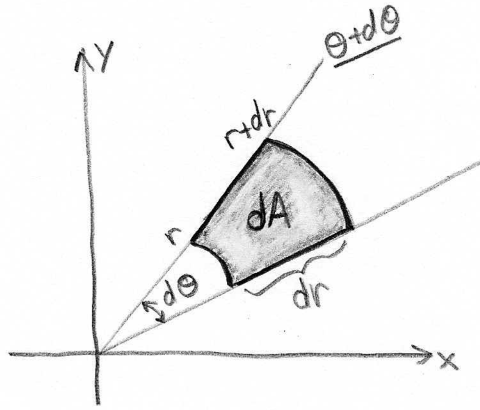
$$= \int_0^2 \int_{-\pi/2}^{\pi/2} r\sqrt{4-r^2}\, d\theta\, dr$$

$$= \int_0^2 \pi r \sqrt{4-r^2}\, dr$$

$$= \frac{-\pi}{2}\frac{2}{3}(4-r^2)^{3/2}\Big|_0^2$$

$$= \frac{-\pi}{3}\left[0 - (2^2)^{3/2}\right] = \boxed{\frac{8\pi}{3}}$$

Remark: We used $dA = r\,d\theta\,dr$, this was derived back in **E104–105** on 344, let me give the infinitesimal argument for this, some of you may find this more logically appealing than the general Jacobian theory. A little "polar rectangle".



Recall from middle school, area of a wedge of radius a of angle $\Delta\theta$ is $\frac{1}{2}a^2\Delta\theta$. You are familiar with $\Delta\theta = 2\pi$ then the wedge is the whole circle and we get πa^2.

• Using 1st order formalism

$$dA = \tfrac{1}{2}(r+dr)^2 d\theta - \tfrac{1}{2}r^2 d\theta = \tfrac{1}{2}\left(r^2 + 2r\,dr + \cancel{(dr)^2} - r^2\right)d\theta = \boxed{r\,dr\,d\theta = dA}$$

$(dr)^2 \to 0$ (3rd order small).

Remark Continued: Another method to derive $dA = r\,dr\,d\theta$ is to use "differential forms". You may do a Bonus project on differential forms if you wish (not a req^{d} topic). In the differential forms set-up we put "wedges" between the dx and dy so $dA = dx \wedge dy = -dy \wedge dx$, the wedge product is antisymmetric. You can talk to me outside class if you'd like to know more, or see my ma 430 notes which are also posted online. For now I give the derivation as a short advertisement for differential forms,

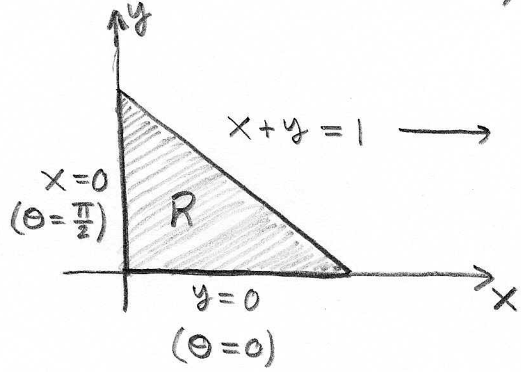
$$\begin{aligned} dA &= dx \wedge dy \\ &= d(r\cos\theta) \wedge d(r\sin\theta) \\ &= [\cos\theta\, dr - r\sin\theta\, d\theta] \wedge [\sin\theta\, dr + r\cos\theta\, d\theta] \\ &= \sin\theta\cos\theta\, \underbrace{dr\wedge dr}_{0} + r\cos^2\theta\, dr\wedge d\theta - r\sin^2\theta\, d\theta\wedge dr - r^2\sin\theta\cos\theta\, \underbrace{d\theta\wedge d\theta}_{0} \\ &= r\cos^2\theta\, dr\wedge d\theta + r\sin^2\theta\, dr\wedge d\theta \\ &= \boxed{r\,dr\wedge d\theta = dA} \end{aligned}$$

here $dA = -r\,d\theta\wedge dr$ also, in contrast to our usual $dA = r\,dr\,d\theta = r\,d\theta\,dr$ the minus sign encodes the "orientation". The "$\wedge$" (wedge product) automatically ignored uninteresting terms.

$\boxed{\text{E115}}$ Let $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ find $\iint_R f(x,y)\,dA$ where R is the region in xy-plane with $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/3$.

$$\begin{aligned} \iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA &= \int_0^{\pi/3}\int_1^2 \frac{1}{r}\, r\,dr\,d\theta \\ &= \left(\int_0^{\pi/3} d\theta\right)\left(\int_1^2 dr\right) \\ &= \boxed{\frac{\pi}{3}} \end{aligned}$$

E116 Find the area of the triangle bounded by $x=0$, $y=0$ and $x+y=1$ using POLAR COORDINATES. Its fairly easy to see that $0 \le \theta \le \pi/2$ on R, however bounding r requires thought,



$$x+y=1 \longrightarrow r\cos\theta + r\sin\theta = 1$$

$$r(\cos\theta+\sin\theta) = 1$$

$$r = \frac{1}{\cos\theta+\sin\theta}$$

$$\Rightarrow \quad 0 \le r \le \frac{1}{\cos\theta+\sin\theta}$$

this is a less trivial polar region, we must put the integration over dr first since its bounds are θ-dependent.

$$\text{Area}(R) = \int_0^{\pi/2}\int_0^{\frac{1}{\cos\theta+\sin\theta}} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(r^2\Big|_0^{1/[\cos\theta+\sin\theta]}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{(\cos\theta+\sin\theta)^2} \quad : \quad \sin\theta+\cos\theta = \sqrt{2}\,\sin(\theta+\pi/4)\ (*)$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{2}\csc^2(\theta+\pi/4)\,d\theta$$

$$= \frac{-1}{4}\cot(\theta+\pi/4)\Big|_0^{\pi/2}$$

$$= \frac{-1}{4}\left(\cot(3\pi/4) - \cot(\pi/4)\right) =$$

$$= \frac{-1}{4}(-1-1) = \boxed{1/2}$$

(*) $\sin(\theta+\pi/4) = \sin\theta\cos\pi/4 + \sin\pi/4\cos\theta = \frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$
thus we find $\sin\theta+\cos\theta = \sqrt{2}\,\sin(\theta+\pi/4)$.

Remark: this is a horrible method to find the area of a triangle. But, it illustrates a general principle which is that coordinates should be chosen to fit the problem. Obviously this problem suggests Cartesians instead.

E117 Evaluate $\iiint_E (x^3+xy^2)\,dV$ where E is the solid in the 1^{st} octants which lies beneath the paraboloid $z = 1-x^2-y^2$. This problem suggests a cylindrical approach, notice

$$x^3+xy^2 = x(x^2+y^2) = xr = r^2\cos\theta$$
$$z = 1-x^2-y^2 = 1-r^2$$

Also, the "first octant" is bounded by $x=0$, $y=0$ and $z=0$. On $z=0$ we find the intersection of $z=1-r^2=0 \Rightarrow r^2=1$ ah ha its just the unit circle. We see

$$E: \quad 0 \le \theta \le \pi/2, \quad 0 \le r \le 1, \quad 0 \le z \le 1-r^2.$$

So calculate, recall $dV = r\,dr\,d\theta\,dz$,

$$\iiint_E (x^3+xy^2)\,dV = \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} (r^2\cos\theta)(r\,dz\,dr\,d\theta)$$

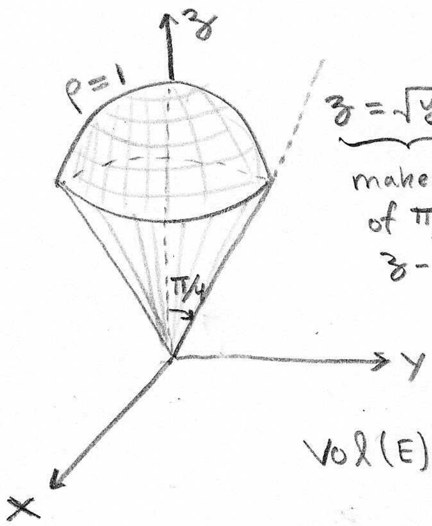
: bounds give the order of the $dz, dr, d\theta$ & vice-versa.

$$= \int_0^{\pi/2}\cos\theta\,d\theta\int_0^1 r^3(1-r^2)\,dr$$
$$= \left(\sin\theta\Big|_0^{\pi/2}\right)\left(\frac{r^4}{4}-\frac{r^6}{6}\Big|_0^1\right)$$
$$= (1-0)\left(\frac{1}{4}-\frac{1}{6}\right)$$
$$= \frac{2}{24} = \boxed{\frac{1}{12}}$$

E118 Consider $f(x,y,z) = \left(e^{\sqrt{x^2+y^2+z^2}}\right)\frac{1}{x^2+y^2+z^2}$. Find $\iiint_E f\,dV$ where $E = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 9\} \Rightarrow 1 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$.

$$\iiint_E \frac{1}{x^2+y^2+z^2}e^{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_1^3 \frac{1}{\rho^2}e^{\rho}\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$
$$= \left(\int_0^{\pi}\sin\varphi\,d\varphi\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_1^3 e^{\rho}\,d\rho\right)$$
$$= (-\cos\pi+\cos 0)(2\pi)(e^3-e^1)$$
$$= \boxed{4\pi(e^3-e)}$$

E119 Find the volume and centroid of the region E which is above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$. We assume that E has a constant density δ.



$\underbrace{z = \sqrt{y^2} = y}$ makes angle of $\pi/4$ with z-axis.

In spherical coordinates the region E is easy to describe,

$0 \le \theta \le 2\pi$
$0 \le \rho \le 1$
$0 \le \varphi \le \pi/4$.

$$\begin{aligned}
Vol(E) &= \int_0^{\frac{\pi}{4}}\int_0^1\int_0^{2\pi} \rho^2 \sin\varphi \, d\theta \, d\rho \, d\varphi \\
&= \int_0^1 \rho^2 d\rho \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^{2\pi} d\theta \\
&= \left(\frac{1}{3}\right)\left(-\cos(\pi/4)+1\right) 2\pi \\
&= \boxed{\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)}
\end{aligned}$$

Now since δ = constant the total mass is $M = vol(E)\cdot\delta = \underline{\frac{2\pi\delta}{3}(1-1/\sqrt{2})}$

The "centroid" is the center of mass it is defined to be $\langle \bar{x}, \bar{y}, \bar{z} \rangle$ where $\bar{x} = M_{yz}/M$, $\bar{y} = M_{zx}/M$ and $\bar{z} = M_{xy}/M$ and M_{yz}, M_{zx}, M_{xy} are the moments about the coordinate planes,

$$M_{yz} \equiv \iiint_E x\delta \, dV = 0$$
$$M_{zx} \equiv \iiint_E y\delta \, dV = 0$$

by the symmetry of the object. Calculate it, if you don't believe me.

$$\begin{aligned}
M_{xy} &= \iiint_E \delta z \, dV = \int_0^{2\pi}\int_0^1\int_0^{\pi/4} (\delta\rho\cos\varphi)(\rho^2\sin\varphi \, d\varphi \, d\rho \, d\theta) \\
&= (2\pi\delta)\left(\rho^4/4 \Big|_0^1\right)\left(\tfrac{1}{2}\sin^2\varphi \Big|_0^{\pi/4}\right) \\
&= \frac{\pi\delta}{4}\left(\frac{1}{\sqrt{2}}\right)^2 = \underline{\pi\delta/8 = M_{xy}}
\end{aligned}$$

Therefore,
$$\begin{aligned}
\langle \bar{x}, \bar{y}, \bar{z} \rangle &= \left\langle 0, 0, \frac{\pi\delta/8}{\frac{2\pi\delta}{3}(1-1/\sqrt{2})} \right\rangle \\
&= \boxed{\left\langle 0, 0, \frac{3}{16(1-1/\sqrt{2})} \right\rangle}
\end{aligned}$$

E120 Find the Kinetic Energy of a ball with radius R and mass m that spins with angular velocity ω and moves linearly with speed V. It is known that $KE_{net} = KE_{rot} + KE_{trans.}$ where $KE_{trans.} = \frac{1}{2}mV^2$ whereas $KE_{rot} = \frac{1}{2}I\omega^2$ and I = moment of intertia, lets say about the (moving) z-axis.

$$I_z = \iiint_E \delta(x,y,z)(x^2+y^2)\,dV \qquad \delta = \text{constant} = \frac{m}{\frac{4}{3}\pi R^3} = \text{density.}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{3m}{4\pi R^3}\left[(\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2\right]\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{3m}{4\pi R^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^4\sin^3\varphi\, d\rho\, d\varphi\, d\theta \qquad : \int \sin^3\varphi\, d\varphi = \int (1-\cos^2\varphi)\sin\varphi\, d\varphi$$

$$= \frac{3m}{4\pi R^3}\left(d\theta\Big|_0^{2\pi}\right)\left(\frac{1}{3}\cos^3\varphi - \cos\varphi\Big|_0^{\pi}\right)\left(\frac{\rho^5}{5}\Big|_0^R\right)$$

$$= \frac{3m}{4\pi R^3}(2\pi)\left(-\frac{1}{3}+1-\frac{1}{3}+\cos 0\right)\left(\frac{1}{5}R^5\right)$$

$$= \frac{\cancel{3}m}{\cancel{4\pi} R^3}(2\cancel{\pi})\left(\frac{4}{\cancel{3}}\right)\left(\frac{1}{5}R^5\right)$$

$$= \boxed{\frac{2}{5}mR^2 = I_{sphere}}$$

Then we can calculate (don't worry this isn't physics you're not expected to know all this by heart.)

$$KE_{total} = \frac{1}{2}mV^2 + \frac{1}{5}mR^2\omega^2$$

If the ball is rolling then $\omega = V/R$ (no slipping)

$$KE_{total} = \frac{1}{2}mR^2\omega^2 + \frac{1}{5}mR^2\omega^2 = \frac{7}{10}mR^2\omega^2$$

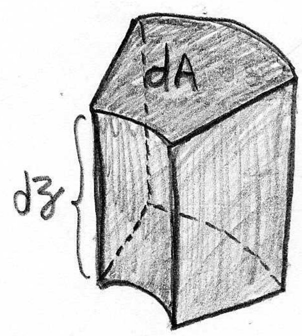
<u>Remark</u>: This type of example is one of my main objections to the standard math notation of $\rho = x^2+y^2+z^2$. I would like to use ρ for density, your text does it anyway.

GEOMETRY OF VOLUME ELEMENTS (§12.8)

The Jacobian gives us the volume element in other coordinate systems if $T(u,v,w) = \langle x, y, z\rangle$ where x, y, z are fncts of u, v, w. then the infinitesimal volume dV

$$\boxed{dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

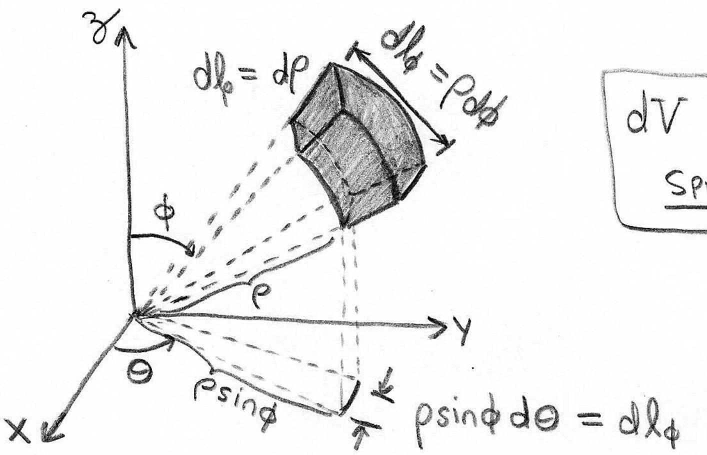
We used this general theory to derive $dV = r\,dr\,d\theta\,dz$ for cylindricals and $dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$. You might wonder if there is an explicit geometric derivation of these volume elements. The answer is yes, I'll show you how, well I'll try



$dA = r\,dr\,d\theta$ (the POLAR rectangle)

$$\boxed{dV = r\,dr\,d\theta\,dz}$$

CYLINDRICAL VOLUME ELEMENT



$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

SPHERICAL VOLUME ELEMENT

Remark: we can also use differential forms to derive these things. Again I mention this is not a req^d topic.

$$dx \wedge dy \wedge dz = d(r\cos\theta) \wedge d(r\sin\theta) \wedge dz$$
$$= \boxed{r\,dr \wedge d\theta \wedge dz = vol_3}$$

where I have used what I already calculated on 352.

(Optional not req^{d}): Volume Element in Spherical Coordinates

Calculate as before, ask me if you'd like to know more about what the calculation below means.

$$\begin{aligned}
\text{vol}_3 &= dx \wedge dy \wedge dz \\
&= d(\rho \sin\phi \cos\theta) \wedge d(\rho \sin\phi \sin\theta) \wedge d(\rho\cos\phi) \\
&= [\sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta] \wedge \\
&\quad \wedge [\sin\phi\sin\theta\, d\rho + \rho\cos\phi\sin\theta\, d\phi + \rho\sin\phi\cos\theta\, d\theta] \wedge \\
&\quad \wedge [\cos\phi\, d\rho - \rho\sin\phi\, d\phi] \\
&= [\sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta] \wedge \\
&\quad \wedge [-\rho\sin^2\phi\sin\theta\, d\rho\wedge d\phi + \rho\cos^2\phi\sin\theta\, d\phi\wedge d\rho + \\
&\quad + \rho^2\sin\phi\cos\phi\cos\theta\, d\theta\wedge d\rho - \rho^2\sin^2\phi\cos\theta\, d\theta\wedge d\phi] \\
&= -\rho^2\sin^3\phi\cos^2\theta\, d\rho\wedge d\theta\wedge d\phi + \rho^2\cos^2\phi\sin\phi\cos^2\theta\, d\phi\wedge d\theta\wedge d\rho \\
&\quad + \rho^2\sin^3\phi\sin^2\theta\, d\theta\wedge d\rho\wedge d\phi - \rho^2\sin\phi\cos^2\phi\sin^2\theta\, d\theta\wedge d\phi\wedge d\rho \\
&= [-\rho^2\sin^3\phi\cos^2\theta - \rho^2\cos^2\phi\sin\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\sin^2\theta]\, d\rho\wedge d\theta\wedge d\phi \\
&= \rho^2\sin\phi\, [\sin^2\phi\cos^2\theta + \cos^2\theta\cos^2\phi + \sin^2\phi\sin^2\theta + \sin^2\theta\cos^2\phi]\, d\rho\wedge d\phi\wedge d\theta \\
&= \rho^2\sin\phi\, [\cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta(\sin^2\phi + \cos^2\phi)]\, d\rho\wedge d\phi\wedge d\theta \\
&= \boxed{\rho^2\sin\phi\, d\rho\wedge d\phi\wedge d\theta = \text{vol}_3}
\end{aligned}$$

In fact the wedge product is just another way to calculate the determinant. You could even define the determinant implicitly by the following formula,

$$\boxed{Ae_1 \wedge Ae_2 \wedge \cdots \wedge Ae_n \equiv \det(A)\, e_1\wedge e_2\wedge\cdots\wedge e_n}$$

For example $A = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$ gives $Ae_1 = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\c\end{bmatrix} = ae_1 + ce_2$ and $Ae_2 = A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}b\\d\end{bmatrix} = be_1 + de_2$ thus

$$\begin{aligned}
Ae_1\wedge Ae_2 &= (ae_1 + ce_2)\wedge(be_1 + de_2) \\
&= ab\, e_1\wedge e_1 + ad\, e_1\wedge e_2 + cb\, e_2\wedge e_1 + cd\, e_2\wedge e_2 \\
&= (ad - cb)\, e_1\wedge e_2 \equiv \det(A)\, e_1\wedge e_2 \Rightarrow \boxed{\det A = ad - bc}
\end{aligned}$$

that is of course the usual formula for a 2x2 determinant.

A comment on n-dim'l integration

Let $\mathbb{R}^n$ have CARTESIAN COORDINATES $(x_1, x_2, \dots, x_n)$ and suppose $T(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$ where y_i is a function of $x_1, \dots, x_n$ for each i and we suppose T has DT invertible over the domain of integration below,

$$\iint_S \cdots \int f(x_1, x_2, \dots, x_n)\, d^n x = \iint_{T(S)} \cdots \int f(y_1, y_2, \dots, y_n) \left| \det(DT) \right| d^n y$$

where $d^n x \equiv dx_1 dx_2 \cdots dx_n$ and $d^n y = dy_1 dy_2 \cdots dy_n$. The meaning of the n-fold integration should be an easy generalization of the cases we've already considered $n = 2$ and 3.

$\boxed{E121}$ The HYPERSPHERE: $(x, y, z, w) \in \mathbb{R}^4$ such that $x^2 + y^2 + z^2 + w^2 \leq R^2$. Generalized Spherical Coordinates are

$$\begin{aligned} x &= r\cos\theta \sin\phi \sin\psi \\ y &= r\sin\theta \sin\phi \sin\psi \\ z &= r\cos\phi \sin\psi \\ w &= r\cos\psi \end{aligned} \qquad \begin{pmatrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi, \psi \leq \pi \\ 0 \leq r \end{pmatrix}$$

you can check that $x^2 + y^2 + z^2 + w^2 = r^2$. Then it's a long but straightforward calculation,

$$\left| \frac{\partial(x, y, z, w)}{\partial(r, \theta, \phi, \psi)} \right| = r^3 \sin^2\psi \sin\phi$$

And consequently if we integrate $dx\,dy\,dz\,dw$ and change to sphericals we'll find,

$$\text{Vol}_4(\text{HYPERSPHERE}) = \int_0^R \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} r^3 \sin^2\psi \sin\varphi \, d\psi\, d\varphi\, d\theta\, dr = \frac{\pi^2 R^4}{2}.$$

(if you like this sort of digression take a look at my ma 430 notes.)