Integration Techniques

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Spring 2023

Abstract

In this supplement it is assumed you have a complete and working knowledge of both basic antidifferentiation theory and the Fundamental Theorems of Calculus parts I and II. You must know the basic integrals as a reflex, if you have to look them up then you are not really ready. In addition, it is assumed you have mastered all the elementary u-substitutions and are ready, willing and able to solve definite and indefinite integrals which require u-subsitution. If this is not the case then you need to review.

We study the major techniques of integration in this article. First we study Integration by Parts (IBP) which is the analog of the product rule for integrals. We'll see IBP allows us to solve integrals of inverse functions as well as a number of expressions which were insolvable by u-substitution alone. Then we turn to review and extend our knowledge of integrals of trigonometric and hyperbolic functions. I'll introduce you to the imaginary exponentials which can be wielded against a vast array of symbolic trigonometric problems. Having established a mastery of trigonometric and hyperbolic integrals we turn to implicit substitutions known as trigonometric or hyperbolic substitution. These implicit substitutions allow integration of many expressions involving radicals. Finally, we turn to our final technique of integration known as the method of partial fractions. This method allows us to take any rational function and rewrite it as a sum of basic rational functions for which integrals are known.

The material in this supplement is challenging and it will stretch your knowledge of algebra, trigonometry and calculus. To master this material is to level-up.

1 Integration by Parts (IBP)

I'm breaking from tradition a little here. I've decided to use U and V as opposed to u and v for integration by parts. Of course you could use other letters if you don't like these.

Theorem 1.1. *integration by parts.*

Suppose that U, V are differentiable functions on some connected subset $J \subseteq \mathbb{R}$. Then,

$$(1.)\quad \int U \frac{dV}{dx}\, dx = UV - \int V \frac{dU}{dx}\, dx.$$

Moreover, if $a, b \in J$ then

$$(2.)\quad \int_a^b U\, \frac{dV}{dx}\, dx = U(b)V(b) - U(a)V(a) - \int_a^b V(x) \frac{dU}{dx}\, dx.$$

Notice that is more common to use the following notation to put this theorem into practice:

$$\int U\, dV = UV - \int V\, dU.$$

here we mean for the expressions to be evaluated at $x \in J$ once the integrations are complete.

Proof: Consider that by linearity of differentiation we find

$$\frac{d}{dx}\left[UV - \int V \frac{dU}{dx}\, dx\right] = \frac{d}{dx}\left[UV\right] - \frac{d}{dx}\left[\int V \frac{dU}{dx}\, dx\right]$$

The rightmost term simplifies to $V\frac{dU}{dx}$ by the definition of the indefinite integral and the other term gives $\frac{d}{dx}[UV] = \frac{dU}{dx}V + U\frac{dV}{dx}$ by the product rule. Thus,

$$\frac{d}{dx}\left[UV - \int V \frac{dU}{dx}\, dx\right] = \frac{dU}{dx}V + U\frac{dV}{dx} - V\frac{dU}{dx} = U\frac{dV}{dx}.$$

On the other hand, by the definition of the indefinite integral,

$$\frac{d}{dx}[\int U \frac{dV}{dx}\, dx] = U\frac{dV}{dx}.$$

Therefore, we find the derivatives of the l.h.s and r.h.s. of (1.) agree hence by the definition of indefinite integration the equality is justified (recall indefinite integration is a shorthand for a whole class of antiderivative functions which differ by at most a constant). The proof of (2.) follows immediately from (1.) together with FTC part II. □

Example 1.2.

$$\begin{aligned}\int x\sin(x)dx = \int \underbrace{x}_{U}\,\underbrace{\sin(x)\,dx}_{dV} &= UV - \int V dU \\ &= -x\cos(x) - \int (-\cos(x))dx \\ &= \boxed{-x\cos(x) + \sin(x) + c.}\end{aligned}$$

I observed that $V = \int dV = \int \sin(x)\,dx = -\cos(x) + c$ *and* $dU = dx$. *You might wonder why I didn't have to add a constant to* V. *Let's discuss this in general:*

Suppose you replace V with $\tilde{V} = V + c$ for some constant c then the formula for IBP changes to:

$$\begin{aligned}\int U d\tilde{V} &= U\tilde{V} - \int \tilde{V} dU \\ &= U(V+c) - \int (V+c)dU \\ &= UV + cU - c\int dU - \int V dU \\ &= UV - \int V dU.\end{aligned}$$

Therefore, either V or $V + c$ will yield the same answer under IBP so we are free to choose c however we do so desire. For example, in the first example I just took $c = 0$ for convenience. It is very likely I will continue to do so for the forseaable future. Another way of stating this result is to comment that when we find V from dV we just need an antiderivative, not the indefinite integral. We take account of the indefinite integral at the end of the calculation when we add c in the final step. You may recall from calculus I that the indefinite integral is technically a whole family of functions whereas the antiderivative is a specific function. Since some of you didn't have me for calculus I, I will elaborate a little further: if we were more technical and less traditional we'd have to write something like $\int x dx = \{\frac{1}{2}x^2 + c \mid c \in \mathbb{R}\}$. But, to be less notationally obtuse we have agreed to write $\int x dx = \frac{1}{2}x^2 + c$ to indicate the set of functions. Ok, now that I have you all properly confused, let's do some more examples. Come back to this paragraph later if at first it doesn't make sense.

Example 1.3.

$$\begin{aligned}\int_0^1 x2^x\,dx = \int_0^1 \underbrace{x}_{U}\underbrace{2^x\,dx}_{dV} &= UV\Big|_0^1 - \int_0^1 V dU \\ &= \frac{1}{\ln(2)}x2^x\Big|_0^1 - \int_0^1 \frac{1}{\ln(2)}2^x dx \\ &= \frac{1}{\ln(2)}x2^x\Big|_0^1 - \frac{1}{[\ln(2)]^2}2^x\Big|_0^1 \\ &= \boxed{\frac{2}{\ln(2)} - \frac{1}{[\ln(2)]^2}(2-1).} \\ &= \boxed{\frac{2}{\ln(2)} - \frac{1}{[\ln(2)]^2}.}\end{aligned}$$

Example 1.4. *In the calculation below we use IBP twice.*

$$\begin{aligned}
\int x^2\sin(x)dx = \int \underbrace{x^2}_{U_1}\underbrace{\sin(x)\,dx}_{dV_1} &= -x^2\cos(x) - \int(-\cos(x))2xdx \\
&= -x^2\cos(x) + \int \underbrace{2x}_{U_2}\underbrace{\cos(x)\,dx}_{dV_2} \\
&= -x^2\cos(x) + 2x\sin(x) - \int \sin(x)2dx \\
&= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + c.}
\end{aligned}$$

The basic idea is that each application of IBP can reduce x^n to x^{n-1}. If we know how to calculate $\int f(x)dx$ then the integral $\int x^n f(x)dx$ is solvable by repeated application of IBP.
Challenge: show that

$$\int x^3\sin(x)dx = -x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x) + c.$$

IBP often allows us to integrate inverse functions: in this type of example there is only one reasonable choice for dV, it must be dx hence $V = x$.

Example 1.5.

$$\begin{aligned}
\int \sin^{-1}(x)dx = \int \underbrace{\sin^{-1}(x)}_{U}\underbrace{dx}_{dV} &= x\sin^{-1}(x) - \int \frac{xdx}{\sqrt{1-x^2}} \\
&= x\sin^{-1}(x) + \frac{1}{2}\int \frac{dw}{\sqrt{w}} \qquad \textit{(here } w = 1-x^2\textit{)} \\
&= x\sin^{-1}(x) + \frac{1}{2}\frac{\sqrt{w}}{\frac{1}{2}} + c \\
&= \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + c.}
\end{aligned}$$

Challenge: follow the method of the example above to find integrals of $\cos^{-1}(x)$ and $\tan^{-1}(x)$. What about $x\sin^{-1}(x)$? Could you integrate that function? You may not have seen the hyperbolic trig. functions in calculus I. If that is the case then you should probably ask me in office hours.

Example 1.6.

$$\begin{aligned}
\int \cosh^{-1}(x)dx = \int \underbrace{\cosh^{-1}(x)}_{U}\underbrace{dx}_{dV} &= x\cosh^{-1}(x) - \int \frac{xdx}{\sqrt{x^2-1}} \\
&= x\cosh^{-1}(x) - \frac{1}{2}\int \frac{dw}{\sqrt{w}} \qquad \textit{(here } w = x^2-1\textit{)} \\
&= x\cosh^{-1}(x) - \frac{1}{2}\frac{\sqrt{w}}{\frac{1}{2}} + c \\
&= \boxed{x\cosh^{-1}(x) - \sqrt{x^2-1} + c.}
\end{aligned}$$

Challenge: follow the method of the example above to find integrals of $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$.

Example 1.7.

$$\int \ln(x)dx = \int \underbrace{\ln(x)}_{U} \underbrace{dx}_{dV} = x\ln(x) - \int \frac{xdx}{x}$$
$$= \boxed{x\ln(x) - x + c.}$$

The pattern we've observed in the last couple examples is something like this:

$$\int g(x)\,dx = \int g(x)\frac{dx}{dx}\,dx = \int \left[\frac{d}{dx}[xg] - x\frac{dg}{dx}\right]dx = xg - \int x\frac{dg}{dx}dx.$$

It's good to remember that all IBP does is to use the product rule. If you forget the formula it should be easy to find it again from the product rule. The examples that follow here are probably more like your homework.

Example 1.8. *I encourage the use of brackets to eliminate careless sign errors.*

$$\int \sin(\ln(x))dx = \int \underbrace{\sin(\ln(x))}_{U_1} \underbrace{dx}_{dV_1} = x\sin(\ln(x)) - \int x\frac{\cos(\ln(x))dx}{x}$$
$$= x\sin(\ln(x)) - \left[\int \underbrace{\cos(\ln(x))}_{U_2} \underbrace{dx}_{dV_2}\right]$$
$$= x\sin(\ln(x)) - \left[x\cos(\ln(x)) - \int x\left(-\frac{\sin(\ln(x))dx}{x}\right)\right]$$
$$= x\sin(\ln(x)) - \left[x\cos(\ln(x)) + \int \sin(\ln(x))dx\right]$$
$$= x\sin(\ln(x)) - x\cos(\ln(x)) - \int \sin(\ln(x))dx.$$

Ok, what's the answer? You tell me. Hint: let $I = \int \sin(\ln(x))dx$ *and solve for* I*. Don't forget to add* c *to the final result since we're calculating an indefinite integral.*

This pattern of looping back to where we start is found in a number of common integrals.

Example 1.9. *I could switch and use the* $\sin(3x+1)dx$ *as the* dV*, but I think it's easier to choose* $dV = e^x dx$ *since* e^x *is trivial to integrate.*

$$\int e^x\sin(3x+1)dx = \int \underbrace{\sin(3x+1)}_{U_1} \underbrace{e^x dx}_{dV_1} = e^x\sin(3x+1) - 3\int \cos(3x+1)e^x dx$$
$$= e^x\sin(3x+1) - 3\left[\int \underbrace{\cos(3x+1)}_{U_2} \underbrace{e^x dx}_{dV_2}\right]$$
$$= e^x\sin(3x+1) - 3\left[e^x\cos(3x+1) - 3\int -\sin(3x+1)e^x dx\right]$$
$$= e^x\sin(3x+1) - 3e^x\cos(3x+1) - 9\int e^x\sin(3x+1)dx$$
$$= e^x\sin(3x+1) - 3e^x\cos(3x+1) - 9\int e^x\sin(3x+1)dx.$$

Therefore,

$$\boxed{\int e^x \sin(3x+1)dx = \frac{1}{10}e^x\big[\sin(3x+1) - \cos(3x+1)\big] + c.}$$

I'm not sure what will happen in the next example. This is an experiment.

Example 1.10.

$$\begin{aligned}\int \sin^2(x)dx = \int \underbrace{\sin(x)}_{U_1}\underbrace{\sin(x)dx}_{dV_1} &= -\sin(x)\cos(x) + \int \cos(x)\cos(x)dx \\ &= -\sin(x)\cos(x) + \int (1-\sin^2(x))dx \\ &= -\sin(x)\cos(x) + x - \int \sin^2(x)dx\end{aligned}$$

Therefore, *by algebra!* ,

$$\boxed{\int \sin^2(x)dx = \frac{1}{2}(x - \sin(x)\cos(x)) + c.}$$

I usually integrate $\sin^2(x)$ via the trigonmetric identity $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ from which it is obvious that $\int \sin^2(x)dx = \frac{x}{2} - \frac{1}{4}\sin(2x) + c$. Can you see the answer we just found from IBP is the same answer?

A formula which is defined by showing how one step goes to the next step is called a *recursive* formula. There are a number of interesting recursive formulas which allow us to integrate arbitrarily high powers of trigonmetric functions through a simple step-by-step procedure. I don't expect you memorize these, but I do expect you could re-derive them if asked. In other words, you should understand the next example.

Example 1.11. *Let $k \in \mathbb{N}$ and consider:*

$$\begin{aligned}\int \sin^k(x)dx = \int \underbrace{\sin^{k-1}(x)}_{U_1}\underbrace{\sin(x)dx}_{dV_1} &= -\sin^{k-1}(x)\cos(x) + \int \cos(x)(k-1)\sin^{k-2}(x)\cos(x)dx \\ &= -\sin^{k-1}(x)\cos(x) + (k-1)\int \sin^{k-2}(x)\big[1-\sin^2(x)\big]dx \\ &= -\sin^{k-1}(x)\cos(x) + (k-1)\int \sin^{k-2}(x)dx - (k-1)\int \sin^k(x)dx\end{aligned}$$

Solve for $\int \sin^k(x)dx$ (note $k \neq 0$ since $k \in \mathbb{N}$)

$$\boxed{\int \sin^k(x)dx = \frac{-1}{k}\sin^{k-1}(x)\cos(x) + \frac{k-1}{k}\int \sin^{k-2}(x)dx + c.}$$

We already have integrals for $\sin^k(x)$ in the cases $k = 1, 2$ so we can use the recursive formula above to calculate the cases $k = 3, 4$.

Example 1.12. *Use Example 1.11 to find for the $k = 4$ case,*

$$\begin{aligned}
\int \sin^4(x)dx &= \frac{-1}{4}\sin^{4-1}(x)\cos(x) + \frac{4-1}{4}\int \sin^{4-2}(x)dx + c \\
&= \frac{-1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left(\frac{1}{2}(x - \sin(x)\cos(x))\right) + c \\
&= \boxed{\frac{3}{8}x - \frac{3}{8}\sin(x)\cos(x) - \frac{1}{4}\sin^3(x)\cos(x) + c.}
\end{aligned}$$

Example 1.13. *Again, use Example 1.11 in the $k = 3$ case,*

$$\begin{aligned}
\int \sin^3(x)dx &= \frac{-1}{3}\sin^{3-1}(x)\cos(x) + \frac{3-1}{3}\int \sin^{3-2}(x)dx + c \\
&= \frac{-1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x) + c \\
&= \frac{-1}{3}\big(1 - \cos^2(x)\big)\cos(x) - \frac{2}{3}\cos(x) + c \\
&= \boxed{\frac{1}{3}\cos^3(x) - \cos(x) + c.}
\end{aligned}$$

2 Integrals of Trigonometric Functions

In this section we return to the problem of integrating trigonometric functions. The tools used here are a combination of basic u-substitution, judiciously chosen trigonometric identities and as a last resort IBP. I'll begin by attacking the problem of $\sin^3(x)$ which we just solved by IBP in the previous section. This is an easier way:

Example 2.1.

$$\begin{aligned}
\int \sin^3(x)dx &= \int \sin^2(x)\sin(x)dx \\
&= \int \big[1 - u^2\big](-du) \qquad \textit{(where } u = \cos(x)\textit{)} \\
&= -u + \frac{1}{3}u^3 + c \\
&= -\cos(x) + \frac{1}{3}\cos^3(x) + c
\end{aligned}$$

The integral of $\sin^4(x)$ is not as easy in my view.

Example 2.2.

$$\begin{aligned}
\int \sin^4(x)dx &= \int \left[\sin^2(x)\right]^2 dx \\
&= \int \left[\frac{1}{2}(1-\cos(2x))\right]^2 dx \\
&= \frac{1}{4}\int \left[1 - 2\cos(2x) + \cos^2(2x)\right] dx \\
&= \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{1}{8}\int (1+\cos(4x))dx \\
&= \frac{x}{4} - \frac{1}{4}\sin(2x) + \frac{x}{8} + \frac{1}{32}\sin(4x) + c \\
&= \boxed{\frac{3x}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c.}
\end{aligned}$$

I invite the reader to verify that Example 1.12 and the example above are consistent. Actually, would you know where to start in comparing these answers?

If you ponder the methods we just used to integrate $\sin^k(x)$ you should be able to integrate any sum or product of $\sin(x)$ and $\cos(x)$. For example, see if you can calculate the integrals $\int \sin(x)\cos(x)dx$ or $\int \sin^2(x)\cos^2(x)dx$. Sums of products and reciprocals of sine and cosine require more thought but, many are not too difficult.

Example 2.3. *Let let $u = \cos(x)$ in the calculation below:*

$$\begin{aligned}
\int \frac{\sin(x)}{\cos(x)}dx &= \int \frac{-du}{u} \\
&= \boxed{-\ln|\cos(x)| + c.}
\end{aligned}$$

Therefore, $\int \tan(x)dx = \ln|\sec(x)| + c$.

I hope you can figure out $\int \cot(x)dx$ with ease. It is important to remember $\tan^2(x)+1 = \sec^2(x)$ and $\int \sec^2(x)dx = \tan(x) + c$ in the examples that follow.

Example 2.4.

$$\begin{aligned}
\int \tan^2(x)dx &= \int (\sec^2(x) - 1)dx \\
&= \boxed{\tan(x) - x + c.}
\end{aligned}$$

Example 2.5. *We let $u = \tan(x)$ so $du = \sec^2(x)dx$,*

$$\begin{aligned}
\int \sec^2(x)\tan^2(x)dx &= \int u^2 du \\
&= \frac{1}{3}u^3 + c \\
&= \boxed{\frac{1}{3}\tan^3(x) + c.}
\end{aligned}$$

Example 2.6.

$$\begin{aligned}
\int \tan^4(x)dx &= \int \tan^2(x)(\sec^2(x)-1)dx \\
&= \int \tan^2(x)\sec^2(x)dx - \int \tan^2(x)dx \\
&= \int \tan^2(x)d(\tan(x)) - \int \tan^2(x)dx \\
&= \boxed{\frac{1}{3}\tan^3(x) + \tan(x) - x + c.}
\end{aligned}$$

The notation used in the third line of the calculation above is a slick implicit notation for indicating a $u = \tan(x)$ substitution. Every so often I make use of this notation. In any event, you should be able to integrals of expressions like $\int \sec^6(x)dx$ or $\int \cot^2(x)dx$ or $\int \cot^2(x)\csc^2(x)dx$ using arguments paralelling the previous triple of examples. What lies beneath is scarier.

Example 2.7. *Observe that if* $u = \sec(x) + \tan(x)$ *then* $\frac{du}{u} = \sec(x)dx$ *(work it out for yourself!). With this bit of trivia in mind note:*

$$\begin{aligned}
\int \sec(x)dx &= \int \frac{du}{u} \\
&= \ln|u| + c \\
&= \boxed{\ln|\sec(x) + \tan(x)| + c.}
\end{aligned}$$

Ok, by now you should expect me to ask if you can integrate $\int \csc(x)dx$ given the patterns above. Given our work thus far it ought to be clear that integrating even powers of secant is actully pretty easy. On the other hand, the first odd power above required a stroke of genious. If you try to convert to a sine/cosine integral it does not help much if you were wondering. Maybe we'll return to secant in a future section and I'll show you a less clever way of calculating the integral. For now we move on to the dreaded $\sec^3(x)$.

Example 2.8. *I use the result of the previous example to go from the* 4^{th} *to the* 5^{th} *line of the calculation below.*

$$\begin{aligned}
\int \sec^3(x)dx &= \int \underbrace{\sec(x)}_{U_1}\underbrace{\sec^2(x)dx}_{dV_1} \\
&= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)dx \\
&= \sec(x)\tan(x) - \int (\sec^2(x)-1)\sec(x)dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx \\
&= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)dx.
\end{aligned}$$

Therefore,

$$\boxed{\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.}$$

Given the example above you ought to have a shot at completing integrals like $\int \sec^5(x)dx$ or $\int \csc^3(x)dx$. The integrals that follow in this section require a review of further trigonometry.

2.1 how to derive trigonometric formulas in a few easy steps

We study two methods to derive identities in trigonometry. Let us begin with the less elegant method. With a little trouble and ingenuity you can use the Law of cosines applied to certain pictures to deduce the fundamental identities which I refer to as the **adding angles identities**

$$\boxed{\cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta}$$

$$\boxed{\sin(\theta+\beta) = \sin\theta\cos\beta + \cos\theta\sin\beta}$$

With these two identities we can derive most anything we want. The examples that follow are in no particular order. I only use the adding angle identities and the definitions of tangent plus a little algebra.

Example 2.9.

$$\begin{aligned} \tan(\theta+\beta) &= \frac{\sin(\theta+\beta)}{\cos(\theta+\beta)} \\ &= \frac{\sin\theta\cos\beta + \cos\theta\sin\beta}{\cos\theta\cos\beta - \sin\theta\sin\beta} \\ &= \frac{\frac{\sin\theta\cos\beta}{\cos\theta\cos\beta} + \frac{\cos\theta\sin\beta}{\cos\theta\cos\beta}}{\frac{\cos\theta\cos\beta}{\cos\theta\cos\beta} - \frac{\sin\theta\sin\beta}{\cos\theta\cos\beta}} \quad \Rightarrow \quad \boxed{\tan(\theta+\beta) = \frac{\tan\theta+\tan\beta}{1-\tan\theta\tan\beta}} \end{aligned}$$

While we are on this example, note if $\theta = \beta$ then we find

$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}}$$

Example 2.10. *The case $\theta = \beta$ gives interesting formulas for sine and cosine,*

$$\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta \quad \Rightarrow \quad \boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta.}$$

Likewise,

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \quad \Rightarrow \quad \boxed{\sin(2\theta) = 2\sin\theta\cos\theta.}$$

Since $\cos^2\theta + \sin^2\theta = 1$ thus $\sin^2\theta = 1 - \cos^2\theta$ it follows that $\cos(2\theta) = 2\cos^2\theta - 1$ hence

$$\boxed{\cos^2\theta = \frac{1}{2}\Big(1+\cos(2\theta)\Big)}$$

Similarly we can solve for $\sin^2\theta$ to obtain,

$$\boxed{\sin^2\theta = \frac{1}{2}\Big(1-\cos(2\theta)\Big)}$$

Naturally, we can continue in this fashion to derive a great variety of trigonometric identities. There is something somewhat unsatisfying about this method. The calculation is indirect. Suppose we wanted to simplify the expression $\sin(\theta)\cos(4\theta)$. How would we do it? To be fair, there are identities for $\sin(\theta)\sin(\beta), \cos(\theta)\cos(\beta)$ and $\sin(\theta)\cos(\beta)$ so we could just look those up and go from there. But, is there a better way to remember all these facts? Is there some elegant formula which encapsulates all these trigonometric identities and reduces these problems to little more than algebra? In fact, yes. However, it comes at the price of understanding a bit of basic complex variables. I would argue that this is a worthy price since most students need to learn more about complex numbers anyway.

Definition 2.11. *imaginary exponential.*

Let θ be a real number then we define the **imaginary exponential** to be the complex number $e^{i\theta}$ given by $e^{i\theta} = \cos\theta + i\sin\theta$.

I invite the reader to verify the following identity hold for the imaginary exponential:

$$e^{i\theta}e^{i\beta} = e^{i(\theta+\beta)}$$

The above identity simultaneously contains both the adding angles formula for sine and cosine. Using $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ we can calculate:

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta \;\Rightarrow\; \boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

and

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta = 2i\sin\theta \;\Rightarrow\; \boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

The boxed formulas above show we can trade sine and cosine for imaginary exponentials. This is of great algebraic advantage since imaginary exponentials obey simple rules of algebra much like real exponentials[1].

Example 2.12. *Suppose you want to derive a nice formula for the square of cosine. Just plug in the boxed formula and use the laws of exponents for imaginary exponentials:*

$$\begin{aligned}
\cos^2\theta &= \left[\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right]^2 \\
&= \frac{1}{4}\left(e^{i\theta}e^{i\theta} + 2e^{i\theta}e^{-i\theta} + e^{-i\theta}e^{-i\theta}\right) \\
&= \frac{1}{4}\left(e^{2i\theta} + 2 + e^{-2i\theta}\right) \\
&= \frac{1}{2} + \frac{1}{2}\frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) \\
&= \frac{1}{2} + \frac{1}{2}\cos 2\theta \\
&= \frac{1}{2}\left(1 + \cos 2\theta\right).
\end{aligned}$$

Naturally, we could also apply the method to calculate formulas for higher powers or products of sine and cosine. Just for a flavor:

Example 2.13.

$$\begin{aligned}
\cos^3\theta &= \left[\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right]^3 \\
&= \frac{1}{8}\left(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}\right) \\
&= \frac{3}{4}\sin(\theta) + \frac{1}{4}\sin(3\theta).
\end{aligned}$$

[1]not exactly the same, the one-to-one property does not hold for the imaginary exponential, instead $e^{i\theta} = e^{i\beta}$ implies $\theta - \beta \in 2\pi\mathbb{Z}$

Example 2.14. *Suppose you want to derive a nice formula for the square of sine. Just plug in the boxed formula and use the laws of exponents for imaginary exponentials*

$$\begin{aligned}
\sin^2\theta &= \left[\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right]^2 \\
&= \frac{-1}{4}\left(e^{i\theta}e^{i\theta} - 2e^{i\theta}e^{-i\theta} + e^{-i\theta}e^{-i\theta}\right) \\
&= \frac{-1}{4}\left(e^{2i\theta} - 2 + e^{-2i\theta}\right) \\
&= \frac{1}{2} - \frac{1}{2}\frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right) \\
&= \frac{1}{2} - \frac{1}{2}\cos 2\theta \\
&= \frac{1}{2}\left(1 - \cos 2\theta\right).
\end{aligned}$$

The identities above you should have memorized anyway, but I don't have to memorize them since I can **derive** them in a pinch. In contrast, the next example is not one for which I could typically quote the answer off the top of my head:

Example 2.15. *Same method again. Covert given functions to imaginary exponentials and do algebra until you see sines and cosines again. Simple as that.*

$$\begin{aligned}
\cos(x)\sin(4x) &= \frac{1}{2}\left(e^{ix} - e^{-ix}\right)\frac{1}{2i}\left(e^{4ix} - e^{-4ix}\right) \\
&= \frac{1}{4i}\left(e^{5ix} - e^{-3ix} - e^{3ix} + e^{-5ix}\right) \\
&= \frac{1}{2}\left[\frac{1}{2i}\left(e^{5ix} - e^{-5ix}\right) + \frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right)\right] \\
&= \frac{1}{2}\sin(5x) + \frac{1}{2}\sin(3x)
\end{aligned}$$

You could calculate identities for $\cos(ax)\cos(bx), \sin(ax)\sin(bx)$ by much the same calculation and you'd find a sum of cosines for each:

$$\boxed{\cos(ax)\cos(bx) = \frac{1}{2}\cos[(a+b)x] + \frac{1}{2}\cos[(a-b)x]}$$

$$\boxed{\sin(ax)\sin(bx) = \frac{1}{2}\cos[(a+b)x] - \frac{1}{2}\cos[(a-b)x]}$$

On the other hand, generally $\cos(ax)\sin(bx)$ yields a sum of sines,

$$\boxed{\cos(ax)\sin(bx) = \frac{1}{2}\sin[(a+b)x] + \frac{1}{2}\sin[(a-b)x]}$$

The product formulas are very important to the study of constructive and destructive inteference in waves. They explain where beats come from among other things. Also, it is worth mentioning that if you remember one of these carefully then you can get others from differentiating. Try differentiating $\sin(a+x)$ to derive the adding angles formula for $\cos(a+x)$.

DeMoivres' theorem in complex notation is simply $(e^{i\theta})^n = e^{in\theta}$. When you unfold this into sines and cosines the result is amazing:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

You can try plugging in $n = 2$ or $n = 3$ and you'll find yet more identities which are less than obvious from other approaches.

Example 2.16.

$$\begin{aligned}
\int \cos(3x)\sin(5x)dx &= \int \left[\frac{1}{2}\sin(8x) + \frac{1}{2}\sin(-2x)\right]dx \\
&= \frac{1}{2}\int \sin(8x)dx - \frac{1}{2}\int \sin(2x)dx \\
&= \boxed{\frac{-1}{16}\cos(8x) - \frac{1}{4}\cos(2x) + c.}
\end{aligned}$$

Example 2.17.

$$\begin{aligned}
\int \cos(3x)\cos(5x)dx &= \int \left[\frac{1}{2}\cos(8x) + \frac{1}{2}\cos(-2x)\right]dx \\
&= \frac{1}{2}\int \cos(8x)dx + \frac{1}{2}\int \cos(2x)dx \\
&= \boxed{\frac{1}{16}\sin(8x) + \frac{1}{4}\sin(2x) + c.}
\end{aligned}$$

Example 2.18.

$$\begin{aligned}
\int \sin(3x)\sin(5x)dx &= \int \left[\frac{1}{2}\cos(8x) - \frac{1}{2}\cos(-2x)\right]dx \\
&= \frac{1}{2}\int \cos(8x)dx - \frac{1}{2}\int \cos(2x)dx \\
&= \boxed{\frac{1}{16}\sin(8x) - \frac{1}{4}\sin(2x) + c.}
\end{aligned}$$

What about $\int \sin(x)\cos(3x)\cos(6x)dx$? How would you attack such a problem?

Example 2.19. *Here we use the adding angles identity for tangent followed by a $u = \cos(4x)$ substitution.*

$$\begin{aligned}
\int \frac{\tan(x) + \tan(3x)}{1 - \tan(x)\tan(3x)}dx &= \int \tan(4x)dx \\
&= \int \frac{\sin(4x)}{\cos(4x)}dx \\
&= \int \frac{-du}{4u} \\
&= \boxed{\frac{-1}{4}\ln|\cos(4x)| + c.}
\end{aligned}$$

Finally, I would just comment that there are many integrations of the hyperbolic trigonometric functions which follow arguments paralell to those given in this section. I'll illustrate a few such calculations in the next subsection.

2.2 hyperbolic functions: calculus and algebra

Hyperbolic sine and cosine are defined by $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ thus $e^x = \cosh x + \sinh x$ and as $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$ we find $e^{-x} = \cosh x - \sinh x$ and observe

$$1 = e^x e^{-x} = (\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x$$

which is the analog of the Pythagorean identity for sine and cosine.

Example 2.20. *Let $u = \sinh x$ in what follows:*

$$\begin{aligned}
\int \cosh^3 x dx = \int (cosh^2 x) \cosh x dx = \int (1 + \sinh^2 x) \cosh x dx \\
&= \int (1 + u^2) du \\
&= u + \frac{1}{3} u^3 \\
&= \boxed{\sinh x + \frac{1}{3}\sinh^3(x) + c}
\end{aligned}$$

Also, we find an analog of the double-angle formula for cosine:

$$\cosh^2 x = \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right)^2 = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) = \frac{1}{2}\left(1 + \cosh(2x)\right).$$

Differentiate with respect to x to find $2\cosh x \sinh x = \sinh(2x)$.

Example 2.21.

$$\int 2\cosh^2 x = \int (1 + \cosh(2x))dx = \boxed{x + \frac{1}{2}\sinh(2x) + c}$$

Also, since $\cosh^2 x - \sinh^2 x = 1$ we find

$$\sinh^2 x = \cosh^2 x - 1 = \frac{1}{2}\left(1 + \cosh(2x)\right) - 1 = \frac{1}{2}\left(\cosh(2x) - 1\right).$$

Moreover, dividing $\cosh^2 x - \sinh^2 x = 1$ by $\cosh^2 x$ reveals $1 - \tanh^2 x = \text{sech}^2 x$. Note

$$\begin{aligned}
\cosh(A + B) &= \frac{1}{2}\left(e^{A+B} + e^{-(A+B)}\right) \\
&= \frac{1}{4}\left(e^A + e^{-A}\right)\left(e^B + e^{-B}\right) + \frac{1}{4}\left(e^A - e^{-A}\right)\left(e^B - e^{-B}\right) \\
&= \cosh(A)\cosh(B) + \sinh(A)\sinh(B).
\end{aligned}$$

Differentiate with respect to A to derive

$$\sinh(A + B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B).$$

Divide the identities above and derive the addiing angles formula for hyperbolic tangent:

$$\tanh(A + B) = \frac{\sinh(A)\cosh(B) + \cosh(A)\sinh(B)}{\cosh(A)\cosh(B) + \sinh(A)\sinh(B)} = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

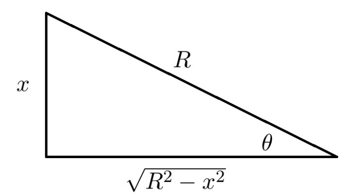
I'll stop here, but it should be apparent by now that to every calculation in trigonometry there is a dual calculation in hyperbolic trigonometry. They're not exactly the same, but if we understand the analogies at play here we understand both trigonometry and hyperbolic trigonometry more deeply. This dualism plays out more powerfully in its application in the next section.

3 implicit substitutions

All of the substitutions made thus far have been explicit. This means the new substituted variable is defined explicitly as a function of the given variable of integration. An implicit substitution may give the change of variables in many other forms. Most common among these is the so-called trig-substitution where we usually either use $x = a\cos(\theta)$ or $x = a\sin(\theta)$ or $x = a\sec(\theta)$ or $x = a\tan(\theta)$. In each of these cases the new variable which is analogous to the u of our earlier work is played by the variable θ. We'll also look at substitutions based on the inverse hyperbolic functions. We will need the techniques of the previous section to properly complete many of these problems. It really is a beautiful chapter in the theory of calculation, an art which is slowly but surely being lost to the ever encroaching hob-goblin of mediocrity in the calculus sequence.

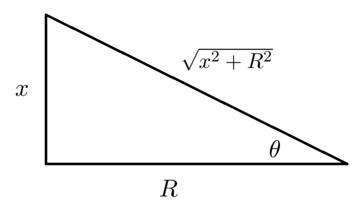
Let me begin by collecting three main trigonometric substitutions:

(1.) If $x = R\sin\theta$ then $dx = R\cos\theta\, d\theta$ and $R^2 - x^2 = R^2\cos^2\theta$ hence $\sqrt{R^2 - x^2} = R\cos\theta$.



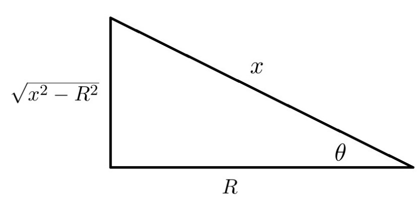
From the diagram above we see $\tan\theta = \dfrac{x}{\sqrt{R^2 - x^2}}$ and $\cos\theta = \dfrac{\sqrt{R^2 - x^2}}{R}$.

(2.) If $x = R\tan\theta$ then $dx = R\sec^2\theta\, d\theta$ and $x^2 + R^2 = R^2\sec^2\theta$ hence $\sqrt{x^2 + R^2} = R\sec\theta$



From the diagram above we observe $\sin\theta = \dfrac{x}{\sqrt{x^2 + R^2}}$ and $\cos\theta = \dfrac{R}{\sqrt{x^2 + R^2}}$.

(3.) If $x = R\sec\theta$ then $dx = R\sec\theta\tan\theta\, d\theta$ and $x^2 - R^2 = R^2\tan^2\theta$ hence $\sqrt{x^2 - R^2} = R\tan\theta$



From the diagram above we find $\tan\theta = \dfrac{\sqrt{x^2 - R^2}}{R}$ and $\sin\theta = \dfrac{\sqrt{x^2 - R^2}}{x}$.

The key concept behind all the substitions is that we wish to use $\cos^2\theta + \sin^2\theta = 1$ or $\tan^2\theta + 1 = \sec^2\theta$ in order to consolidate two terms into a single term. Let's see how the trigonometry and calculus aids our solution to a number of tricky integrals.

Example 3.1. *We wish to calculate* $\int \sqrt{4-x^2}\; dx$*. The squareroot is trouble. One sneaky way to eliminate it is to let* $x = 2\sin(\theta)$ *thus* $4-x^2 = 4-4\sin^2(\theta) = 4\cos^2(\theta)$ *and* $dx = 2\cos(\theta)d\theta$*. Hence,*

$$\begin{aligned}
\int \sqrt{4-x^2}dx &= \int \sqrt{4\cos^2(\theta)}2\cos(\theta)d\theta \\
&= \int 4\cos^2(\theta)d\theta \\
&= \int \big[2+\cos(2\theta)\big]d\theta \\
&= 2\theta + \frac{1}{2}\sin(2\theta) + c \\
&= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left[\frac{x}{2}\right]\right) + c.}
\end{aligned}$$

The careful student will have questioned how I knew that $\sqrt{4\cos^2(\theta)} = 2\cos(\theta)$. After all, algebraically we ought to have $\sqrt{4\cos^2(\theta)} = 2|\cos(\theta)|$. The answer is simply that we insist θ be chosen such that $\cos(2\theta) \geq 0$, otherwise we'd have to introduce a sign in that step. Similar comments apply to future examples, but I will not belabor this point further.

Example 3.2. *We wish to calculate* $\int \sqrt{x^2-9}\; dx$*. The squareroot is trouble. One sneaky way to eliminate it is to let* $x = 3\sec(\theta)$ *thus* $x^2-9 = 9\sec^2(\theta)-9 = 9\tan^2(\theta)$ *and* $dx = 3\sec(\theta)\tan(\theta)d\theta$*. Hence,*

$$\begin{aligned}
\int \sqrt{x^2-9}dx &= \int \sqrt{9\tan^2(\theta)}3\sec(\theta)\tan(\theta)d\theta \\
&= \int 9\sec(\theta)\tan^2(\theta)d\theta \\
&= \int 9\sec(\theta)(\sec^2(\theta)-1)d\theta \\
&= 9\int \sec^3(\theta)d\theta - 9\int \sec(\theta)d\theta \\
&= \frac{9}{2}\sec(\theta)\tan(\theta) + \frac{9}{2}\ln|\sec(\theta)+\tan(\theta)| - 9\ln|\sec(\theta)+\tan(\theta)| + c \\
&= \frac{9}{2}\sec(\theta)\tan(\theta) - \frac{9}{2}\ln|\sec(\theta)+\tan(\theta)| + c
\end{aligned}$$

We can simplify this answer nicely if we think about the substitution in terms of a triangle. Note that if $x = 3\sec(\theta)$ *then* $\sec(\theta) = \frac{x}{3} = \frac{hyp}{adj}$ *hence opp* $= \sqrt{x^2-9}$ *and so* $\tan(\theta) = \frac{\sqrt{x^2-9}}{3}$*. We find,*

$$\boxed{\int \sqrt{x^2-9}\; dx = \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + c.}$$

Example 3.3. *The fundamental identity for hyperbolic trigonometry is* $\cosh^2(\phi) - \sinh^2(\phi) = 1$*. We can rearrange this to give* $\cosh^2(\phi) - 1 = \sinh^2(\phi)$*. This suggests we might be able to work the previous example by making a* $x = 3\cosh(\phi)$ *substitution. If* $x = 3\cosh(\phi)$ *then* $dx = 3\sinh(\phi)d\phi$ *and* $x^2-9 = 9\sinh^2(\phi)$*. Hence,*

$$\int \sqrt{x^2-9}\; dx = \int 9\sinh^2(\phi)d\phi.$$

I need an identity for $\sinh^2(\phi)$*, let's derive it from scratch,*

$$\sinh^2(\phi) = \left[\frac{1}{2}(e^{\phi} - e^{-\phi})\right]^2 = \frac{1}{4}\left[e^{2\phi} - 2 + e^{-2\phi}\right] = \frac{1}{2}\cosh(2\phi) - \frac{1}{2}$$

Returning to our integration with this new-found insight,

$$\int \sqrt{x^2-9}\ dx = \int \left(\frac{9}{2}\cosh(2\phi) - \frac{9}{2}\right)d\phi = \frac{9}{4}\sinh(2\phi) - \frac{9\phi}{2} + c.$$

Note that $2\sinh(\phi)\cosh(\phi) = \frac{1}{2}(e^{\phi} - e^{-\phi})(e^{\phi} + e^{-\phi}) = \frac{1}{2}(e^{2\phi} - e^{-2\phi}) = \sinh(2\phi)$*. Furthermore, we began by supposing that* $x = 3\cosh(\phi)$ *hence* $\cosh(\phi) = \frac{x}{3}$ *and* $\sinh(\phi) = \sqrt{x^2-9}$*. We find that* $\frac{9}{4}\sinh(2\phi) = \frac{9}{2}\frac{x}{3}\frac{\sqrt{x^2-9}}{3} = \frac{1}{2}x\sqrt{x^2-9}$*. On the other hand, in our current formalism we are led to write* $\frac{9\phi}{2} = \frac{9}{2}\cosh^{-1}(\frac{x}{3})$*. Thus,*

$$\boxed{\int \sqrt{x^2-9}\ dx = \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\cosh^{-1}\left(\frac{x}{3}\right) + c.}$$

I should mention there are fascinating expressions which recast the inverse hyperbolic functions as the composite of a logarithm and an algebraic function. In particular:

$$\begin{aligned} \cosh^{-1}(x) &= \ln\left(x + \sqrt{x^2-1}\right) \qquad \text{for } x \geq 1. \\ \sinh^{-1}(x) &= \ln\left(x + \sqrt{x^2+1}\right) \qquad \text{for } x \in \mathbb{R}. \\ \tanh^{-1}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \text{for } |x| < 1. \end{aligned} \tag{1}$$

We can understand the previous pair of examples are in agreement if we sort through how the identity above for inverse hyperbolic cosine connects to the log of an algebraic function.

Example 3.4. *We wish to calculate* $\int \sqrt{4x^2+9}\ dx$*. The squareroot is trouble. One sneaky way to eliminate it is to let* $2x = 3\tan(\theta)$ *thus* $4x^2 + 9 = 9\tan^2(\theta) + 9 = 9\sec^2(\theta)$ *and* $2dx = 3\sec^2(\theta)d\theta$*. Hence,*

$$\begin{aligned} \int \sqrt{4x^2+9}\ dx &= \int \sqrt{9\sec^2(\theta)}\ \frac{3\sec^2(\theta)d\theta}{2} \\ &= \frac{9}{2}\int \sec^3(\theta)d\theta \\ &= \frac{9}{4}\sec(\theta)\tan(\theta) + \frac{9}{4}\ln|\sec(\theta) + \tan(\theta)| + c \end{aligned}$$

We can simplify this answer nicely if we think about the substitution in terms of a triangle. Note that if $2x = 3\tan(\theta)$ *then* $\tan(\theta) = \frac{2x}{3} = \frac{opp}{adj}$ *hence* $hyp = \sqrt{4x^2+9}$ *and so* $\sec(\theta) = \frac{\sqrt{4x^2+9}}{3}$*. We find,*

$$\boxed{\int \sqrt{4x^2+9}\ dx = \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\ln\left|\frac{\sqrt{4x^2+9}}{3} + \frac{2x}{3}\right| + c.}$$

Notice, we can simplify this answer by factoring out a $1/3$ *in the natural log argument,*

$$\boxed{\int \sqrt{4x^2+9}\ dx = \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\ln\left|\sqrt{4x^2+9} + 2x\right| + c.}$$

Example 3.5. *Another way to calculate* $\int \sqrt{4x^2+9}\ dx$ *is to make a hyperbolic substitution. Observe that* $\cosh^2(\phi) - \sinh^2(\phi) = 1$ *gives* $\cosh^2(\phi) = \sinh^2(\phi) + 1$. *This suggests we can make a* $2x = 3\sinh(\phi)$ *substitution. If* $2x = 3\sinh(\phi)$ *then* $4x^2 + 9 = 9\sinh^2(\phi) + 9 = 9\cosh^2(\phi)$ *and* $2dx = 3\cosh(\phi)d\phi$.

$$\begin{aligned}\int \sqrt{4x^2+9}\ dx &= \int \sqrt{9\cosh^2(\phi)}\ \frac{3\cosh(\phi)d\phi}{2} \\ &= \frac{9}{2}\int \cosh^2(\phi)d\phi\end{aligned}$$

I need an identity for $\cosh^2(\phi)$*, let's derive it from scratch,*

$$\cosh^2(\phi) = \left[\frac{1}{2}\big(e^{\phi} + e^{-\phi}\big)\right]^2 = \frac{1}{4}\left[e^{2\phi} + 2 + e^{-2\phi}\right] = \frac{1}{2}\cosh(2\phi) + \frac{1}{2}$$

Returning to our integration with this new-found insight,

$$\begin{aligned}\int \sqrt{4x^2+9}\ dx &= \frac{9}{4}\int \big(\cosh(2\phi) + 1\big)d\phi \\ &= \frac{9}{8}\sinh(2\phi) + \frac{9}{4}\phi + c. \\ &= \frac{9}{4}\frac{2x}{3}\frac{\sqrt{4x^2+9}}{3} + \frac{9}{4}\phi + c. \\ &= \boxed{\frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\sinh^{-1}\left(\frac{2x}{3}\right) + c.}\end{aligned}$$

I used the identity $\sinh(2\phi) = 2\sinh(\phi)\cosh(\phi)$ *to help simplify the answer.*

Again, you can consult Equation 1 to help see why the last two examples are in fact consistent. Also, by now you should start to appreciate that both the hyperbolic and the trig. substitutions have pros and cons. I believe a healthy approach is to be ready to apply either approach to a given problem. Knowledge of this mathematics will also aid you in interpreting Mathemtatica or other C.A.S. outputs. Mathematica might give the answer in terms of hyperbolics whereas you know the answer for other reasons in terms of logarithms. Hopefully as you work the problems in this section you'll start to understand that hyperbolic functions and trigonmetric functions are intrically linked. The explicit connection and ultimate synthesis for these functions is largely covered in the complex variables course. The following triple of examples all invoke a bit of algebra which is commonly called "completing the square[2]".

Example 3.6. *To begin I simplify the quadratic by completing the square:*

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}.$$

Remember that $\tan^2(\theta) + 1 = \sec^2(\theta)$ *so this makes me think a* $x + 2 = \tan(\theta)$ *substitution will simplify this problem. If* $x+2 = \tan(\theta)$ *then* $(x+2)^2+1 = \tan^2(\theta)+1 = \sec^2(\theta)$ *and* $dx = \sec^2(\theta)d\theta$.

[2] if you don't already know this algebra, now is the time to learn it

Thus,

$$\begin{aligned}\int \frac{dx}{x^2+4x+5} &= \int \frac{\sec^2(\theta)d\theta}{\sec^2(\theta)} \\ &= \theta + c \\ &= \boxed{\tan^{-1}(x+2)+c.}\end{aligned}$$

Example 3.7. *To begin I simplify the quadratic:*

$$\int \frac{dx}{4x^2+8x+9} = \int \frac{dx}{4(x+1)^2+5}.$$

Remember that $\tan^2(\theta)+1=\sec^2(\theta)$ *so this makes me think a* $4(x+1)^2 = 5\tan^2(\theta)$ *substitution will simplify this problem. If* $2(x+1)=\sqrt{5}\tan(\theta)$ *then* $4(x+1)^2+5 = 5\tan^2(\theta)+5 = 5\sec^2(\theta)$ *and* $2dx = \sqrt{5}\sec^2(\theta)d\theta$*. Thus,*

$$\begin{aligned}\int \frac{dx}{4x^2+8x+9} &= \int \frac{\sqrt{5}\sec^2(\theta)d\theta}{2(5\sec^2(\theta))} \\ &= \frac{1}{2\sqrt{5}}\theta + c \\ &= \boxed{\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2(x+1)}{\sqrt{5}}\right)+c.}\end{aligned}$$

Example 3.8. *To begin I simplify the quadratic by completing the square:*

$$\int \frac{dx}{x^2+6x+5} = \int \frac{dx}{(x+3)^2-4}.$$

Remember that $\tan^2(\theta) = \sec^2(\theta)-1$ *so this makes me think a* $(x+3)^2 = 4\sec^2(\theta)$ *substitution will simplify this problem. If* $x+3 = 2\sec(\theta)$ *then* $(x+3)^2-4 = 4\sec^2(\theta)-4 = 4\tan^2(\theta)$ *and* $dx = 2\sec(\theta)\tan(\theta)d\theta$*. Thus,*

$$\begin{aligned}\int \frac{dx}{x^2+6x+5} &= \int \frac{2\sec(\theta)\tan(\theta)d\theta}{4\tan^2(\theta)} \\ &= \int \frac{\sec(\theta)d\theta}{2\tan(\theta)} \\ &= \frac{1}{2}\int \csc(\theta)d\theta \\ &= \frac{-1}{2}\int \frac{du}{u} \qquad \text{let } u = \csc(\theta)+\cot(\theta) \text{ hence } -du/u = \csc(\theta)d\theta \\ &= \frac{-1}{2}\ln|\csc(\theta)+\cot(\theta)| + c\end{aligned}$$

We let $x+3 = 2\sec(\theta)$ *thus* $\sec(\theta) = \frac{x+3}{2} = \frac{hyp}{adj}$*. The other side of the substitution triangle is found by the pythagorean theorem; opp* $= \sqrt{(x+3)^2-4}$*. Observe that* $\cot(\theta) = \frac{adj}{opp} = \frac{2}{\sqrt{(x+3)^2-4}}$

and $\csc(\theta) = \frac{hyp}{opp} = \frac{x+3}{\sqrt{(x+3)^2-4}}$. *Therefore,*

$$\int \frac{dx}{x^2+6x+5} = \boxed{\frac{-1}{2}\ln\left|\frac{x+3}{\sqrt{(x+3)^2-4}} + \frac{2}{\sqrt{(x+3)^2-4}}\right| + c.}$$

Let's attack the preceding example once more, but this time from the angle of a hyperbolic substitution.

Example 3.9. *If* $\cosh^2(\phi) - \sinh^2(\phi) = 1$ *then dividing by* $\sinh^2(\phi)$ *reveals the hyperbolic cotangent/cosecant identity:* $\coth^2(\phi) - 1 = csch^2(\phi)$ *where by definition* $\coth(\phi) = \frac{\cosh(\phi)}{\sinh(\phi)}$ *and* $csch(\phi) = \frac{1}{\sinh(\phi)}$. *In view of the identity above, the integral* $\int \frac{dx}{x^2+6x+5} = \int \frac{dx}{(x+3)^2-4}$ *is likely simplified by an* $(x+3)^2 = 4\coth^2(\phi)$ *substitution. If* $x+3 = 2\coth(\phi)$ *then* $(x+3)^2 - 4 = 4\coth^2(\phi) - 4 = 4csch^2(\phi)$ *and* [3] $dx = -2csch^2(\phi)d\phi$.

$$\begin{aligned}\int \frac{dx}{x^2+6x+5} &= \int \frac{dx}{(x+3)^2-4}.\\ &= \int \frac{-2csch^2(\phi)d\phi}{4csch^2(\phi)}.\\ &= \frac{-1}{2}\phi + c.\\ &= \boxed{\frac{-1}{2}\coth^{-1}\left(\frac{x+3}{2}\right) + c.}\end{aligned}$$

I show how $\int \frac{dx}{x^2+6x+5} = \frac{-1}{2}\tanh^{-1}\left(\frac{x+3}{2}\right) + c$ *in the next example.*

Example 3.10. *If* $\cosh^2(\phi) - \sinh^2(\phi) = 1$ *then dividing by* $\cosh^2(\phi)$ *reveals the hyperbolic cotangent/cosecant identity:* $1 - \tanh^2(\phi) = sech^2(\phi)$ *where by definition* $\tanh(\phi) = \frac{\sinh(\phi)}{\cosh(\phi)}$ *and* $sech(\phi) = \frac{1}{\cosh(\phi)}$. *In view of the identity above, the integral* $\int \frac{dx}{x^2+6x+5} = -\int \frac{dx}{4-(x+3)^2}$ *is likely simplified by an* $(x+3)^2 = 4\tanh^2(\phi)$ *substitution. If* $x+3 = 2\tanh(\phi)$ *then* $4-(x+3)^2 = 4 - 4\tanh^2(\phi) = 4sech^2(\phi)$ *and*[4] $dx = 2sech^2(\phi)d\phi$.

$$\begin{aligned}\int \frac{dx}{x^2+6x+5} &= -\int \frac{dx}{4-(x+3)^2}.\\ &= -\int \frac{2sech^2(\phi)d\phi}{4sech^2(\phi)}.\\ &= \frac{-1}{2}\phi + c.\\ &= \boxed{\frac{-1}{2}\tanh^{-1}\left(\frac{x+3}{2}\right) + c.}\end{aligned}$$

[3]just use the quotient rule; $\coth(\phi)' = \frac{\cosh(\phi)}{\sinh(\phi)}' = \frac{\sinh^2(\phi)-\cosh^2(\phi)}{\sinh^2(\phi)} = \frac{-1}{\sinh^2(\phi)} = -csch^2(\phi)$

[4]just use the quotient rule; $\tanh(\phi)' = \frac{\sinh(\phi)}{\cosh(\phi)}' = \frac{\cosh^2(\phi)-\sinh^2(\phi)}{\cosh^2(\phi)} = \frac{1}{\cosh^2(\phi)} = sech^2(\phi)$

Example 3.11. *We wish to calculate* $\int \frac{x^3 dx}{\sqrt{9-x^2}}\ dx$*. The squareroot is trouble. One sneaky way to eliminate it is to let* $x = 3\sin(\theta)$ *thus* $9-x^2 = 9-9\sin^2(\theta) = 9\cos^2(\theta)$ *and* $dx = 3\cos(\theta)d\theta$*. Hence,*

$$\begin{aligned}
\frac{x^3 dx}{\sqrt{9-x^2}} &= \int \frac{27\sin^3(\theta)(3\cos(\theta)d\theta)}{3\cos(\theta)} \\
&= \int 27\sin^3(\theta)d\theta \\
&= \int 27\big(1-\cos^2(\theta)\big)\sin(\theta)d\theta \\
&= \int [27u^2 - 27]du \qquad \textit{let } u = \cos(\theta) \\
&= 9\cos^3(\theta) - 27\cos(\theta) + c
\end{aligned}$$

Note that $x = 3\sin(\theta)$ *gives* $\sin(\theta) = \frac{x}{3} = \frac{opp}{hyp}$ *where we have in mind a triangle which represents the subsitution. Applying the pythagorean theorem we calculate* $adj = \sqrt{9-x^2}$ *hence* $\cos(\theta) = \frac{adj}{hyp} = \frac{\sqrt{9-x^2}}{3}$ *and we can simplify our answer nicely:* $9\cos^3(\theta) = \frac{9(\sqrt{9-x^2})^3}{27} = \frac{1}{3}(9-x^2)\sqrt{9-x^2}$ *and* $9\cos^3(\theta) - 27\cos(\theta) = \frac{1}{3}(9-x^2)\sqrt{9-x^2} - \frac{1}{3}27\sqrt{9-x^2}$,

$$\boxed{\int \frac{x^3 dx}{\sqrt{9-x^2}} = \frac{-1}{3}(18+x^2)\sqrt{9-x^2} + c.}$$

Remark 3.12.

If we have an integral with bounds then we change the integrand, measure and the bounds under an implicit substitution. We have not faced that difficulty in this section so far because I have focused on indefinite integration. Also, once bounds enter in then one may hope for an explicit geometric interpretation of the substitution. I have emphasized an algebraic understanding in this section because I believe it will lead less students astray. In my experience, few students understand geometry well enough to appropriately modify it for the solution of nonstandard problems. That said, I will invoke a geometry-based substitution in the final example that follows.

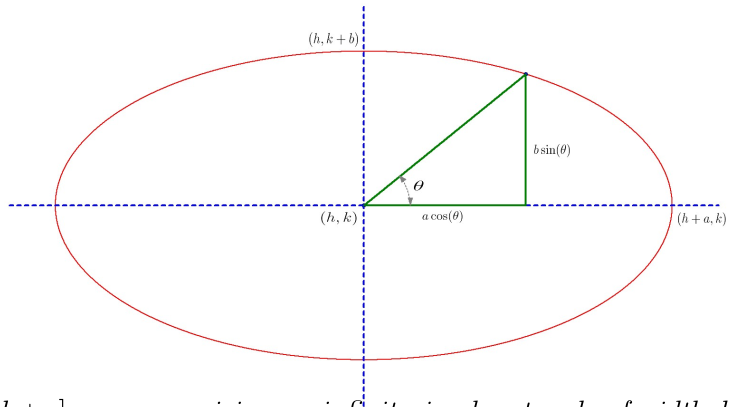
Example 3.13. *Find the area bounded by* $b^2(x-h)^2 + a^2(y-k)^2 = a^2b^2$ *where* $a, b, h, k \in \mathbb{R}$ *and it is given that* $a, b > 0$*. This is an ellipse. Note if* (x, y) *is a solution to the given equation then it is also a solution to*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

and we identify that the area is an ellipse centered at (h, k)*. We can view this ellipse as two graphs pasted together; note* $(y-k)^2 = b^2\big[1 - \frac{(x-h)^2}{a^2}\big] = \frac{b^2}{a^2}\big[a^2 - (x-h)^2\big]$ *has solutions*

$$y_T = k + \frac{b}{a}\sqrt{a^2-(x-h)^2} \qquad \textit{and} \qquad y_B = k - \frac{b}{a}\sqrt{a^2-(x-h)^2}.$$

These solutions have domains which are governed by the inequality $a^2 - (x-h)^2 \geq 0$ *hence* $a^2 \geq |x-h|^2$ *and as* $a > 0$ *we find* $|x-h| \leq a$ *which is equivalent to* $h - a \leq x \leq h + a$*.*



At each $x \in [h-a, h+a]$ we can envision an infinitesimal rectangle of width dx and height $y_T - y_B = 2\frac{b}{a}\sqrt{a^2-(x-h)^2}$ hence the typical infinitesimal area is given by $dA = 2\frac{b}{a}\sqrt{a^2-(x-h)^2}\ dx$ (I'll let you draw in the dA) and we can calculate the total area by integration:

$$A = \int_{ellipse} dA = \int_{h-a}^{h+a} 2\frac{b}{a}\sqrt{a^2-(x-h)^2}\ dx$$

This is a nontrivial integration. However, we can do it. We let $x = h + a\cos(\theta)$ hence $x - h = a\cos(\theta)$ and $a^2-(x-h)^2 = a^2 - a^2\cos(\theta) = a^2\sin^2(\theta)$ and $dx = -a\sin(\theta)d\theta$. Consider the bounds:

(1.) *if $x = h + a = h + a\cos(\theta)$ then we find $\cos(\theta) = 1$ hence $\theta = 0$ (our choice)*

(2.) *if $x = h - a = h + a\cos(\theta)$ then we find $\cos(\theta) = -1$ hence $\theta = \pi$ (our choice)*

I say "our choice" because once I state these bounds I am clarifying that we made the substitution for $0 \leq \theta \leq \pi$. Other intervals are possible, however this one will suffice. We calculate,

$$A = \int_{\pi}^{0} 2\frac{b}{a}\sqrt{a^2\sin^2(\theta)}\ (-a\sin(\theta)d\theta)$$

Now, our choice of $0 \leq \theta \leq \pi$ indicates the sine function is non-negative and as $a > 0$ was given it follows $\sqrt{a^2\sin^2(\theta)} = a\sin(\theta)$. Flipping the bounds to remove the minus we find,

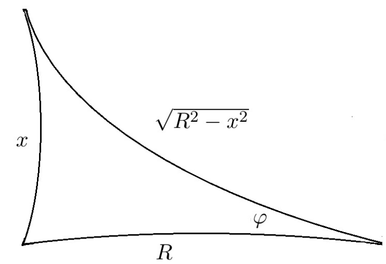
$$A = \int_0^{\pi} 2ab\sin^2(\theta)d\theta = \int_0^{\pi} ab\big[1-\cos(2\theta)\big]d\theta = ab\big[\theta - \frac{1}{2}\sin(2\theta)\big]\Big|_0^{\pi} = \boxed{\pi ab.}$$

In the case $a = b$ we usually say $a = r$ and we find $A = \pi r^2$. Only in that special case is the "angle" θ technically an angle in the geometric sense. For $a \neq b$ the quantity θ is simply a parameter which does not match the polar angle as measured from the center of the ellipse.

There are more exotic trigonometric substitutions. For example, there is a well-known substitution that allows one to integrate a fraction of sines and cosines: $\int \frac{a\cos(x)+b\sin(x)+c}{d\cos(x)+e\sin(x)+g}dx$. The fact that this is solvable means that there are corresponding algebraic functions which become solvable under a substitution which brings the integral to the fraction of sines and cosines. I may assign such a problem concerning the Weierstrauss substitution in homework as a challenge.

Remark 3.14. *To conclude this section I illustrate the hyperbolic substitutions using a triangle on hyperbolic space which satisfies $a^2 - b^2 = c^2$ where a is adjacent the hyperbolic angle φ and b is opposite and c serves as the hypotenuse. There is a whole theory of hyperbolic trigonometry where the triangles have less than 180^o because the space has negative curvature. This is an example of a* **non-Euclidean geometry**. *Ask me more in office hours if you wish.*

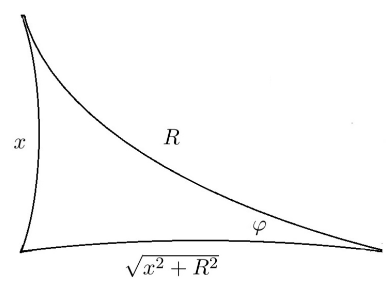
(1.) If $x = R\tanh\varphi$ then $dx = \text{sech}^2\varphi\, d\theta$ and $R^2 - x^2 = R^2\,\text{sech}^2\varphi$ hence $\sqrt{R^2 - x^2} = R\,\text{sech}\varphi$.



Here we have $\cosh\varphi = \dfrac{R}{\sqrt{R^2 - x^2}}$ and $\sinh\varphi = \dfrac{x}{\sqrt{R^2 - x^2}}$. As a check on my claims, notice

$$\cosh^2\varphi - \sinh^2\varphi = \frac{R^2}{R^2 - x^2} - \frac{x^2}{R^2 - x^2} = 1.$$

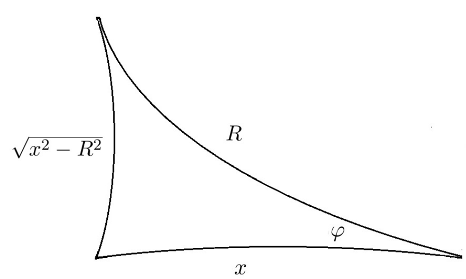
(2.) If $x = R\sinh\varphi$ then $dx = R\cosh\varphi\, d\varphi$ and $x^2 + R^2 = R^2\cosh^2\varphi$ hence $\sqrt{x^2 + R^2} = R\cosh\varphi$.



Here we have $\tanh\varphi = \dfrac{x}{\sqrt{x^2 + R^2}}$ and $\text{sech}\,\varphi = \dfrac{R}{\sqrt{x^2 + R^2}}$. As a check on my claims, notice

$$\tanh^2\varphi + \text{sech}^2\varphi = \frac{x^2}{x^2 + R^2} + \frac{R^2}{x^2 + R^2} = 1.$$

(3.) If $x = R\cosh\varphi$ then $dx = R\sinh\varphi\, d\varphi$ and $x^2 - R^2 = R^2\sinh^2\varphi$ hence $\sqrt{x^2 - R^2} = R\sinh\varphi$.



Identify $\tanh\varphi = \dfrac{\sqrt{x^2 - R^2}}{x}$ and $\text{sech}\,\varphi = \dfrac{R}{x}$. As a check on my claims, notice

$$\tanh^2\varphi + \text{sech}^2\varphi = \frac{x^2 - R^2}{x^2} + \frac{R^2}{x^2} = 1.$$

The key concept behind all the substitions is that we wish to use $\cosh^2\varphi - \sinh^2\varphi = 1$ or $\tanh^2\varphi - 1 = \text{sech}^2\varphi$ in order to consolidate two terms into a single term. Of course, in application it's a bit more tricky since we also have to face various multipliers and it may be necessary to complete the square to see clearly the correct course of action.

4 partial fractions

If we can find a formula in terms of finitely many elementary functions for $\int f(x)\,dx$ then I say the integral is *solvable*. In constrast, I would say f is integrable if there exists an antiderivative function $F'(x) = f(x)$ for all $x \in dom(f)$. This criteria of *integrable* is much weaker than solvable. You should recall that the FTC part I proves that any continuous function is integrable. However, the FTC just provides the existence of the antiderivative, it does not explain how to find its formula. While it is not at all an easy thing to prove, there are many continuous functions which have insolvable integrals. For example, $\int e^{-x^2}dx$ or $\int \frac{\sin(x)}{x}dx$ cannot be solved in terms of elementary functions alone[5]. Given all of this, you might be surprised by the following claim:

THE INTEGRAL OF ANY RATIONAL FUNCTION IS SOLVABLE.

I will attempt to illustrate a partial proof of this claim in this section. After many examples I'll outline the general procedure which proves the claim above.

Example 4.1. *The calculation of* $\int \frac{dx}{x^2+6x+5}$ *is greatly simplified by the algebra below:*

$$\frac{1}{x^2+6x+5} = \frac{1}{(x+1)(x+5)} = \frac{A}{x+1} + \frac{B}{x+5}$$

$$\Rightarrow \qquad 1 = A(x+5) + B(x+1)$$

$$\textit{evaluating at } x = -1 \quad \Rightarrow \quad 1 = -4B \quad \Rightarrow \quad B = -1/4$$

$$\textit{evaluating at } x = -5 \quad \Rightarrow \quad 1 = 4A \quad \Rightarrow \quad A = 1/4$$

$$\Rightarrow \qquad \frac{1}{x^2+6x+5} = \frac{1/4}{x+1} - \frac{1/4}{x+5}$$

We say the algebraic technique above is performing the partial fractal decomposition of the rational function. In other words, I take the given rational function and try to write it as a sum of more basic rational functions which happen to have nice integrals. Observe that

$$\int \frac{dx}{x^2+6x+5} = \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+5}$$

Therefore,

$$\boxed{\int \frac{dx}{x^2+6x+5} = \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x+5| + c.}$$

Remark 4.2.

Why is it ok to plug in $x = -1$ and $x = -5$? Shouldn't this lead to an error since the integrand is not even defined at those values? But, there is no error, the algebra is correct, why?

A rational function f is by definition the quotient of two polynomials p and q; f is rational iff there exist polynomials p, q such that $f = p/q$. Moreover, we say $f = \frac{p}{q}$ is proper iff $deg(p) < deg(q)$. On the other hand, we say $f = \frac{p}{q}$ is improper iff $deg(p) \geq deg(q)$. In the case f is improper we can always find a proper rational function r and a polynomial g such that $f = \frac{p}{q} = g + \frac{r}{q}$. The polynomial

[5] there are techniques to find approximations for the antiderivatives, this is one of the reasons we cover power series expansions later in this course.

r is sometimes called the remainder and it is calculated in general by long-division of polynomials. If you can do long-division for integers then you should be able to do it for polynomials. I do not use synthetic division because it is neither necessary or general. The algorithm and logic for polynomial long-division is essentially the same as the process we were taught as children. We systematically subtract multiples of the divisor from the numerator until we can no longer subtract whole copy from what remains. For example,

$$
\begin{array}{r|l}
 & 2x^2 \quad -5x+18 \\
\hline
x+3 & 2x^3 \ +x^2 \ +3x \ +5 \\
 & -2x^3-6x^2 \\
\hline
 & -5x^2 \ +3x \\
 & 5x^2+15x \\
\hline
 & 18x \ +5 \\
 & -18x-54 \\
\hline
 & -49
\end{array}
$$

The example below makes use of the division above.

Example 4.3.

$$\int \frac{2x^3+x^2+3x+5}{x+3}dx = \int \left(2x^2-5x+18-\frac{49}{x+3}\right)dx = \boxed{\frac{2}{3}x^3-\frac{5}{2}x^2+18x-49\ln|x+3|+c.}$$

We can have higher degree polynomials in the denominator and still perform long-division.

$$
\begin{array}{r|l}
 & 2x \quad -7 \\
\hline
x^2+4x+3 & 2x^3 \ +x^2 \ +3x \ +5 \\
 & -2x^3-8x^2 \ -6x \\
\hline
 & -7x^2 \ -3x \ +5 \\
 & 7x^2+28x+21 \\
\hline
 & 25x+26
\end{array}
$$

Again, I use this long-division to begin the next example.

Example 4.4.

$$\int \frac{2x^3+x^2+3x+5}{x^2+4x+3}dx \ = \int \left(2x-7+\frac{25x+26}{x^2+4x+3}\right)dx \ = x^2-7x \ + \int \frac{25x+26}{x^2+4x+3}dx.$$

Calculation of the remaining integral is accomplished with the partial fractions idea once more:

$$\frac{25x+26}{x^2+4x+3} = \frac{25x+26}{(x+1)(x+3)} = \frac{A}{x+1}+\frac{B}{x+3}$$

$$\Rightarrow \qquad 25x+26 = A(x+3)+B(x+1)$$

$$\textit{evaluating at } x=-3 \quad \Rightarrow \quad -75+26=-2B \quad \Rightarrow \quad B=49/2$$

$$\textit{evaluating at } x=-1 \quad \Rightarrow \quad -25+26=2A \quad \Rightarrow \quad A=1/2$$

$$\Rightarrow \qquad \frac{25x+26}{x^2+4x+3} = \frac{1/2}{x+1}+\frac{49/2}{x+3}$$

Hence,

$$\int \frac{25x+26}{x^2+4x+3}dx = \frac{1}{2}\int \frac{dx}{x+1} + \frac{49}{2}\int \frac{dx}{x+3} = \underline{\frac{1}{2}\ln|x+1| + \frac{49}{2}\ln|x+3| + c.}$$

Therefore,

$$\boxed{\int \frac{2x^3+x^2+3x+5}{x^2+4x+3}dx \;= x^2 - 7x + \frac{1}{2}\ln|x+1| + \frac{49}{2}\ln|x+3| + c.}$$

Example 4.5. *Some problems are sufficiently simple that I don't have to do the long-division.*

$$\begin{aligned}\int \frac{1-2x}{3-2x}dx &= \int \frac{3-2x-2}{3-2x}dx.\\ &= \int \Big(1 - \frac{2}{3-2x}\Big)dx.\\ &= \boxed{x + \ln|3-2x| + c.}\end{aligned}$$

Again, I avoid long division with an appropriately added zero:

Example 4.6.

$$\begin{aligned}\int \frac{3x^2}{x^2+1}dx &= \int \frac{3(x^2+1)-3}{x^2+1}dx\\ &= \int \Big(3 - \frac{3}{x^2+1}\Big)dx.\\ &= \boxed{3x - 3\tan^{-1}(x) + c.}\end{aligned}$$

What if we have denominators of larger degree than two? I'll begin with a long division just to emphasize once more it's importance.

$$\begin{array}{r|rrrrrrr}
 & & & & x^3 & -4x^2 & +13x & -38\\ \hline
x^3+4x^2+3x & x^6 & & & +2x^3 & & +2x & +7\\
 & -x^6 & -4x^5 & -3x^4 & & & & \\ \cline{2-4}
 & & -4x^5 & -3x^4 & +2x^3 & & & \\
 & & 4x^5 & +16x^4 & +12x^3 & & & \\ \cline{3-5}
 & & & 13x^4 & +14x^3 & & & \\
 & & & -13x^4 & -52x^3 & -39x^2 & & \\ \cline{4-6}
 & & & & -38x^3 & -39x^2 & +2x & \\
 & & & & 38x^3 & +152x^2 & +114x & \\ \cline{5-7}
 & & & & & 113x^2 & +116x & +7
\end{array}$$

Example 4.7. *We wish to calculate* $\int \frac{x^6+2x^3+2x+7}{x^3+4x^2+3x}dx$. *Note from the long-division above,*

$$\begin{aligned}\int \frac{x^6+2x^3+2x+7}{x^3+4x^2+3x}dx &= \int \left(x^3 - 4x^2 + 13x - 38 + \frac{113x^2+116x}{x^3+4x^2+3x}\right)dx\\ &= \frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{13}{2}x^2 - 38x + \int \frac{113x^2+116x}{x^3+4x^2+3x}dx\end{aligned}$$

The denominator factors into three linear factors $x^3 + 4x^2 + 3x = x(x+1)(x+3)$. This factoring is what leads us to the decomposition beneath:

$$\frac{113x^2 + 116x}{x^3 + 4x^2 + 3x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}$$

Multiply by the denominator, nice cancellations occur:

$$113x^2 + 116x = A(x+1)(x+3) + Bx(x+3) + Cx(x+1)$$

Now, since all three linear factors correspond to distinct roots we can plug in the roots to obtain solutions for A, B, C, if we had repeated roots then we'll see this system breaks down, but let's finish this example for now,

$$x = 0 \quad \Rightarrow \quad 0 = 3A$$

$$x = -1 \quad \Rightarrow \quad 113 - 116 = -2B$$

$$x = -3 \quad \Rightarrow \quad 113(9) - 3(116) = 6C$$

Hence, $A = 0$, $B = 3/2$ and $C = 223/2$ (the fact that $A = 0$ was obvious from the outset)

$$\begin{aligned}
\int \frac{x^6 + 2x^3 + 2x + 7}{x^3 + 4x^2 + 3x} dx &= \frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{13}{2}x^2 - 38x + \int \frac{113x^2 + 116x}{x^3 + 4x^2 + 3x} dx \\
&= \frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{13}{2}x^2 - 38x + \frac{3}{2}\int \frac{dx}{x+1} + \frac{223}{2}\int \frac{dx}{x+3} \\
&= \boxed{\frac{1}{4}x^4 + \frac{4}{3}x^3 + \frac{13}{2}x^2 - 38x + \frac{3}{2}\ln|x+1| + \frac{223}{2}\ln|x+3| + c.}
\end{aligned}$$

Ok, I think we've seen enough long-division examples for this section. Keep in mind that we could add that wrinkel to the examples that follow, but it just adds one extra step at the beginning of the calculation.

Example 4.8. *We wish to calculate the integral below:*

$$\int \frac{2x^2 + 3x - 9}{x^4 + 4x^3 + 5x^2 + 9x^2 + 36x + 45} dx.$$

This is a proper rational function so we don't have to do any long division. Our first task is to factor the denominator. Note $x^4 + 4x^3 + 5x^2 + 9x^2 + 36x + 45 = x^2(x^2 + 4x + 5) + 9(x^2 + 4x + 5)$ hence

$$\int \frac{2x^2 + 3x - 9}{x^4 + 4x^3 + 5x^2 + 9x^2 + 36x + 45} dx = \int \frac{2x^2 + 3x - 9}{(x^2 + 9)(x^2 + 4x + 5)} dx.$$

I cannot factor further over the real numbers because these quadratic factors have complex roots. The partial fractions requires two unknown coefficients for each quadratic factor:

$$\frac{2x^2 + 3x - 9}{(x^2 + 9)(x^2 + 4x + 5)} = \frac{Ax + B}{x^2 + 9} + \frac{Cx + D}{x^2 + 4x + 5}$$

$$\Rightarrow \qquad 2x^2 + 3x - 9 = (Ax + B)(x^2 + 4x + 5) + (Cx + D)(x^2 + 9)$$

$$\Rightarrow \qquad 2x^2 + 3x - 9 = Ax^3 + Bx^2 + 4Ax^2 + 4Bx + 5Ax + 5B + Cx^3 + Dx^2 + 9Cx + 9D$$

$$\Rightarrow \qquad 2x^2 + 3x - 9 = x^3(A+C) + x^2(B+4A+D) + x(4B+5A+9C) + 5B + 9D$$

Two polynomials are equal iff they have matching coefficients. We equate coefficients of $x^3, x^2, x^1 and x^0$:

$$(i.)\ 0 = A + C \qquad (ii.)\ 2 = B + 4A + D \qquad (iii.)\ 3 = 4B + 5A + 9C \qquad (iv.)\ -9 = 5B + 9D$$

We have four linear equations and four unknowns. We procede by elimination: using i. and ii.,

$$A = -C, \qquad A = (2 - B - D)/4, \Rightarrow (2 - B - D)/4 = -C \Rightarrow \underline{2 - B - D + 4C = 0 \ \ (v.)}$$

Next eliminate A from ii. and iii.

$$A = (2-B-D)/4, \qquad A = (3-4B-9C)/5 \Rightarrow (2-B-D)/4 = (3-4B-9C)/5 \Rightarrow 5(2-B-D) = 4(3-4B-9C)$$

Simplifying gives $\underline{11B + 36C - 5D - 2 = 0 \ \ (vi.)}$ so equations iv., v and vi. involve only the variables B, C, D. We have reduced the problem to 3 equations and 3 unknowns:

$$5B + 9D = -9, \qquad -B + 4C - D = -2, \qquad 11B + 36C - 5D = 2$$

We can finish by elimination, but for a change of pace I'll use Kramer's rule from high-school algebra II, we write the system as a matrix problem:

$$\begin{bmatrix} 5 & 0 & 9 \\ -1 & 4 & -1 \\ 11 & 36 & -5 \end{bmatrix} \begin{bmatrix} B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -9 \\ -2 \\ 2 \end{bmatrix}$$

Then, Kramer's rule says:

$$B = \frac{det \begin{bmatrix} -9 & 0 & 9 \\ -2 & 4 & -1 \\ 2 & 36 & -5 \end{bmatrix}}{det \begin{bmatrix} 5 & 0 & 9 \\ -1 & 4 & -1 \\ 11 & 36 & -5 \end{bmatrix}} = \frac{-9(-20+36) + 9(-72-8)}{5(-20+36) + 9(-36-44)} = \frac{-864}{-640} = \frac{27}{20}.$$

At which point finding D is easy from iv.,

$$D = -1 - \frac{5}{9}B = -1 - \frac{5}{9} \cdot \frac{27}{20} = \frac{-7}{4}$$

Find C with ease from v. next,

$$C = \frac{1}{4}(-2 + B + D) = \frac{1}{4}(-2 + \frac{27}{20} + \frac{-7}{4}) = \frac{-3}{5}$$

Hence, returning to i., clearly $A = \frac{3}{5}$. As a check on the algebra, I'll solve the problem again from the start using technology: the row reduced echelon form of the augmented coefficient matrix will reveal the solution (if it exists) of any linear system: the first row comes from $A + C = 0$ then equations ii.,iii. and iv. follow in their matrix notation:

$$rref \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 9 & -9 \\ 0 & -1 & 4 & -1 & -2 \\ 0 & 11 & 36 & -5 & 2 \end{array}\right] = rref \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & 0 & 27/20 \\ 0 & 0 & 1 & 2 & -3/5 \\ 0 & 0 & 0 & 1 & -7/4 \end{array}\right]$$

Which confirms the algebra of the last page. Pragmatically speaking, if you are not taking a test and don't need to practice your algebra then the reduced row echelon form (a.k.a. Gaussian elimination) is hard to beat. We examine the logical underpinnings of both Kramers rule and Gaussian elimination in the linear algebra course[6] *(Math 321). Let's collect our thoughts. We have shown that*

$$\frac{2x^2+3x-9}{x^4+4x^3+5x^2+9x^2+36x+45} = \frac{\frac{3}{5}x+\frac{27}{20}}{x^2+9} + \frac{\frac{-3}{5}x-\frac{7}{4}}{x^2+4x+5}$$

Observe we can integrate using u-substitution on the A-term and trig. substitution on the B-term[7]

$$\begin{aligned}
\int \frac{\frac{3}{5}x+\frac{27}{20}}{x^2+9}dx &= \frac{3}{10}\int\frac{2xdx}{x^2+9} + \frac{27}{20}\int\frac{dx}{x^2+9} \\
&= \frac{3}{10}\int\frac{du}{u} + \frac{27}{20}\int\frac{3\sec^2(\theta)d\theta}{9\sec^2(\theta)} \\
&= \frac{3}{10}\ln|u| + \frac{9}{20}\theta + c_1 \qquad (\textit{where } u = x^2+9 \textit{ and } x = 3\tan(\theta)) \\
&= \underline{\frac{3}{10}\ln|x^2+9| + \frac{9}{20}\tan^{-1}\left(\frac{x}{3}\right) + c_1}\ \star
\end{aligned}$$

Next, turn to the C and D terms, a little creative algebra from the outset makes life easier,

$$\begin{aligned}
\int \frac{\frac{-3}{5}x-\frac{7}{4}}{x^2+4x+5}dx &= \int\frac{\frac{-3}{5}(x+2)+\frac{6}{5}-\frac{7}{4}}{x^2+4x+5}dx \\
&= \frac{-3}{5}\int\frac{(x+2)dx}{(x+2)^2+1} - \frac{11}{20}\int\frac{dx}{(x+2)^2+1} \\
&= \frac{-3}{10}\int\frac{du}{u} - \frac{11}{20}\int\frac{dw}{w^2+1} \\
&= \frac{-3}{10}\ln|u| - \frac{11}{20}\tan^{-1}(w) + c_2 \qquad (\textit{where } u = (x+2)^2+1 \textit{ and } w = x+2) \\
&= \underline{\frac{-3}{10}\ln|x^2+4x+5| - \frac{11}{20}\tan^{-1}(x+2) + c_2}\ \star\star
\end{aligned}$$

Therefore, using $\star$ *and* $\star\star$ *we find* $\int \frac{2x^2+3x-9}{x^4+4x^3+5x^2+9x^2+36x+45}dx$ *is*

$$\boxed{\frac{3}{10}\ln|x^2+9| + \frac{-3}{10}\ln|x^2+4x+5| - \frac{11}{20}\tan^{-1}(x+2) + \frac{9}{20}\tan^{-1}\left(\frac{x}{3}\right) + c.}$$

At this point, we have seen examples of how to integrate rational functions with denominators that consist of a linear factor, a pair of linear factors, a triple of linear factors or a pair irreducible quadratic factors. I know consider an example where we have a mixture of linear and irreducible quadratic terms. Again, the heart of the solution is the partial fractal decomposition.

[6] however, linear algebra is about a lot more than just solving linear equations, you are really supposed to know what I've shown here from highschool. Granted many of you have forgotten. Now is time to remember. As you work the homework you should not use technology to solve all the algebra, instead, use technology to check your answers. I have links on my website to free online calculators which will do partial fractions and/or the rref calculation. Make use of them wisely.

[7] I do not expect you to memorize these formulas, I expect you to be able to derive them!

Example 4.9. *We wish to calculate* $\int \frac{3x^2-3x-8}{x^3-3x^2+x-3}dx$*. Note that* $x^3 - 3x^2 + x - 3 = (x-3)(x^2+1)$ *hence we propose the following decomposition:*

$$\frac{3x^2 - 3x - 8}{x^3 - 3x^2 + x - 3} = \frac{A}{x-3} + \frac{Bx + C}{x^2 + 1}$$

Multiply by the denominator, nice cancellations occur:

$$3x^2 - 3x - 8 = A(x^2 + 1) + (Bx + C)(x - 3)$$

I'll plug in the real root and one of the complex roots[8] *to obtain:*

$$x = 3 \quad \Rightarrow \quad 3(9) - 3(3) - 8 = 10A \quad \Rightarrow \quad A = 1.$$

$$x = i = \sqrt{-1} \quad \Rightarrow \quad 3i^2 - 3i - 8 = (iB + C)(i - 3) \quad \Rightarrow \quad -11 - 3i = -B - 3C + i(C - 3B)$$

The beautiful thing about complex equations is they do twice the work of real equations. This is due to the fact that the real and imaginary parts of a complex equation must separately balance. In particular,

$$-11 - 3i = -B - 3C + i(C - 3B) \quad \Rightarrow \quad -11 = -B - 3C \quad \text{and} \quad -3 = C - 3B$$

Multiply $-11 = -B - 3C$ *by* -3 *to find* $33 = 3B + 9C$ *then add this to* $-3 = C - 3B$ *to obtain* $30 = 10C$ *hence* $C = 3$ *and it follows* $B = 11 - 3C = 2$*. In summary,* $A = 1$*,* $B + 2$ *and* $C = 3$*. Finish it.*

$$\int \frac{3x^2 - 3x - 8}{x^3 - 3x^2 + x - 3}dx = \int \frac{dx}{x-3} + \int \frac{2xdx}{x^2+1} + 3\int \frac{dx}{x^2+1}$$
$$= \boxed{\ln|x-3| + \ln|x^2+1| + \tan^{-1}(x) + c.}$$

Example 4.10. *We wish to calculate* $\int \frac{2}{x^4-1}dx$*. Well, this is easy to factor:* $x^4 - 1 = (x^2+1)(x^2-1)$*. The partial fractions decomposition with respect to this factoring is*

$$\frac{2}{x^4 - 1} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 - 1}$$

Multiply by the denominator, nice cancellations occur:

$$2 = (Ax + B)(x^2 - 1) + (Cx + D)(x^2 + 1)$$

I'll plug in the real roots and one of the complex roots to obtain:

$$x = 1 \quad \Rightarrow \quad 2 = 2C + 2D.$$

$$x = -1 \quad \Rightarrow \quad 2 = -2C + 2D.$$

$$x = i = \sqrt{-1} \quad \Rightarrow \quad 2 = (Ai + B)(i^2 - 1) \quad \Rightarrow \quad 2 = -2B - 2Ai$$

The real and imaginary parts of a complex equation must separately balance. In particular,

$$2 = -2B - 2Ai \quad \Rightarrow \quad 2 = -2B \quad \text{and} \quad 0 = -2A$$

Therefore, $A = 0$ *and* $B = -1$*. Adding and subtracting the* $x = \pm 1$ *equations yields* $C = 0$ *and* $D = 1$*.*

$$\boxed{\int \frac{2}{x^4 - 1}dx = \int \frac{dx}{x^2 - 1} - \int \frac{dx}{x^2 + 1} = -\tanh^{-1}(x) - \tan^{-1}(x) + c.}$$

[8] yes, you could also multiply the polynomials out and equate coefficients as we did in a previous example.

In the preceding example I made use of the inverse hyperbolic tangent integral. If you don't like the inverse hyperbolic function then you could just as well break the $1/(x^2-1)$ term into a sum of reciprocals of $(x-1)$ and $(x+1)$. You can show that

$$\frac{1}{x^2-1} = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right].$$

Or, you could at least check that I am correct by making a common denominator of the RHS to recover the LHS. In any event, integration will yield natural logs from this algebraic approach:

$$\begin{aligned}\int \frac{dx}{x^2-1} &= \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c.\end{aligned}$$

There are identities which connect this expression and $-\tanh^{-1}(x)$. If nothing else, we know they have to be different formulas for the same function since

1. $\frac{d}{dx}[-\tanh^{-1}(x)] = \frac{d}{dx}[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|] = \frac{1}{x^2-1}$
2. $\tanh^{-1}(0) = 0$ and $\frac{1}{2}\ln|0-1| - \frac{1}{2}\ln|0+1| = 0$

Recall that if $f'(x) = g'(x)$ for all $x \in J$ (a connected interval) then $f(x) = g(x) + c$ on J. If we also know $f(0) = g(0)$ then it follows $c = 0$ hence $f = g$, provided the domain of both functions is J. Now perhaps my argument is less than satisfying, after all I didn't really explain how these rather different looking formulas are the same. See Stewart around page 466 for more on how to connect these algebraically.

Example 4.11. *We wish to calculate* $\int \frac{x^2+5}{x^4+8x^2+16}dx$. *This integral contains a new difficulty, it has a irreducible quadratic repeated in the denominator since* $x^4+8x^2+16 = (x^2+4)^2$. *The partial fractions decomposition for a repeated quadratic is as follows:*

$$\frac{x^2+5}{x^4+8x^2+16} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}.$$

I invite the reader to verify that the usual algebra yields $A = 0, B = 1, C = 0$ *and* $D = 1$. *I know this will happen since the following algebra leads to the same conclusion:*

$$\frac{x^2+5}{x^4+8x^2+16} = \frac{x^2+4+1}{(x^2+4)^2} = \frac{1}{x^2+4} + \frac{1}{(x^2+4)^2}$$

If $x = 2\tan(\theta)$ *then* $x^2+4 = 4\tan^2(\theta)+4 = 4\sec^2(\theta)$ *and* $dx = 2\sec^2(\theta)d\theta$. *We will make use of this substitution in the integral below:*

$$\begin{aligned}\int \frac{x^2+5}{x^4+8x^2+16}dx &= \int\left[\frac{1}{x^2+4} + \frac{1}{(x^2+4)^2}\right]dx \\ &= \int\left[\frac{1}{4\sec^2(\theta)} + \frac{1}{(4\sec^2(\theta))^2}\right](2\sec^2(\theta)d\theta) \\ &= \int\left[\frac{1}{2} + \frac{1}{8}\cos^2(\theta)\right]d\theta \\ &= \int\left[\frac{1}{2} + \frac{1}{16} + \frac{1}{16}\cos(2\theta)\right]d\theta \\ &= \frac{9\theta}{16} + \frac{1}{32}\sin(2\theta) + c\end{aligned}$$

Recall that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ *and note* $\tan(\theta) = \frac{x}{2}$ *hence* $\sin(\theta) = \frac{x}{\sqrt{x^2+4}}$ *and* $\cos(\theta) = \frac{2}{\sqrt{x^2+4}}$ *thus,* $\sin(2\theta) = \frac{4x}{x^2+4}$. *Therefore,*

$$\boxed{\int \frac{x^2+5}{x^4+8x^2+16}dx = \frac{9}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{8(x^2+4)} + c.}$$

Example 4.12. *We wish to calculate* $\int \frac{2x^3+3x^2-x+1}{(x-2)(x+1)^3}dx$. *To begin we need to work out the partial fractions decomposition. Given the factoring of the denominator we propose:*

$$\frac{2x^3+3x^2-x+1}{(x-2)(x+1)^3} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

Multiply by the denominator to find:

$$2x^3+3x^2-x+1 = A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2)$$

We can evaluate at $x=-1$ *to find that* $-2+3+1+1 = -3D$ *hence* $\boxed{D=-1}$. *Likewise, evaluate at* $x=2$ *to obtain that* $16+12-2+1 = 27A$ *hence* $\boxed{A=1}$. *To find* B *and* C *we need additional equations. One easy choice is to evaluate at* $x=0$ *which gives* $1 = A-2B-2C-2D = 1-2B-2C+2$ *hence* $\underline{-2=-2B-2C}$. *Next, I evaluate*[9] *at* $x=1$ *to obtain* $2+3-1+1 = 8A-4B-2C-D$ *hence* $5 = 8-4B-2C+1$ *thus* $4 = 4B+2C$ *and dividing by two gives* $\underline{2=2B+C}$. *Collecting our thoughts,*

$$-2=-2B-2C \quad \textit{added to} \quad 2=2B+C \quad \Rightarrow \quad 0=-C \quad \Rightarrow \quad \boxed{C=0} \textit{ and } \boxed{B=1}.$$

Therefore,

$$\begin{aligned}
\int \frac{2x^3+3x^2-x+1}{(x-2)(x+1)^3}dx &= \int \left[\frac{1}{x-2} + \frac{1}{x+1} - \frac{1}{(x+1)^3}\right]dx \\
&= \int \frac{du}{u} + \int \frac{dw}{w} - \int \frac{dw}{w^3} \textit{let } u = x-2 \textit{ and } w = x+1 \\
&= \boxed{\ln|x-2| + \ln|x+1| + \frac{1}{2(x+1)^2} + c.}
\end{aligned}$$

Example 4.13. *We wish to calculate* $\int \frac{dx}{(x^2+6x+13)^3}$. *Note that* $x^2+6x+13 = (x+3)^2+4$ *hence the denominator cannot be further reduced. This is already a basic rational function. We make a* $x+3 = 2\tan(\theta)$ *substitution hence* $(x+3)^2+4 = 4\sec^2(\theta)$ *and* $dx = 2\sec^2(\theta)d\theta$. *Observe*

$$\begin{aligned}
\int \frac{dx}{(x^2+6x+13)^3} &= \int \frac{2\sec^2(\theta)d\theta}{64\sec^6(\theta)} \\
&= \frac{1}{32}\int \cos^4(\theta)d\theta \qquad \textit{(the next step is nontrivial)} \\
&= \frac{1}{32}\left[\frac{3\theta}{8} + \frac{1}{8}\sin(\theta)\cos^3(\theta) - \frac{1}{8}\sin^3(\theta)\cos(\theta) + \frac{1}{2}\sin(\theta)\cos(\theta)\right] + c
\end{aligned}$$

[9] because the arithmetic is easy and multiplying out to equate coefficients looks like more work here

We know $\tan(\theta) = \frac{x+3}{2}$ *hence* $\sin(\theta) = \frac{x+3}{\sqrt{(x+3)^2+4}}$ *and* $\cos(\theta) = \frac{2}{\sqrt{(x+3)^2+4}}$. *Thus,*

$$\begin{aligned}
\int \frac{dx}{(x^2+6x+13)^3} &= \frac{1}{32}\left[\frac{3\theta}{8} + \frac{1}{8}\sin(\theta)\cos^3(\theta) - \frac{1}{8}\sin^3(\theta)\cos(\theta) + \frac{1}{2}\sin(\theta)\cos(\theta)\right] + c \\
&= \frac{1}{32}\Bigg(\frac{3}{8}\tan^{-1}\left(\frac{x+3}{2}\right) \\
&\qquad + \frac{1}{8}\frac{x+3}{\sqrt{(x+3)^2+4}}\left[\frac{2}{\sqrt{(x+3)^2+4}}\right]^3 \\
&\qquad - \frac{1}{8}\left[\frac{x+3}{\sqrt{(x+3)^2+4}}\right]^3 \frac{2}{\sqrt{(x+3)^2+4}} \\
&\qquad + \frac{1}{2}\frac{x+3}{\sqrt{(x+3)^2+4}}\frac{2}{\sqrt{(x+3)^2+4}} \Bigg) + c
\end{aligned}$$

After a some algebra the expressions above ought to simply to the following:

$$\boxed{\int \frac{dx}{(x^2+6x+13)^3} = \frac{282+202x+54x^2+6x^3}{(13+6x+x^2)^2} + \frac{3}{256}\tan^{-1}\left(\frac{3+x}{2}\right) + c.}$$

Remark 4.14.

I think we've seen enough. At this point we can make a few observations:

- **(1.)** when integrating a rational function we can always reduce the problem to integrating a proper rational function and a polynomial because we know long division for polynomials.
- **(2.)** any proper rational function can be decomposed into a sum of basic rational functions which either have the form $\frac{1}{(x-a)^k}$ or $\frac{1}{[(x-\alpha)^2+\beta^2]^k}$ for $k = 1, 2, \dots$
- **(3.)** we can integrate any basic rational function, we have the methods, we just need paper and perservance.
- **(4.)** the partial fractions decomposition of a proper rational function with denominator of degree N will have N-unknown coefficients in its decomposition.

I seek to sketch a proof of (2.) and (4.) in the section that follows. The proofs of (1.) and (3.) are left to the reader.

Beyond these comments, we also observed that the inverse hyperbolic functions allow us to treat reducible reciprocal quadratic factors in much the same way as we treat the irreducible reciprocals. This led to a great diversity in the appearance of answers.

4.1 existence of partial fractions decomposition

Remark 4.15.

Read if interested. Ask me questions if want further detail. Pragmatically speaking, if you understand the examples before this subsection you should do just fine. This section does contain useful comments about algebra in general and a few complex calculations that all electrical engineers should be eager to master.

Suppose we have a rational function $f(x) = \frac{P(x)}{Q(x)}$. Suppose further that $deg(Q) = N$ and $deg(P) = n$ and $n < N$ so this rational function is proper. A fundamental theorem of algebra states that any polynomial with real coefficients can be written as a product of linear factors which may be repeated and/or correspond to complex roots. However, the complex roots must come in complex conjugate pairs which once multiplied give irreducible quadratic factors. That said, I will give an argument which allows complex notation. We know that Q factors as

$$Q(x) = A(x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_r)^{k_r}$$

where $A \neq 0$ and $a_j \in \mathbb{C}$ and $k_1 + k_2 + \cdots + k_r = N$. We claim that f can be written as a sum of the reciprocals $(x-a_1)^{k_1}, (x-a_2)^{k_2}, \cdots, (x-a_r)^{k_r}$. That is, we claim that the following equation has a solution for some $A_1, A_2, \dots, A_r \in \mathbb{C}$,

$$f(x) = \frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_{k_1}}{(x-a_1)^{k_1}} + \cdots + \frac{A_{N-k_r}}{x-a_r} + \frac{A_{N-k_r+1}}{(x-a_r)^2} + \cdots + \frac{A_N}{(x-a_r)^{k_r}}$$

Multiply both sides by $Q(x)$ to obtain $\star$:

$$\begin{aligned} P(x) = A_1&\big[(x-a_1)^{k_1-1}(x-a_2)^{k_2}\cdots(x-a_r)^{k_r}\big] + \cdots \\ &\cdots + A_{k_1}\big[(x-a_2)^{k_2}\cdots(x-a_r)^{k_r}\big] + \cdots \\ &\cdots + A_{N-k_r}\big[(x-a_1)^{k_1-1}(x-a_2)^{k_2}\cdots(x-a_r)^{k_r-1}\big] + \cdots \\ &\cdots + A_{N-k_r}\big[(x-a_1)^{k_1-1}(x-a_2)^{k_2}\cdots(x-a_{r-1})^{k_{r-1}}\big] \end{aligned}$$

Observe that if we plug in $x = a_1$ into $\star$ then we find

$$P(a_1) = A_{k_1}(a_1-a_2)^{k_2}\cdots(a_1-a_r)^{k_r} \qquad \Rightarrow \qquad A_{k_1} = \frac{P(a_1)}{(a_1-a_2)^{k_2}\cdots(a_1-a_r)^{k_r}}$$

Next, we can plug in $x = a_2$ into $\star$ and all terms drop to zero except one

$$P(a_2) = A_{k_2}(a_2-a_1)^{k_1}(a_2-a_3)^{k_3}\cdots(a_2-a_r)^{k_r} \qquad \Rightarrow \qquad A_{k_2} = \frac{P(a_2)}{(a_2-a_1)^{k_1}(a_2-a_3)^{k_3}\cdots(a_2-a_r)^{k_r}}$$

Continue in this fashion and we find solutions for A_{k_j} for $j = 3, 4, \dots, r$. However, this is only a subset of the coefficients since we allow $k_j > 1$ since we want to treat the case of repeated roots. Differentiate $\star$ and call it $\frac{d\star}{dx}$. Plug $x = a_1$ into $\frac{d\star}{dx}$ and note that just one term remains. The derivative of the term with coefficient A_{k_1-1} produces many terms, however only one remains after the evaluation. The term that remains is the one which $(x-a_1)$ was differentiated hence

$$P'(a_1) = A_{k_1-1}(a_1-a_2)^{k_2}\cdots(a_1-a_r)^{k_r} \qquad \Rightarrow \qquad A_{k_1-1} = \frac{P'(a_1)}{(a_1-a_2)^{k_2}\cdots(a_1-a_r)^{k_r}}$$

Then, if we differentiate twice then evaluate more we find a formula for A_{k_1-2}. Continuing in this fashion we find formulas for $A_1, A_2, \dots, A_{k_1}$. To complete the argument simply apply the procedure for the first root to the remaining roots $a_2, a_3, \dots a_r$. Finally, note in the case of complex roots we must have a conjugate pair. Observe that for such a pair we can reproduce the form we've used in this section:

$$\frac{A}{x-\alpha-i\beta}+\frac{B}{x-\alpha+i\beta}=\frac{A(x-\alpha+i\beta)+B(x-\alpha-i\beta)}{(x-\alpha-i\beta)(x-\alpha+i\beta)}=\frac{(A+B)x-\alpha(A+B)+i\beta(A-B)}{(x-\alpha)^2+\beta^2}$$

This is precisely the form we used, just written with ugly coefficients. You might worry that the rational function is complex so that will not match the manifestly real partial fractions decomposition. However, the fact that $f=P/Q$ is real implies certain reality conditions on A and B hence the imaginary term will vanish. Rather than work this out in general let me illustrate how the real and complex form of the partial fractions work out in a specific case:

$$\frac{2x+6}{x^2+4x+5}=\frac{x+3}{(x+2)^2-i^2}=\frac{A}{x+2-i}+\frac{B}{x+2+i}$$

Multiply both sides by the denominator to find $2x+6=A(x+2+i)+B(x+2-i)$. Thus,

$$2x+6=(A+B)x+2(A+B)+i(A-B)$$

Equating coefficients gives $2=A+B$ and $6=2(A+B)+i(A-B)$ which yields $6=4+i(A-B)$. It follows that $2=i(A-B)$ thus $A-B=2/i=-2i$. Add $A-B=-2i$ and $A+B=2$ to find $2A=2-2i$ thus $A=1-i$. Likewise, subtract $A-B=-2i$ from $A+B=2$ to find $2B=2+2i$ thus $B=1+i$. We have derived the complex partial fractions decomposition

$$\frac{2x+6}{x^2+4x+5}=\frac{1-i}{x+2-i}+\frac{1+i}{x+2+i}.$$

In general, one can prove that if $A+B\in\mathbb{R}$ and $A-B\in i\mathbb{R}$ and it follows that expression on the far right below is manifestly real once the imaginary terms are cancelled

$$\frac{A}{x-\alpha-i\beta}+\frac{B}{x-\alpha+i\beta}=\frac{(A+B)x-\alpha(A+B)+i\beta(A-B)}{(x-\alpha)^2+\beta^2}.$$

In any event, I include these general comments for those students who wonder about why this method works in general. The algebraic technique of partial fractions extends beyond calculus, but it is seldom proved in any course. It is doubtful I lecture on this stretch of the notes.

Lemma 4.16. *equating coefficients.*

Suppose F, G and Q are continuous functions and $Q(x)=0$ has finitely many solutions. If $\frac{F(x)}{Q(x)}=\frac{G(x)}{Q(x)}$ for all $x\in\mathbb{R}$ such that $Q(x)\neq 0$ then $F(x)=G(x)$.

Proof: Note that if $x\in\mathbb{R}$ such that $Q(x)\neq 0$ then it is clear that $F(x)=G(x)$. Suppose $x_o\in\mathbb{R}$ such that $Q(x_o)=0$ and consider that $F(x)=G(x)$ for all $x\in B_\delta(x_o)_o$ (a deleted nbhd of radius $\delta>0$ must exist on which $Q(x)\neq 0$ throughout since the zeros of Q are finite in number so we can separate them) but then

$$\lim_{x\to x_o}F(x)=\lim_{x\to x_o}G(x)$$

since the limit considers only points near the limit point. But, F and G are continuous hence

$$\lim_{x\to x_o}G(x)=G(x_o)\qquad\text{and}\lim_{x\to x_o}F(x)=F(x_o).$$

Therefore, $F(x_o)=G(x_o)$ and as this was an arbitrary zero of Q the lemma follows. □

This was ≈ exercise 69 of page 519 of Stewart's Ed. 6, however I proved it in more generality. Note that polynomials have finitely many zeros so they are covered by the lemma.