Introduction to Mechanics

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2024 and beyond

preface

format of my notes

These notes were prepared with LaTeX. You'll notice a number of standard conventions in my notes:

(1.) definitions are in green.

(2.) remarks are in red.

(3.) theorems, propositions, lemmas and corollaries are in blue.

(4.) proofs start with a **Proof:** and are concluded with a $\square$. However, we also use the discuss... theorem format where a calculation/discussion leads to a theorem and the formal proof is left to the reader.

James Cook, May 27, 2017.

version 3.01

Contents

Chapter 1

Geometry and Vectors

1.1 Euclidean Space as a Model

Euclidean space is a mathematical abstraction which we have been taught to think of as a concrete reality. We call $\mathbb{R}^2$ the **two dimensional Euclidean space** and $\mathbb{R}^3$ is known as **three dimensional Euclidean space**. Basic to our discussion is the idea that we can arrange a one-to-one correspondence between Euclidean space of an appropriate dimension and a given physical system. Almost always this is an idealization, it is at best an approximation of physical reality.

- **one-dimensional space** An interstate highway corresponds to $\mathbb{R}$, we could use mile markers to correspond to the whole number tick marks on a number line. If the highway runs North/South then we might take North-directed travel as corresponding to increasing values on the corresponding number line. Of course, every highway is finite and so the correspondence is not technically to $\mathbb{R} = (-\infty, \infty)$ rather the highway corresponds to $(0, N)$ where N is the length of the highway. What does $\mathbb{R}$ fail to capture about an actual physical highway ? Many things, it is a very limited model which captures only one aspect of an actual highway. In this course, when we solve a one-dimensional problem, we are often making a similar conceptual slight of hand. We focus on one-direction at the exclusion of the others. Fortunate for us, God has created nature in such a way that in many regards we may understand it one part at a time. Our overall understanding is then a synthesis of many little bites.

- **two-dimensional space** Any page in book corresponds to $\mathbb{R}^2$, or, to be more precise, if we set the origin $(0,0)$ of the plane at the lower left corner of the page then the upper right corner of the page is (w,h) where w is the width and h is the height of the page. If we use inches, then a typical page corresponds naturally to the mathematical set[1]

$$\{(x,y) \mid 0 \leq x \leq 7,\ 0 \leq y \leq 11\} \subset \mathbb{R}^2$$

However, if we use centimeters, it is known $2.54\,cm = 1.00''$ so the page also corresponds to the mathematical set

$$\{(x,y) \mid 0 \leq x \leq 21.59,\ 0 \leq y \leq 27.94\} \subset \mathbb{R}^2$$

[1] if you don't understand this notation, feel free to ask about it in office hours. Generally $\mathbb{R}^n$ is the set of n-tuples of real numbers, (x,y) is a typical element of $\mathbb{R}^2$ wheres (x,y,z) is a typical element of $\mathbb{R}^3$ and on occasion we might need $(x_1, x_2, \dots, x_n)$ is a typical element of $\mathbb{R}^n$. Physics we study concerns mostly $n = 1, 2, 3$. The main thing we must remember about an n-tuple is that it is an ordered list of n-real numbers thus equality of two n-tuples requires that every entry of both lists equate; $(x_1, \dots, x_n) = (y_1, \dots, y_n)$ means $x_j = y_j$ for $j = 1, 2, \dots, n$.

Is the page either of these sets ? Certainly not. Think about what might be written on the page, its color, its texture, its smell, so many other things. When we say a page is a two dimensional space we are ignoring all the aspects of the page which fail to be captured by the simplistic model of the page as a mere mathematical set. Furthermore, for a given page, we can choose many different measurements of distance.

- **three-dimensional space** A rectangular room with a flat floor and ceiling and square walls corresponds naturally to

$$[0, L] \times [0, W] \times [0, H] = \{(x, y, z) \mid 0 \leq x \leq L,\ 0 \leq y \leq W,\ 0 \leq x \leq H\} \subset \mathbb{R}^3$$

  where L is the length of the room, W is the width of the room and H is the height of the room. We could measure L, W, H using meters, or feet or even cubits if you want to get Biblical here. The choice of where $(0, 0, 0)$ is found in the room is by no means unique. We can imagine setting up coordinates in many different ways. However, modulo these choices, once we set-up a coordinate system then each point in the room uniquely correpsonds to a particular triple of numbers which give a road map on how to get from the origin to that point. I'll probably illustrate this further in lecture. Once more, I don't think it is fair to say the room is the subset of $\mathbb{R}^3$.

Consider this, if a store sells between 200 and 500 kilograms of fish and between $5,000$ and $10,000$ kilograms of carrots in a typical year then we could use $[200, 500] \times [5,000, 10,000]$ as fish-carrot space where a point in this space models the amount of fish and carrots sold in a given year. In a mathematical sense, this is a two-dimensional space. However, this is a different usage of the word **dimension** than in my three examples. There is a difference between **spatial dimension** and dimension in mathematics. The difference is a physical distinction, not a mathematical one. Only in a spatial dimension are we able to **move**. There is no way for a hypothetical fishcarrotist to walk from one market point to another. It's just plain nonsense. On the other hand, a car on the highway, an ant on your page, or a student trying to leave class early (I see you), these all refer to actual physical motions.

Another gross simplification we make almost universally in the Physics course is we relegate people, cars, cats, etc. to mathematical points. This is the **point-mass** idealization. Of course the actual physical dimensions (length, width, height etc.) of people, cars, cats etc. do matter, but when we want to study their motion then it turns out we can treat them as if they were a mass at a single point. This is why the diagrams we draw in examples in this course are often more of a caricature than a portrait. Eventually we will face the reality that this point-mass model is too simplistic for interesting things like yo-yo's or throwing stars or other non-ninja related extended objects. Even so, the point-mass simplification is one we use throughout this course.

1.1.1 basic units of measurement and common derived units

Physics seeks to describe the natural world mathematically. In particular, Physics is largely shaped by the reductionist paradigm which holds to desribing everything in nature with as small as set of laws and physical variables as possible. At the present day there are seven basic units of measurement.

- Length - meter (m)

- Time - second (s)
- Amount of substance - mole (mole)
- Electric current - ampere (A)
- Temperature - kelvin (K)
- Luminous intensity - candela (cd)
- Mass - kilogram (kg)

The SI (System International, or "metric system") was redefined as recently as 2019. I will not get into the details, if you thirst for such trivia, I recommend the textbook or Wikipedia to be honest. From the above list we can create derived units for other physical observables such as:

- Speed - (m/s)
- Accelertion - (m/s^2)
- Force - ($kgm/s^2 = N$ for Newton)
- Momentum - $(kgm/s\ )$
- Angular Momentum - (kgm^2/s)
- Energy - ($kgm^2/s^2 = Nm = J$ for Joule)
- Pressure - $(kg/(ms^2) = N/m^2\ )$

This list is by no means complete, there is more to learn in the Electromagnetism course. We discuss more about what is missing from mechanics at the conclusion of this course. Mechanics is important and it is by far the part of physics which invites the most natural intuition. This is both a blessing and a curse. It's a blessing if you have sense on where to start. It's a curse if you know the answer but can't justify it. How do we justify our work ? As a general rule, solving a physics problem goes something like this:

- read the problem, define variables, draw a picture
- understand what the question is, what it is you need to find
- apply physical laws as appropriate, use calculus where needed,
- use algebra, geometry, trigonometry, vectors to answer the question.

Other guidelines are relevant, be careful with units. If your answer has incorrect units then you probably wrote a physical law which is bogus. If your answer is a vector but it should be a number or vice-versa then this is a big problem. Learning to use vectors to formulate and solve physical problems is a major emphasis of this course. Is the answer physically reasonable ? In this course, feel free to ask me if in doubt, I try to make physically reasonable questions, but I am not as wise as some of the other instructors on this point. I mean, if you happen to solve a problem and find a human running 30mph then I would not read too much into that. I am very open minded about how fast humans can run. Remember, in Bible times, at least until right after the flood, there were giants. Maybe they could run fast. If in doubt about that sort of thing, just ask. I should also mention, if you don't know scientific notation then please read the textbook. I do expect you write answers using proper written notation like 3.45×10^{15}. If your calculator displays something else and you just write that down then I will probably take off points.

I am not terribly invested in teaching significant figures in this course. As a general custom I would like 4 significant digits in the answer. As in, the answer might be 2345 or 2.345 or 0.2345 or 2.345×10^{-642} or 23.45×10^4. Typically if you keep 5 digits for calculations then obtaining 4 digits for the answer without numerical error is plausible.

Enough with the pleasantries. Let us begin.

1.2 Distance Between Points

Sometimes the term **euclidean** is added to emphasize that we suppose distance between points is measured in the usual manner. Recall that in the one-dimensional case the distance between $x, y \in \mathbb{R}$ is given by the absolute value function; $d(x,y) = |y-x| = \sqrt{(y-x)^2}$. Likewise:

Definition 1.2.1. *euclidean distance.*

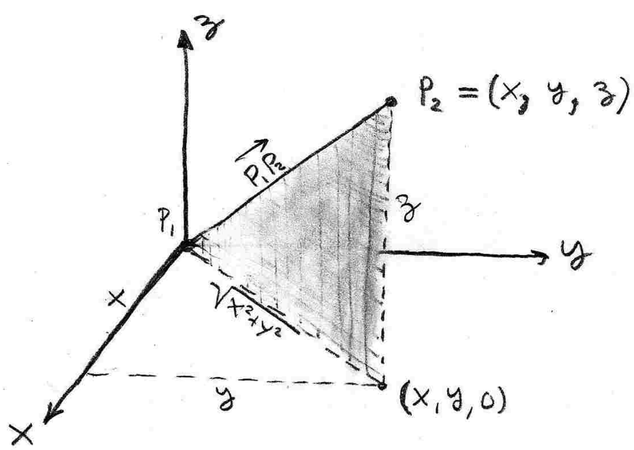
(1.) distance in two-dimensional euclidean space: if $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{R}^2$ then the distance between points p_1 and p_2 is

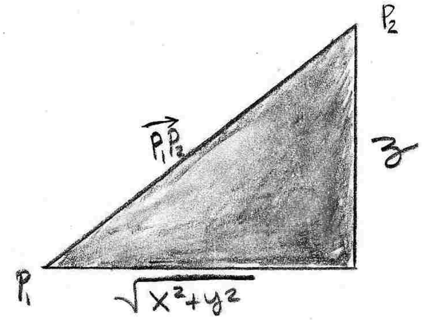
$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(2.) distance in three-dimensional euclidean space: if $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ then the distance between points p_1 and p_2 is

$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

It is simple to verify that the definition above squares with our traditional ideas about distance from previous math courses. In particular, notice these follow from the Pythagorean theorem applied to appropriate triangles. The picture below shows the three dimensional distance formula is consistent with the two dimensional formula.





1.3 Vectors in Two or Three Dimensions

The directed line-segment from P_1 to P_2 is denoted $\overrightarrow{P_1P_2}$ in the above diagram. Directed line-segments are called **vectors**. In contrast to points, a nonzero directed line-segment has an extent in one-direction.

Definition 1.3.1. *Two Dimensional Vectors:*

If $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ then $\overrightarrow{PQ}$ is the vector from P to Q given by:

$$\overrightarrow{PQ} = Q - P = \langle Q_1 - P_1, Q_2 - P_2 \rangle$$

If $P = (P_1, P_2)$ then $\vec{P} = \langle P_1, P_2 \rangle$; we write $\vec{P}$ for the vector from the origin to the point P. The arrow notation is used to emphasize the object is a directed-line segment. If $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then we define **addition** and **scalar multiplication** by $c \in \mathbb{R}$ as follows:

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle, \qquad \& \qquad c\vec{v} = \langle cv_1, cv_2 \rangle.$$

Furthermore, the **length** or **magnitude** of the vector $\vec{v} = \langle v_1, v_2 \rangle$ is defined by:

$$\|\vec{v}\| = v = \sqrt{v_1^2 + v_2^2}.$$

If $\vec{v} \neq 0$ then $\hat{v} = \frac{1}{v}\vec{v}$ and we call $\hat{v}$ the **direction-vector** or **unit-vector** of $\vec{v}$.

Notice $\vec{v} \neq 0$ can be written as the product of its magnitude and direction; $\vec{v} = v\hat{v}$. Moreover, our definition of vector length makes the length of $\overrightarrow{PQ}$ simply the distance from P to Q.

Example 1.3.2. *If $P = (-2, 4)$ and $Q = (8, 7)$ then $\overrightarrow{PQ} = \langle 8 - (-2), 7 - 4 \rangle = \langle 10, 3 \rangle$. The magnitude $\|\overrightarrow{PQ}\| = \sqrt{10^2 + 3^2} = \sqrt{109} \cong 10.44$[2] is the distance from P to Q.*

Example 1.3.3. *Let $\vec{A} = \langle 1, 3 \rangle$ and $\vec{B} = \langle -1, 0 \rangle$ then*

$$\vec{A} + \vec{B} = \langle 1, 3 \rangle + \langle -1, 0 \rangle = \langle 1 - 1, 3 + 0 \rangle = \langle 0, 3 \rangle.$$

We find magnitudes $A = \sqrt{1^2 + 3^2} = \sqrt{10} \cong 3.162$ and $B = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$. Thus unit-vectors in the $\vec{A}$ and $\vec{B}$ directions are given by:

$$\widehat{A} = \frac{1}{A}\vec{A} = \frac{1}{3.162}\langle 1, 3 \rangle = \langle 0.3162, 0.9487 \rangle \qquad \& \qquad \widehat{B} = \frac{1}{B}\vec{B} = \langle -1, 0 \rangle.$$

Example 1.3.4. *Let $\vec{A} = \langle 3, 4 \rangle$ then $||A|| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. Therefore, $\widehat{A} = \langle 0.6, 0.8 \rangle$.*

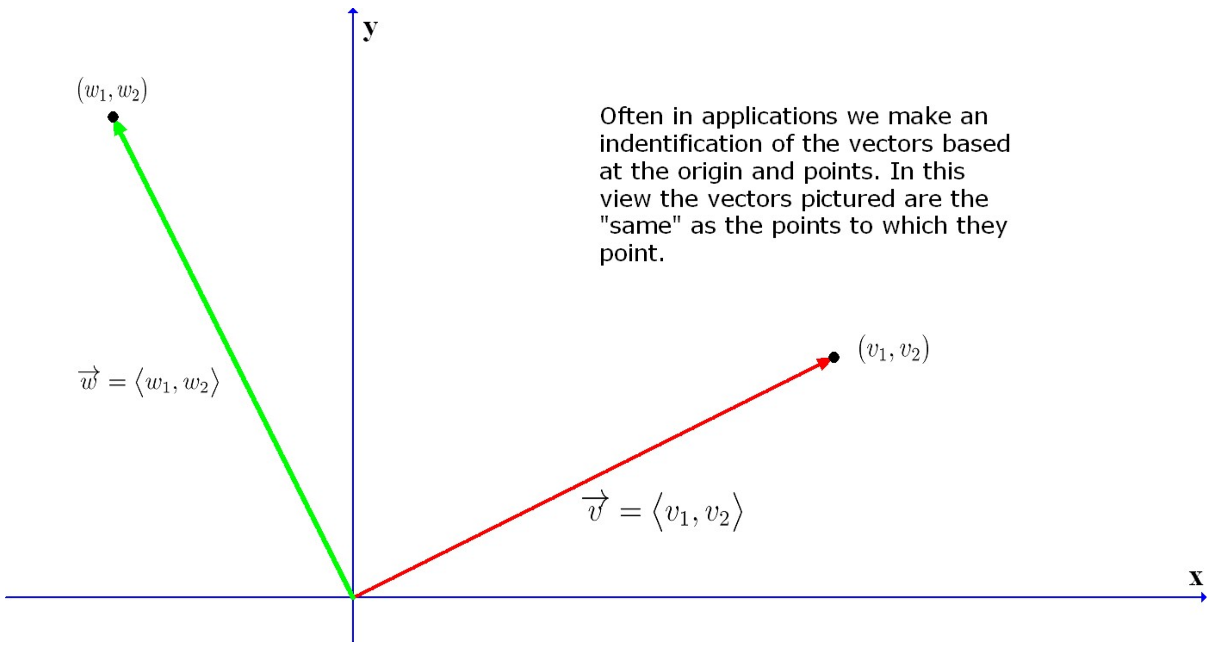
Example Problem 1.3.5. *Find a vector $\vec{B}$ with length 7 and the same direction as $\vec{A} = \langle 1, 1 \rangle$.*

Solution: *Observe $\widehat{A} = \frac{1}{\sqrt{2}}\langle 1, 1 \rangle$ hence $\vec{B} = B\widehat{B} = \frac{7}{\sqrt{2}}\langle 1, 1 \rangle = \langle 4.95, 4.95 \rangle$.*

[2]one notable distinction between Math 231 and Physics 231 is that $\sqrt{109}$ would be a perfectly acceptable answer in Math 231 whereas it just earns partial credit in Physics 231. As a matter of custom, when the answer is numerical we prefer decimal answers. I use unsimplified numbers in the middle of calculations, especially where I know they will cancel out anyway, but, the final result should not require further calculation on the part of the reader. We should finish what we start.

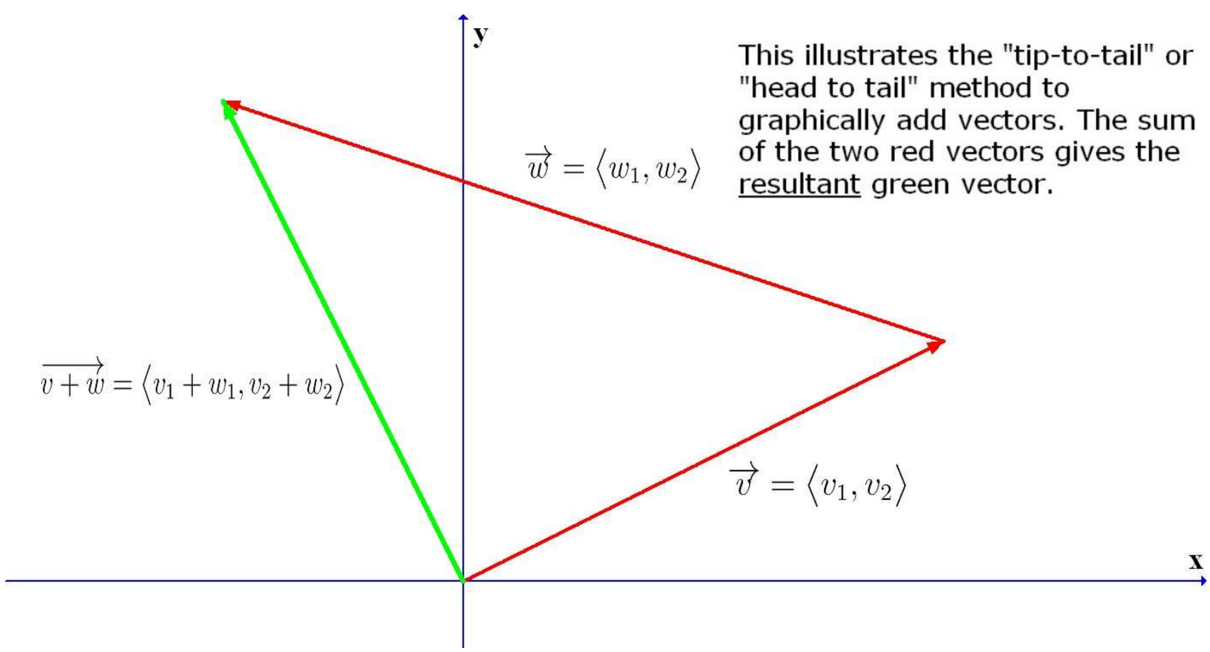
The solution given in the preceding example is *geometrically* motivated. An alternative *algebraic* approach would be to solve $\vec{B} = k\vec{A}$ and $B = 7$ for k. Both approaches have merit. I used the geometric approach to induce insight for the direction vector concept.

There is a natural correspondence between points and directed line-segments from the origin.

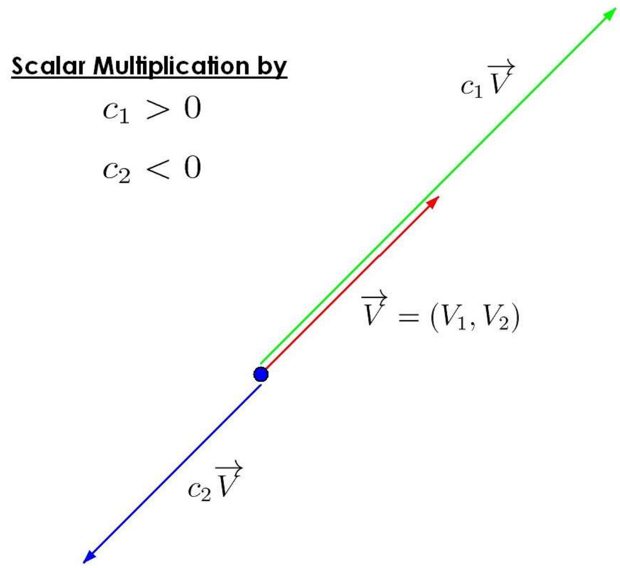


We will use the notation $\vec{p}$ for vectors throughout the remainder of these notes to emphasize the fact that $\vec{p}$ is a vector. Some texts use **bold** to denote vectors, but I prefer the over-arrow notation which is easily duplicated in hand-written work.

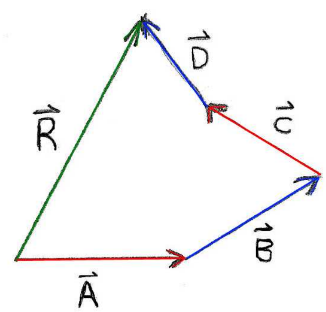
We add vectors geometrically by the tip-to-tail method as illustrated below.



Also, we rescale them by shrinking or stretching their length by a scalar multiple:

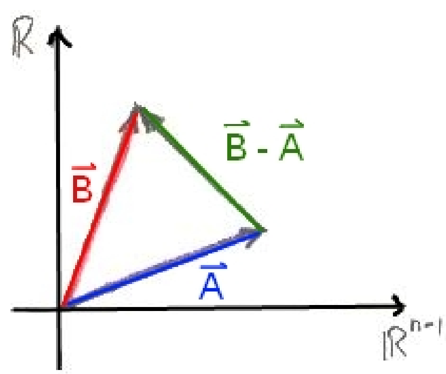


In the diagram below we illustrate the geometry behind the vector equation $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$.



Continuing in this way we can add any finite number of vectors in the same *tip-2-tail* fashion. I used $\vec{R}$ in the diagram above because the result of a vector addition is called the **resultant** vector.

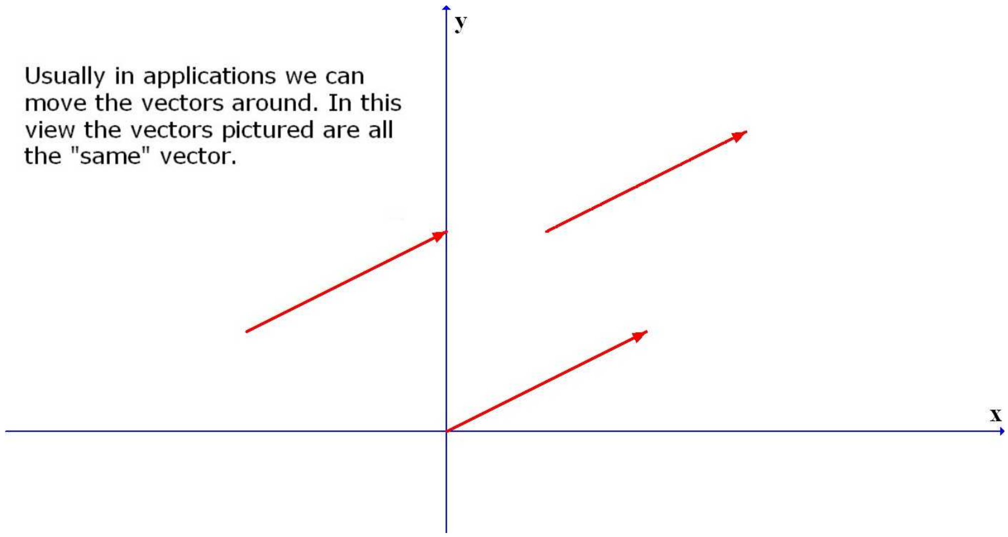
It is sometimes useful to see how $\vec{A}$ and $\vec{B}$ are connected by the vector $\vec{B} - \vec{A}$:



Notice that $\vec{A} + (\vec{B} - \vec{A}) = \vec{B}$ by the tip-2-tail diagram above[3].

[3]the picture above is one of my exceedingly silly methods for graphing n-dimensions

In most applications of vectors we are free to move a given vector around the plane in such a way that we maintain its direction and length:



If we wish to keep track of the base point of vectors then additional comment is required. I think of vectors as based at the origin unless there is reason from the context to think of them based elsewhere. For example, if I think about a force applied to a lever arm then I imagine the force as acting on its point of application.

I have mostly emphasized two-dimensional vectors up to this point, but we can easily extend the discussion to three-dimensional vectors.

Definition 1.3.6. *Three Dimensional Vectors:*

If $P = (P_1, P_2, P_3)$ and $Q = (Q_1, Q_2, Q_3)$ then $\overrightarrow{PQ}$ is the vector from P to Q given by:

$$\overrightarrow{PQ} = Q - P = \langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \rangle$$

If $P = (P_1, P_2, P_3)$ then $\vec{P} = \langle P_1, P_2, P_3 \rangle$; we write $\overrightarrow{P}$ for the vector from the origin to the point P. The arrow notation is used to emphasize the object is a directed-line segment. If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then we define **addition** and **scalar multiplication** by $c \in \mathbb{R}$ as follows:

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle, \qquad \& \qquad c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle.$$

Furthermore, the **length** or **magnitude** of the vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is defined by:

$$||\vec{v}|| = v = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

If $\vec{v} \neq 0$ then $\hat{v} = \frac{1}{v}\vec{v}$ and we call $\hat{v}$ the **direction-vector** or **unit-vector** of $\vec{v}$.

The example below illustrates a nice trick for constructing vectors.

Example 1.3.7. *If* $\vec{A} = \langle 1, 2, -2 \rangle$ *then* $A = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$ *thus* $\widehat{A} = \langle 1/3, 2/3, -2/3 \rangle$. *If you want to construct a vector* $\vec{B}$ *of length* 18 *in the direction of* $\vec{A}$ *then simply use* $\vec{B} = 18\widehat{A} = 18\langle 1/3, 2/3, -2/3 \rangle = \langle 6, 12, -12 \rangle$.

1.4 Decomposing Vectors into Components

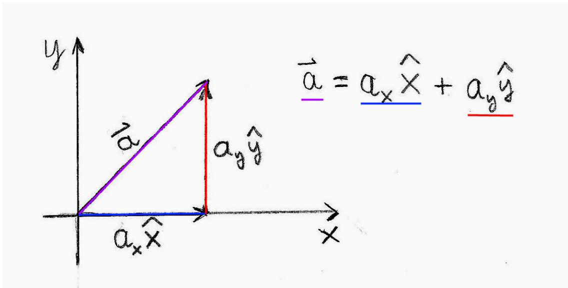
For $\mathbb{R}^2$, **define**[4] $\hat{\mathbf{x}} = \langle 1, 0\rangle$ **and** $\hat{\mathbf{y}} = \langle 0, 1\rangle$ hence:

$$\begin{aligned}\langle a, b\rangle &= \langle a, 0\rangle + \langle 0, b\rangle \\ &= a\langle 1, 0\rangle + b\langle 0, 1\rangle \\ &= a\hat{\mathbf{x}} + b\hat{\mathbf{y}}\end{aligned}$$

Definition 1.4.1. *vector and scalar components of two-vectors.*

The **vector component** of $\langle a, b\rangle$ in the x-direction is simply $a\hat{\mathbf{x}}$ whereas the **vector component** of $\langle a, b\rangle$ in the y-direction is simply $b\hat{\mathbf{y}}$. In contrast, a, b are the **scalar components** of $\langle a, b\rangle$ in the x, y-directions respective.

Scalar components are scalars whereas vector components are vectors. These are entirely different objects if $n \neq 1$, please keep clear this distinction in your mind. Notice that the vector components are what we use to reproduce a given vector by the tip-to-tail sum:



Example 1.4.2. *Let $\vec{v} = \langle 2, -3\rangle$ then $2\hat{\mathbf{x}}$ is the x-vector component of $\vec{v}$ and 2 is the scalar component of $\vec{v}$ in the x-direction. Likewise, $-3\hat{\mathbf{y}}$ is the y-vector component of $\vec{v}$.*

Example Problem 1.4.3. *find a vector $\vec{A}$ of length 10 which has $6\hat{\mathbf{x}}$ as its x-vector component.*

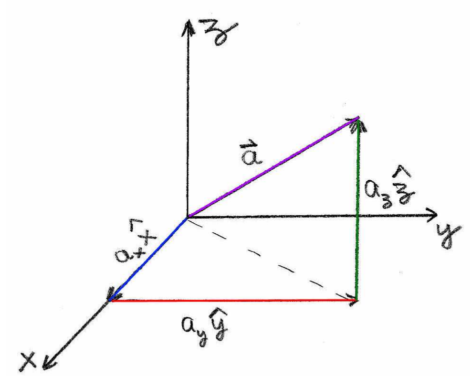
Solution: *we seek to find y such that $\vec{A} = \langle 6, y\rangle$ has length 10. Notice $A^2 = 6^2 + y^2 = 10^2$ hence $y^2 = 64$ which gives $y = \pm 8$. We find two vectors which solve this problem, $\vec{A} = \langle 6, \pm 8\rangle$.*

For $\mathbb{R}^3$ we define the following notation[5]: $\hat{\mathbf{x}} = \langle 1, 0, 0\rangle$**,** $\hat{\mathbf{y}} = \langle 0, 1, 0\rangle$**, and** $\hat{\mathbf{z}} = \langle 0, 0, 1\rangle$ hence:

$$\begin{aligned}\langle a, b, c\rangle &= \langle a, 0, 0\rangle + \langle 0, b, 0\rangle + \langle 0, 0, c\rangle \\ &= a\langle 1, 0, 0\rangle + b\langle 0, 1, 0\rangle + c\langle 0, 0, 1\rangle \\ &= a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}\end{aligned}$$

[4] I should mention that often $\hat{\imath}$ is used for $\hat{\mathbf{x}}$ and $\hat{\jmath}$ is used for $\hat{\mathbf{y}}$, I choose this less popular notation because it is far more descriptive than the traditional notation, I trust the reader can adapt in future studies if need be. Incidentially, another popular notation in linear algebra is that $e_1 = (1, 0)$ and $e_2 = (0, 1)$ in the context of $\mathbb{R}^2$.

[5] yes, in the context of $\mathbb{R}^3$ we have $\hat{\mathbf{x}} = \hat{\imath} = e_1 = (1, 0, 0)$ whereas $\hat{\mathbf{y}} = \hat{\jmath} = e_2 = (0, 1, 0)$ and $\hat{\mathbf{z}} = \hat{k} = e_3 = (0, 0, 1)$, notice the number of zeros depends on the context.



Definition 1.4.4. *vector and scalar components of three-vectors.*

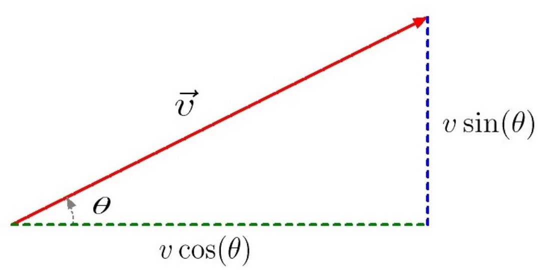
> The **vector components** of $\langle a, b, c\rangle$ are: $a\hat{\mathbf{x}}$ in the x-direction, $b\hat{\mathbf{y}}$ in the y-direction and $c\hat{\mathbf{z}}$ in the z-direction. In contrast, a, b, c are the **scalar components** of $\langle a, b, c\rangle$ in the x, y, z-directions respective.

Example 1.4.5. *Observe,* $\langle 1, 2, 3\rangle = \langle 1, 0, 0\rangle + \langle 0, 2, 0\rangle + \langle 0, 0, 3\rangle = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$.

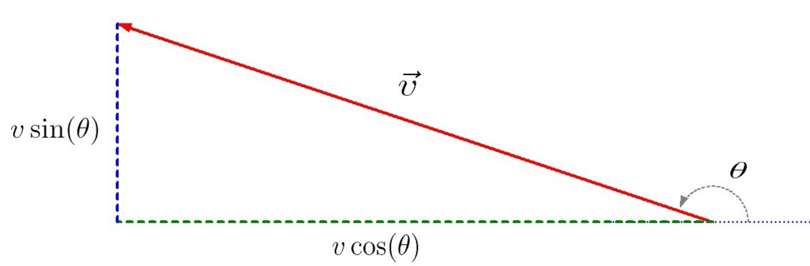
Example Problem 1.4.6. *find a vector $\vec{A}$ of length 13 which has $4\hat{\mathbf{y}}$ as its y-vector component and $-3\hat{\mathbf{z}}$ as its z-vector component.*

Solution: *we seek to find x such that $\vec{A} = \langle x, 4, -3\rangle$ has length 5. Notice $A^2 = x^2+4^2+(-3)^2 = 13^2$ hence $x^2 = 169 - 25 = 144$ which gives $x = \pm 12$. We find two solutions $\vec{A} = \langle \pm 12, 4, -3\rangle$.*

We conclude this section by discussing how trigonometry is often applied to the study of vectors in the plane. It is not uncommon to be faced with vectors which are described by a length and a direction in the plane. In such a case we need to rely on trigonometry to *break-down* the vector into it's Cartesian components.



Example 1.4.7. *Suppose a vector $\vec{v}$ has a length $v = 5$ at $\theta = 60^o$ then $v\cos\theta = 5\cos(60^o) = 2.5$ and $v\sin\theta = 5\sin(60^o) \cong 4.33$. Therefore, $\vec{v} \cong \langle 2.5,\ 4.33\rangle$.*



Example 1.4.8. *Suppose a vector $\vec{v}$ has a length $v = 2$ at $\theta = 150^o$ then $v\cos\theta = 2\cos(150^o) \cong -1.732$ and $v\sin\theta = 2\sin(150^o) = 1$. Therefore, $\vec{v} = \langle -1.732, 1\rangle$. Notice, $\theta = 150^o$ is in Quadrant II and our result is consistent with the figure above.*

In general, for $\vec{v} = \langle v_1, v_2\rangle \neq 0$ we can describe $\vec{v}$ in terms of its magnitude $v = \sqrt{v_1^2 + v_2^2}$ and standard angle θ. Place $\vec{v}$ at the origin then following the diagrams given in this section,

$$v_1 = v\cos\theta \qquad \& \qquad v_2 = v\sin\theta$$

Consequently, $\vec{v} = \langle v\cos\theta, v\sin\theta\rangle = v\langle\cos\theta, \sin\theta\rangle$. However, we also know $\vec{v} = v\hat{v}$ hence we find:

$$\boxed{\hat{v} = \langle\cos\theta, \sin\theta\rangle}$$

Notice, $\|\langle\cos\theta, \sin\theta\rangle\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$. Thus, $\langle\cos\theta, \sin\theta\rangle$ is a unit-vector. You should remember from previous math coursework that $(\cos\theta, \sin\theta)$ is a typical point on the unit-circle. Now we're simply observing $\langle\cos\theta, \sin\theta\rangle$ is the vector of length one which points from the origin to the point $(\cos\theta, \sin\theta)$.

Example 1.4.9. *Suppose $\vec{v}$ has length 7 and is directed at the standard angle $\theta = 295^o$. Then the unit-vector in the direction of $\vec{v}$ is simply $\hat{v} = \langle\cos(295^o), \sin(295^o)\rangle = \langle 0.4226, \ -0.9063\rangle$. Thus, $\vec{v} = 7\langle 0.4226, \ -0.9063\rangle = \langle 2.958, \ -6.344\rangle$*

Example 1.4.10. *If $\vec{v} \neq 0$ has $\theta = -30^o$ then $\widehat{v} = \langle\cos(-30^o), \sin(-30^o)\rangle = \langle 0.866, \ -0.5\rangle$.*

When we describe the direction of a two-dimensional vector we can either use a unit-vector or a standard angle. Only two dimensions allows for vector direction to be specified by a single angle.

1.5 The Dot Product

The dot-product of two vectors gives a number which relates to whether the given pair of vectors is parallel or perpendicular or somewhere in-between.

Definition 1.5.1. *dot product.*

The **dot-product** is a useful operation on vectors. In $\mathbb{R}^2$ we define,

$$\langle V_1, V_2\rangle \bullet \langle W_1, W_2\rangle = V_1W_1 + V_2W_2.$$

In $\mathbb{R}^3$ we define,

$$\langle V_1, V_2, V_3\rangle \bullet \langle W_1, W_2, W_3\rangle = V_1W_1 + V_2W_2 + V_3W_3.$$

It is important to notice that the dot-product takes in two *vectors* and outputs a *scalar*. You can easily verify the following identities hold for the dot-product:

$$\vec{A}\bullet\vec{B} = \vec{B}\bullet\vec{A}. \qquad \& \qquad \vec{A}\bullet(\vec{B}+\vec{C}) = \vec{A}\bullet\vec{B} + \vec{A}\bullet\vec{C} \qquad \& \qquad \vec{A}\bullet(c\vec{B}) = c\vec{A}\bullet\vec{B}.$$

Example 1.5.2. *Let $\vec{A} = \langle 3, 4\rangle$ and $\vec{B} = \langle 7, -2\rangle$. We calculate,*

$$\vec{A}\bullet\vec{B} = \langle 3, 4\rangle \bullet \langle 7, -2\rangle = (3)(7) + (4)(-2) = 13.$$

The next example illustrates an important use of dot-products:

Example 1.5.3. *Let* $\vec{A} = \langle A_1, A_2 \rangle$ *then* $\vec{A} \bullet \hat{\mathbf{x}} = \langle A_1, A_2 \rangle \bullet \langle 1, 0 \rangle = A_1(1) + A_2(0) = A_1$ *whereas* $\vec{A} \bullet \hat{\mathbf{y}} = \langle A_1, A_2 \rangle \bullet \langle 0, 1 \rangle = A_1(0) + A_2(1) = A_2$. *We can use the dot-product of* $\vec{A}$ *against the unit-vectors to find the components of* $\vec{A}$.

Example 1.5.4. *Let* $\vec{A} = \langle 1, 2, 3 \rangle$ *and* $\vec{B} = \langle 1, -1, 5 \rangle$. *We calculate,*

$$\vec{A} \bullet \vec{B} = \langle 1, 2, 3 \rangle \bullet \langle 1, -1, 5 \rangle = 1 - 2 + 15 = 14.$$

If you understood the Example 1.5.3 then this example will be totally unsurprising:

Example 1.5.5. *Let* $\vec{A} = \langle A_1, A_2, A_3 \rangle$ *then* $\vec{A} \bullet \hat{\mathbf{x}} = \langle A_1, A_2, A_3 \rangle \bullet \langle 1, 0, 0 \rangle = A_1(1) + A_2(0) + A_3(0) = A_1$ *whereas* $\vec{A} \bullet \hat{\mathbf{y}} = \langle A_1, A_2, A_3 \rangle \bullet \langle 0, 1, 0 \rangle = A_1(0) + A_2(1) + A_3(0) = A_2$ *and lastly* $\vec{A} \bullet \hat{\mathbf{z}} = \langle A_1, A_2, A_3 \rangle \bullet \langle 0, 0, 1 \rangle = A_1(0) + A_2(0) + A_3(1) = A_3$.

What happens when we take the dot-product of a vector with itself ? Consider:

$$\vec{A} \bullet \vec{A} = \langle A_1, A_2, A_3 \rangle \bullet \langle A_1, A_2, A_3 \rangle = A_1^2 + A_2^2 + A_3^2 = A^2 \quad \Rightarrow \quad \boxed{A = \sqrt{\vec{A} \bullet \vec{A}}}.$$

A unit-vector $\hat{\mathbf{u}}$ must satisfy $\hat{\mathbf{u}} \bullet \hat{\mathbf{u}} = 1$. Of course, we could have easily seen this for $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ or $\hat{\mathbf{z}}$, but this identity has nothing to do with the particular x, y or z direction.

In the previous section we learned for $\vec{B} = \langle B_1, B_2 \rangle$ we could write $B_1 = B \cos\theta$ and $B_2 = B \sin\theta$. Geometrically this is based on measuring the angle θ off the positive x-axis in the CCW (Counter-ClockWise) sense. Notice:

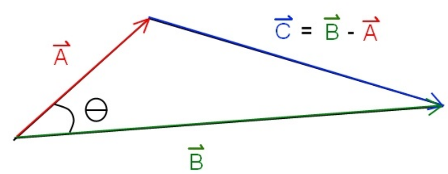
$$\hat{\mathbf{x}} \bullet \vec{B} = B_1 = B \cos\theta$$

Consider $\vec{A} \neq 0$ and **define** the x-axis to point in the $\vec{A}$-direction. Then $\vec{A} = A\hat{\mathbf{x}}$. But then[6],

$$\vec{A} \bullet \vec{B} = A\hat{\mathbf{x}} \bullet \vec{B} = AB \cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. It is customary to use $0 \leq \theta \leq 180^o$. The argument just given can easily be extended to the three dimensional context and it follows we have derived the following formula for the dot-product:

$$\boxed{\vec{A} \bullet \vec{B} = AB \cos\theta}$$



where θ is the angle between[7] $\vec{A}$ and $\vec{B}$. Let's examine the triangle formed by $\vec{A}, \vec{B}$ and their difference $\vec{C} = \vec{B} - \vec{A}$. Let θ be the angle opposite C. Since we already know $A^2 = \vec{A} \bullet \vec{A}$ and $B^2 = \vec{B} \bullet \vec{B}$ and $C^2 = \vec{C} \bullet \vec{C}$ gives:

$$C^2 = (\vec{B} - \vec{A}) \bullet (\vec{B} - \vec{A}) = \vec{B} \bullet \vec{B} - \vec{B} \bullet \vec{A} - \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{A} = A^2 + B^2 - 2\vec{A} \bullet \vec{B}$$

Thus, as $\vec{A} \bullet \vec{B} = AB \cos\theta$, we find $C^2 = A^2 + B^2 - 2AB \cos\theta$ which is the **Law of Cosines**.

Example Problem 1.5.6. *If* $\vec{A} = \langle 2, 2, 1 \rangle$ *and* $\vec{B} = \langle 3, 0, -4 \rangle$ *the find the angle between* $\vec{A}$ *and* $\vec{B}$.

Solution: *calculate* $A = \sqrt{4+4+1} = 3$ *and* $B = \sqrt{9+16} = 5$ *and* $\vec{A} \bullet \vec{B} = 2(3)+2(0)+1(-4) = 2$. *Since* $\vec{A} \bullet \vec{B} = AB \cos\theta$ *we find* $2 = 3(5)\cos\theta$. *Thus* $\theta = \cos^{-1}(2/15) = 82.34^o$.

[6] we can easily show $c(\vec{A} \bullet \vec{B}) = (c\vec{A}) \bullet \vec{B}$ directly from the definition of dot-product given earlier, perhaps this will be a homework question

[7] you have to understand the context of θ, this is not a standard angle in this context, unless it just happens that one of the vectors points in the positive x-direction and $0 \leq \theta \leq 180^o$. This issue has caused some consternation in my Math 231 course, so beware. You must understand both the formula and its context.

Dot-products of the coordinate unit-vectors in $\mathbb{R}^3$ are very easy to remember[8]:

$$\hat{\mathbf{x}} \bullet \hat{\mathbf{x}} = 1, \qquad \hat{\mathbf{y}} \bullet \hat{\mathbf{y}} = 1, \qquad \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = 1,$$

$$\hat{\mathbf{x}} \bullet \hat{\mathbf{y}} = 0, \qquad \hat{\mathbf{x}} \bullet \hat{\mathbf{z}} = 0, \qquad \hat{\mathbf{y}} \bullet \hat{\mathbf{z}} = 0.$$

Formally, this makes $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ an *orthonormal set* of vectors. Let me give a proper definition:

Definition 1.5.7. *orthogonal vectors.*

> We say $\vec{A}$ is **orthogonal** to $\vec{B}$ if and only if $\vec{A} \bullet \vec{B} = 0$. A set of vectors which is both orthogonal and all of unit length is said to be an **orthonormal set** of vectors. We also call orthogonal vectors **perpendicular**. If $\vec{A} \bullet \vec{B} = \pm AB$ then $\vec{A}, \vec{B}$ are **colinear**. If $\vec{A} \bullet \vec{B} = AB$ then $\vec{A}, \vec{B}$ are **parallel**. If $\vec{A} \bullet \vec{B} = -AB$ then $\vec{A}, \vec{B}$ are **anti-parallel**.

Equivalently, nonzero vectors $\vec{A}$ and $\vec{B}$ are parallel if the angle between $\vec{A}$ and $\vec{B}$ is zero whereas $\vec{A}$ and $\vec{B}$ are anti-parallel if the angle between them is 180^o. Two nonzero vectors are perpendicular if the angle between them is 90^o. The equivalency of the concepts is seen from $\vec{A} \bullet \vec{B} = AB\cos\theta$. Orthonormality makes for beautiful formulas. Behold:

$$\vec{A} \bullet \hat{\mathbf{x}} = (A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}}) \bullet \hat{\mathbf{x}} = A_1\hat{\mathbf{x}} \bullet \hat{\mathbf{x}} + A_2\hat{\mathbf{y}} \bullet \hat{\mathbf{x}} = A_1$$

and

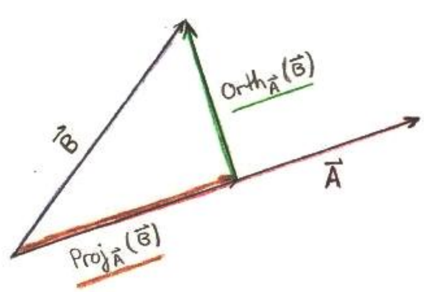
$$\vec{A} \bullet \hat{\mathbf{y}} = (A_1\hat{\mathbf{x}} + A_2\hat{\mathbf{y}}) \bullet \hat{\mathbf{y}} = A_1\hat{\mathbf{x}} \bullet \hat{\mathbf{y}} + A_2\hat{\mathbf{y}} \bullet \hat{\mathbf{y}} = A_2$$

Thus, $\vec{A} = (\vec{A} \bullet \hat{\mathbf{x}})\hat{\mathbf{x}} + (\vec{A} \bullet \hat{\mathbf{y}})\hat{\mathbf{y}}$ for any two dimensional vector. We can use the dot-product to select the scalar *Cartesian* components of a given vector. This might not seem particularly interesting at first glance, but it goes to show we can use the dot-product to cast a shadow of one vector upon another. The dot-product of $\vec{B}$ with the unit-vector of $\vec{A}$ gives us the length of the vector-component of $\vec{B}$ which is parallel to $\vec{A}$.

Definition 1.5.8. *vector projection*

> Let $\vec{A} \neq 0, \vec{B}$ be vectors, then the parallel projection of $\vec{B}$ onto $\vec{A}$ is $\text{Proj}_{\vec{A}}(\vec{B}) = (\vec{B} \bullet \hat{A})\hat{A}$. Likewise, we define $\text{Orth}_{\vec{A}}(\vec{B}) = \vec{B} - (\vec{B} \bullet \hat{A})\hat{A}$ thus $\text{Proj}_{\vec{A}}(\vec{B}) + \text{Orth}_{\vec{A}}(\vec{B}) = \vec{B}$.

We can picture the definition above as follows:



I invite the reader to check for themselves, $\text{Proj}_{\vec{A}}(\vec{B}) \bullet \text{Orth}_{\vec{A}}(\vec{B}) = 0$. I find the use of the projection operation is very helpful in solving nontrivial geometric problems. Some Physics books might skip it, but that's their loss not ours. I will illustrate its use in the examples.

[8]a more elegant method is to denote $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1$ and $\hat{\mathbf{y}} = \hat{\mathbf{x}}_2$ and $\hat{\mathbf{z}} = \hat{\mathbf{x}}_3$ in order to see that $\hat{\mathbf{x}}_i \bullet \hat{\mathbf{x}}_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta. We define $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Suppose $\vec{V}$ and $\vec{W}$ are perpendicular then we calculate

$$(\vec{V}+\vec{W})\bullet(\vec{V}+\vec{W}) = \vec{V}\bullet\vec{V} + 2\vec{V}\bullet\vec{W} + \vec{W}\bullet\vec{W} = V^2 + W^2$$

In other words, if $\vec{B} = \vec{V}+\vec{W}$ where $\vec{V} \perp \vec{W}$ then the Pythagorean Theorem holds for the triangle with sides $\vec{V}, \vec{W}$ and $\vec{B}$;

$$B^2 = V^2 + W^2$$

This is especially interesting when we think about what it means for the perpendicular $\text{Proj}_{\vec{A}}(\vec{B}), \text{Orth}_{\vec{A}}(\vec{B})$ where $\vec{B} = \text{Proj}_{\vec{A}}(\vec{B}) + \text{Orth}_{\vec{A}}(\vec{B})$. We have:

$$B^2 = \|\text{Proj}_{\vec{A}}(\vec{B})\|^2 + \|\text{Orth}_{\vec{A}}(\vec{B})\|^2$$

where I'm using the notation $\|\vec{V}\| = V$.

1.5.1 examples to showcase dot-product based calculation

Let's introduce some nice short notation into the mix. $\angle(\vec{A}, \vec{B})$ denotes the angle between $\vec{A}$ and $\vec{B}$. Also, $\vec{A} \perp \vec{B}$ means $\vec{A}$ is perpendicular to $\vec{B}$.

Example 1.5.9. *Consider $\vec{A} = \langle 1, 2, -3\rangle$ and $\vec{B} = \langle 3, 0, 1\rangle$. Since $\vec{A}\bullet\vec{B} = 1(3) + 2(0) - 3(1) = 0$ we find $\angle(\vec{A}, \vec{B}) = 90^o$. That is, $\vec{A} \perp \vec{B}$.*

Example 1.5.10. *Let $\vec{A} = \langle -5, 3, 7\rangle$ and $\vec{B} = \langle 6, -8, 2\rangle$. Are these vectors parallel, antiparallel or orthogonal? We can calculate the dot-product to answer this question. Observe,*

$$\vec{A}\bullet\vec{B} = \langle -5, 3, 7\rangle\bullet\langle 6, -8, 2\rangle = -30 - 24 + 14 = -40 \neq 0.$$

Thus, we know $\vec{A}$ and $\vec{B}$ are not orthogonal. Furthermore, they cannot be parallel as the dot-product's sign indicates they point in directions more than 90^o oppposed. Are they antiparallel?

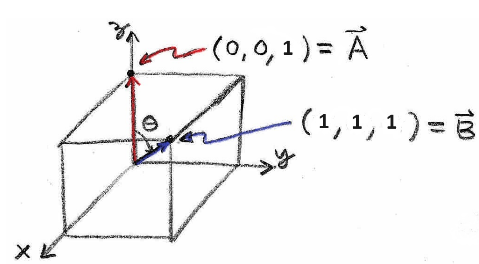
$$-AB = -\sqrt{25+9+49}\sqrt{36+64+4} = -\sqrt{8632} = 92.91 \neq -40$$

Therefore, the given pair of vectors is neither parallel, antiparallel nor orthogonal. Of course, we could have ascertained all these comments by simply calculating the angle between the given vectors:

$$\angle(\vec{A}, \vec{B}) = \cos^{-1}\left(\frac{\vec{A}\bullet\vec{B}}{AB}\right) = \cos^{-1}\left(\frac{-40}{\sqrt{8932}}\right) = 115.5^o.$$

Example Problem 1.5.11. *Consider a cube of side-length 1. What is the angle between the interior diagonal of the cube and the edge of the cube?*

Solution: *We place the cube at the origin and envision the diagonal from $(0,0,0)$ to $(1,1,1)$. The edge goes from $(0,0,0)$ to $(0,0,1)$. Let us label the diagonal and edge by $\vec{B}$ and $\vec{A}$ respectively. Observe $A = 1$ and $B = \sqrt{3}$ whereas $\vec{A}\bullet\vec{B} = 1$. Therefore,*
$\angle(\vec{A}, \vec{B}) = \cos^{-1}\left(\frac{\vec{A}\bullet\vec{B}}{AB}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \boxed{54.74^o}$.



The reason the angle is not 45^o in the example above is that the vectors $\vec{A}$ and $\vec{B}$ lie on the edge and diagonal of a nonsquare-rectangle. The larger point here: **use vectors** to escape wrong intuition in three-dimensional geometry. The mathematics of vectors allows us to solve problems step-by-step which defy direct geometric methods.

Example Problem 1.5.12. *Consider $\vec{A} = \langle a, b\rangle$ with $ab \neq 0$. Find all vectors perpendicular to $\vec{A}$.*

Solution: *Let $\vec{B} = \langle x, y\rangle$ and suppose $\vec{A} \bullet \vec{B} = 0$. Thus*

$$\langle a, b\rangle \bullet \langle x, y\rangle = ax + by = 0.$$

If $ab \neq 0$ then both a and b are nonzero. Solve for y, and substitute that into $\vec{B} = \langle x, y\rangle$,

$$y = -\frac{ax}{b} \quad \Rightarrow \quad \vec{B} = \left\langle x, -\frac{ax}{b}\right\rangle = \frac{x}{b}\langle b, -a\rangle$$

So for any choice of x the vector $\vec{B} = \frac{x}{b}\langle b, -a\rangle$ is perpendicular to $\vec{A}$.

Is the example above disturbing ? Sometimes there is more than one answer to a problem. In retrospect, a wise student could easily have guessed that $\langle b, -a\rangle \perp \langle a, b\rangle$. That is fairly obvious once you see it once. Let's use this new-found wisdom on the next problem.

Example Problem 1.5.13. *Consider $\vec{A} = \langle 3, 4\rangle$. Find all unit vectors perpendicular to $\vec{A}$.*

Solution: *notice $\vec{B} = \langle -4, 3\rangle$ has $\vec{A} \bullet \vec{B} = 0$, however $B = 5$. Hence one of the answers is clearly $\boxed{\hat{B} = \langle -0.8, 0.6\rangle}$. Naturally, $-\hat{B}$ is also a unit vector which is perpendicular to $\vec{A}$ hence $\boxed{\langle 0.8, -0.6\rangle}$ is the other possible answer. The fact that there are just two answers is geometrically clear; the only angles which yield a zero dot-product are $\pm 90^o$ if we envision the angle being based on the $\vec{A}$-axis.*

Example Problem 1.5.14. *Consider $\vec{A}$ with $A = 3$ at standard angle α. Find all unit vectors perpendicular to $\vec{A}$.*

Solution: *picture $\vec{A}$ pointing at standard angle α then the unit-vectors $\hat{U}_\pm$ with standard angles $\alpha \pm 90^o$ respective have $\angle(\hat{U}_\pm, \vec{A}) = 90^o$. Thus, the desired perpendicular unit vectors are:*

$$\hat{U}_+ = \langle \cos(\alpha + 90^o), \sin(\alpha + 90^o)\rangle \qquad \& \qquad \hat{U}_- = \langle \cos(\alpha - 90^o), \sin(\alpha - 90^o)\rangle$$

Trigonometry[9] *simplifies the results above to:*

$$\hat{U}_+ = \langle -\sin\alpha, \cos\alpha\rangle \qquad \& \qquad \hat{U}_- = \langle \sin\alpha, -\cos\alpha\rangle.$$

Of course, since $\vec{A} = 3\langle \cos\alpha, \sin\alpha\rangle$ the results above are to be expected.

One more take on this problem. This time with a creative use of the projection concept.

[9] I hope you have memorized $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$ since that is the trigonometry in play here, well, that and the fact that sine is odd, cosine is even and $\cos(90^o) = 0$ and $\sin(90^o) = 1$ both of which we should know without touching a calculator. If you need help with strategies to remember less by deriving more, then by all means ask me in office hours. I know things.

Example Problem 1.5.15. *Consider $\vec{A} = \langle 2, 1\rangle$. Find all unit vectors perpendicular to $\vec{A}$.*

Solution: *let us create a vector not parallel to $\vec{A}$. There are infinitely many choices possible, let's just use $\vec{B} = \langle 1, 0\rangle$. Then*

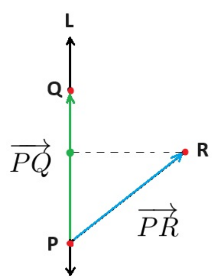
$$Proj_{\vec{A}}(\vec{B}) = \frac{\vec{A} \bullet \vec{B}}{\vec{A} \bullet \vec{A}}\vec{A} = \frac{2}{5}\langle 2, 1\rangle = \langle 0.8, 0.4\rangle$$

Thus, $Orth_{\vec{A}}(\vec{B}) = \vec{B} - Proj_{\vec{A}}(\vec{B}) = \langle 1, 0\rangle - \langle 0.8, 0.4\rangle = \langle 0.2, -0.4\rangle$. Notice $\vec{A} \perp \langle 0.2, -0.4\rangle$ as we should hope. Notice $\|\langle 0.2, -0.4\rangle\| = \sqrt{0.2^2 + 0.4^2} = 0.44721$ thus dividing $\langle 0.2, -0.4\rangle$ by its length yields unit vector $\langle \frac{0.2}{0.44721}, \frac{-0.4}{0.44721}\rangle = \boxed{\langle 0.4472, -0.8944\rangle}$. The other answer is $\boxed{\langle -0.4472, 0.8944\rangle}$.

The example above is a bit strange, if you find it weird, you've probably got a lot of company. Let's look at better use of the projection.

Example Problem 1.5.16. *Let L be the line which contains the points $P = (-1, 2, 3)$ and $Q = (4, 5, 5)$. Find the point on the line L which is closest to the point $R = (7, 7, 7)$.*

Solution: *let us define $\overrightarrow{PQ} = Q - P = \langle 5, 3, 2\rangle$ and $\overrightarrow{PR} = R - P = \langle 8, 5, 4\rangle$. Calculate*



$$\begin{aligned} Proj_{\overrightarrow{PQ}}(\overrightarrow{PR}) &= \left(\frac{\overrightarrow{PQ} \bullet \overrightarrow{PR}}{\overrightarrow{PQ} \bullet \overrightarrow{PQ}}\right)\overrightarrow{PQ} \\ &= \left(\frac{5(8) + 3(5) + 2(4)}{25 + 9 + 4}\right)\langle 5, 3, 2\rangle \\ &= \langle 8.289,\ 4.974,\ 3.316\rangle \end{aligned}$$

The point on L which is closest to R can be reached by adding $Proj_{\overrightarrow{PQ}}(\overrightarrow{PR})$ to the vector based at the origin which terminates at P. That is,

$$S = P + Proj_{\overrightarrow{PQ}}(\overrightarrow{PR}) = (-1, 2, 3) + \langle 8.289,\ 4.974,\ 3.316\rangle = (7.289,\ 6.974,\ 6.316)$$

Notice $\|Orth_{\overrightarrow{PQ}}(\overrightarrow{PR})\| = \|(-0.289,\ 0.026,\ 0.684)\| = 0.743$ is the distance from S to R. Would you be able to find S without vectors ? I cannot.

Example Problem 1.5.17. *Suppose you're given perpendicular unit vectors $\vec{A}$ and $\vec{B}$ and suppose $\vec{C} = 3\vec{A} + \alpha\vec{B}$ then α for which $\vec{C}$ is perpendicular to $\vec{A} + \vec{B}$.*

Solution: *we need $\vec{C} \bullet (\vec{A} + \vec{B}) = 0$. Hence consider,*

$$0 = (3\vec{A} + \alpha\vec{B}) \bullet (\vec{A} + \vec{B}) = 3\vec{A} \bullet \vec{A} + (3 + \alpha)\vec{A} \bullet \vec{B} + \alpha\vec{B} \bullet \vec{B} = 3A^2 + \alpha B^2 = 3 + \alpha \quad \Rightarrow \quad \boxed{\alpha = -3}$$

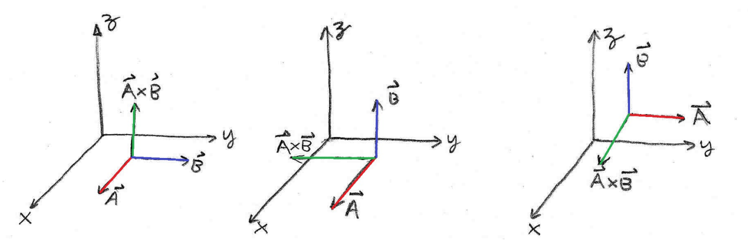
There is more to say about projections. If $\hat{U}$ and $\hat{V}$ are unit vectors which are tangent to a given plane $\mathcal{M}$ then if we envision $\vec{A}$ as a vector based at a point on the plane then we can project $\vec{A}$ onto the plane by the simple formula

$$Proj_{\mathcal{M}}(\vec{A}) = (\vec{A} \bullet \hat{U})\hat{U} + (\vec{A} \bullet \hat{V})\hat{V}.$$

Think about how this formula works for the xy-plane; $\text{Proj}_{xy\text{-plane}}(\langle a, b, c\rangle) = \langle a, b, 0\rangle$.

1.6 The Cross Product

We saw that the dot-product gives us a natural way to check if a pair of vectors is orthogonal. You should remember: $\vec{A}, \vec{B}$ are orthogonal iff $\vec{A} \bullet \vec{B} = 0$. We turn to a slightly different goal in this section: given a pair of nonzero, nonparallel vectors $\vec{A}, \vec{B}$ how can we find another vector $\vec{A} \times \vec{B}$ which is perpendicular to both $\vec{A}$ and $\vec{B}$? Geometrically, in $\mathbb{R}^3$ it's not too hard to picture it:



My intent in this section is to motivate the standard formula for this product and to prove some of the standard properties of this cross product. These calculations are special to $\mathbb{R}^3$. The material from here to Definition 1.6.1 is simply to give some insight into where the mysterious formula for the cross product arises. If you insist on remaining unmotivated, feel free to skip to the definition.

Suppose $\vec{A}, \vec{B}$ are nonzero, nonparallel vectors in $\mathbb{R}^3$. I'll calculate conditions on $\vec{A} \times \vec{B}$ which insure it is perpendicular to both $\vec{A}$ and $\vec{B}$. Let's denote $\vec{A} \times \vec{B} = \vec{C}$. We should expect $\vec{C}$ is some function of the components of $\vec{A}$ and $\vec{B}$. I'll use $\vec{A} = \langle A_1, A_2, A_3 \rangle$ and $\vec{B} = \langle B_1, B_2, B_3 \rangle$ whereas $\vec{C} = \langle C_1, C_2, C_3 \rangle$

$$0 = \vec{C} \bullet \vec{A} = C_1A_1 + C_2A_2 + C_3A_3$$

$$0 = \vec{C} \bullet \vec{B} = C_1B_1 + C_2B_2 + C_3B_3$$

Suppose $A_1 \neq 0$, then we may solve $0 = \vec{C} \bullet \vec{A}$ as follows,

$$C_1 = -\frac{A_2}{A_1}C_2 - \frac{A_3}{A_1}C_3$$

Suppose $B_1 \neq 0$, then we may solve $0 = \vec{C} \bullet \vec{B}$ as follows,

$$C_1 = -\frac{B_2}{B_1}C_2 - \frac{B_3}{B_1}C_3$$

It follows, given the assumptions $A_1 \neq 0$ and $B_1 \neq 0$,

$$\frac{A_2}{A_1}C_2 + \frac{A_3}{A_1}C_3 = \frac{B_2}{B_1}C_2 + \frac{B_3}{B_1}C_3$$

Multiply by A_1B_1 to obtain:

$$B_1A_2C_2 + B_1A_3C_3 = A_1B_2C_2 + A_1B_3C_3$$

Thus,

$$(A_1B_2 - B_1A_2)C_2 + (A_1B_3 - B_1A_3)C_3 = 0$$

One solution is simply $C_2 = A_3B_1 - A_1B_3$ and $C_3 = A_1B_2 - B_1A_2$ and it follows that $C_1 = A_2B_3 - B_2A_3$. Of course, generally we could have vectors which are nonzero and yet have $A_1 = 0$ or $B_1 = 0$. The point of the calculation is not to provide a general derivation. Instead, my intent is simply to show you how you might be led to make the following definition:

Definition 1.6.1. *cross product.*

Let $\vec{A}, \vec{B}$ be vectors in $\mathbb{R}^3$. The vector $\vec{A} \times \vec{B}$ is called the **cross product** of $\vec{A}$ with $\vec{B}$ and is defined by

$$\vec{A} \times \vec{B} = \langle\, A_2B_3 - A_3B_2,\; A_3B_1 - A_1B_3,\; A_1B_2 - A_2B_1 \,\rangle.$$

We say $\vec{A}$ cross $\vec{B}$ is $\vec{A} \times \vec{B}$.

It is a simple exercise to verify that

$$\vec{A} \bullet (\vec{A} \times \vec{B}) = 0 \qquad \text{and} \qquad \vec{B} \bullet (\vec{A} \times \vec{B}) = 0.$$

Both of these identities should be utilized to check your calculation of a given cross product. Let's think about the formula for the cross product a bit more. We have

$$\vec{A} \times \vec{B} = (A_2B_3 - A_3B_2)\hat{\mathbf{x}}_1 + (A_3B_1 - A_1B_3)\hat{\mathbf{x}}_2 + (A_1B_2 - A_2B_1)\hat{\mathbf{x}}_3$$

distributing,

$$\vec{A} \times \vec{B} = A_2B_3\hat{\mathbf{x}}_1 - A_3B_2\hat{\mathbf{x}}_1 + A_3B_1\hat{\mathbf{x}}_2 - A_1B_3\hat{\mathbf{x}}_2 + A_1B_2\hat{\mathbf{x}}_3 - A_2B_1\hat{\mathbf{x}}_3$$

The pattern is clear. Each term has indices $1, 2, 3$ without repeat and we can generate the signs via the antisymmetric symbol ϵ_{ijk} which is defined be zero if any indices are repeated and

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \qquad \text{whereas} \qquad \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1.$$

With this convenient shorthand we find the nice formula for the cross product that follows:

$$\boxed{\vec{A} \times \vec{B} = \sum_{i,j,k=1}^{3} A_iB_j\epsilon_{ijk}\hat{\mathbf{x}}_k}$$

Interestingly the Cartesian unit-vectors $\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3$ satisfy the simple relation:

$$\boxed{\hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j = \sum_{k=1}^{3} \epsilon_{ijk}\hat{\mathbf{x}}_k,}$$

which is just a fancy way of saying that

$$\boxed{\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}}$$

There are many popular mnemonics to remember these. The basic properties of the cross product together with these formula allow us to quickly calculate some cross products (see Example 1.6.7)

Proposition 1.6.2. *basic properties of the cross product.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$

(1.) anticommutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$,

(2.) distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$,

(3.) distributive: $(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$,

(4.) scalars factor out: $\vec{A} \times (c\vec{B}) = (c\vec{A}) \times \vec{B} = c\vec{A} \times \vec{B}$, Remark: I left these proofs here to help you understand why I care about the funny ϵ_{ijk} notation. I omitted the more sophisticated proofs later in this section for the sake of brevity. You can look at my Calculus III notes for all the missing details if you're curious.

Proof: once more, the proof is easy with the right notation. Begin with (1.),

$$\vec{A}\times\vec{B} = \sum_{i,j,k=1}^{3} A_iB_j\epsilon_{ijk}\hat{\mathbf{x}}_k = -\sum_{i,j,k=1}^{3} A_iB_j\epsilon_{jik}\hat{\mathbf{x}}_k = -\sum_{i,j,k=1}^{3} B_jA_i\epsilon_{jik}\hat{\mathbf{x}}_k = -\vec{B}\times\vec{A}.$$

The key observation was that $\epsilon_{ijk} = -\epsilon_{jik}$ for all i,j,k. If you don't care for this argument then you could also give the brute-force argument below:

$$\begin{aligned}
\vec{A}\times\vec{B} &= \langle\, A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1\,\rangle \\
&= -\langle\, A_3B_2 - A_2B_3,\ A_1B_3 - A_3B_1,\ A_2B_1 - A_1B_2\,\rangle \\
&= -\langle\, B_2A_3 - B_3A_2,\ B_3A_1 - B_1A_3,\ B_1A_2 - B_2A_1\,\rangle \\
&= -\vec{B}\times\vec{A}.
\end{aligned}$$

Next, to prove (2.) we once more use the compact notation,

$$\begin{aligned}
\vec{A}\times(\vec{B}+\vec{C}) &= \sum_{i,j,k=1}^{3} A_i(B_j + C_j)\epsilon_{ijk}\hat{\mathbf{x}}_k \\
&= \sum_{i,j,k=1}^{3} (A_iB_j\epsilon_{ijk}\hat{\mathbf{x}}_k + A_iC_j\epsilon_{ijk}\hat{\mathbf{x}}_k) \\
&= \sum_{i,j,k=1}^{3} A_iB_j\epsilon_{ijk}\hat{\mathbf{x}}_k + \sum_{i,j,k=1}^{3} A_iC_j\epsilon_{ijk}\hat{\mathbf{x}}_k \\
&= \vec{A}\times\vec{B} + \vec{A}\times\vec{C}.
\end{aligned}$$

The proof of (3.) follows naturally from (1.) and (2.), note:

$$(\vec{A}+\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}+\vec{B}) = -\vec{C}\times\vec{A} - \vec{C}\times\vec{B} = \vec{A}\times\vec{C} + \vec{B}\times\vec{C}.$$

I leave the proof of (4.) to the reader. $\square$

The properties above basically say that the cross product behaves the same as the usual addition and multiplication of numbers with the caveat that the order of factors matters. If we switch the order then we must include a minus due to the anticommutivity of the cross product.

Example 1.6.3. *Consider,* $\vec{A}\times\vec{A} = -\vec{A}\times\vec{A}$ *hence* $2\vec{A}\times\vec{A} = 0$. *Consequently,* $\vec{A}\times\vec{A} = 0$.

We often use the result of the example above in future work. For example:

Example 1.6.4. *Let* $\vec{A},\vec{B}$ *be two three dimensional vectors. Simplify* $(\vec{A}-\vec{B})\times(\vec{A}+\vec{B})$.

$$\begin{aligned}
(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) &= \vec{A}\times(\vec{A}+\vec{B}) - \vec{B}\times(\vec{A}+\vec{B}) \\
&= \vec{A}\times\vec{A} + \vec{A}\times\vec{B} - \vec{B}\times\vec{A} - \vec{B}\times\vec{B} \\
&= 2\vec{A}\times\vec{B}.
\end{aligned}$$

There are a number of popular tricks to remember the rule for the cross-product. Let's look at a particular example a couple different ways:

Example 1.6.5. *Let $\vec{A} = \langle 1,2,3\rangle$ and $\vec{B} = \langle 4,5,6\rangle$. Calculate $\vec{A}\times\vec{B}$ directly from the definition:*

$$\begin{aligned}\vec{A}\times\vec{B} &= \langle 1,2,3\rangle \times \langle 4,5,6\rangle \\ &= \langle 2(6)-3(5),\ 3(4)-1(6),\ 1(5)-2(4)\rangle \\ &= \langle -3,\ 6,\ -3\rangle.\end{aligned}$$

There are at least 6 opportunities to make an error in the calculation of a cross product. It is important to check our work before we continue. A simple check is that $\vec{A}$ and $\vec{B}$ must be orthogonal to the cross product. We can easily calculate that $\vec{A}\bullet(\vec{A}\times\vec{B}) = 0$ and $\vec{B}\bullet(\vec{A}\times\vec{B}) = 0$. This almost guarantees we have correctly calculated the cross product.

The other popular method to calculate the cross product is based on an abuse of notation with the **determinant**. A determinant can be calculated for any $n\times n$ matrix A. The significance of the determinant is that it gives the signed-volume of the n-piped with edges taken as the rows or columns of A. A simple formula for the determinant in general is given by:

$$\det(A) = \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n}$$

Ok, I jest. This formula takes a bit of work to really appreciate. So, typically we introduce the determinant in terms of the **expansion by minors** due to Laplace. We begin with a 2×2 matrix:

$$\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad - bc.$$

Next, a 3×3 can be calculated by an expansion across the top-row,

$$\begin{aligned}\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i\end{bmatrix} &= a\cdot\det\begin{bmatrix} e & f \\ h & i\end{bmatrix} - b\cdot\det\begin{bmatrix} d & f \\ g & i\end{bmatrix} + c\cdot\det\begin{bmatrix} d & e \\ g & h\end{bmatrix} \\ &= a(ei-fh) - b(di-fg) + c(dh-eg).\end{aligned}$$

The minus sign in the middle term is part of the structure of the expansion. It is also one of the most common places where students make an error in their computation of a determinant [10]. We can express the cross product by following the patterns introduced for the 3×3 case. In particular,

$$\begin{aligned}\langle A_1,A_2,A_3\rangle \times \langle B_1,B_2,B_3\rangle &= \det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3\end{bmatrix} \\ &= \hat{\mathbf{x}}(A_2B_3 - A_3B_2) - \hat{\mathbf{y}}(A_1B_3 - A_3B_1) + \hat{\mathbf{z}}(A_1B_2 - A_2B_1) \\ &= (A_2B_3 - A_3B_2)\hat{\mathbf{x}} + (A_3B_1 - A_1B_3)\hat{\mathbf{y}} + (A_1B_2 - A_2B_1)\hat{\mathbf{z}}.\end{aligned}$$

I invite the reader to verify this aligns perfectly with Definition 1.6.1.

[10] If we go on, a 4×4 matrix breaks into a signed-weighted-sum of 4 determinants of 3×3 submatrices. More generally, an $n\times n$ martrix has a determinant which requires on the order of $n!$ arithmetic steps. You'll learn more in your linear algebra course, I merely initiate the discussion here. Fortunately, we only need $n=2$ and $n=3$ for the majority of the topics in this course.

Example 1.6.6. *Let $\vec{A} = \langle 1,2,3\rangle$ and $\vec{B} = \langle 4,5,6\rangle$. Calculate $\vec{A}\times\vec{B}$ via the determinant formula:*

$$\begin{aligned}\langle 1,2,3\rangle \times \langle 4,5,6\rangle &= det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\ &= \hat{\mathbf{x}}(2(6)-3(5)) - \hat{\mathbf{y}}(1(6)-3(4)) + \hat{\mathbf{z}}(1(5)-2(4)) \\ &= -3\hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 3\hat{\mathbf{z}}.\end{aligned}$$

This result matches $\vec{A}\times\vec{B} = \langle -3,6,-3\rangle$ as we found in Example 1.6.5.

Technically, this formula is not really a determinant since genuine determinants are formed from matrices filled with objects of the same type. In the hybrid expression above we actually have one row of vectors and two rows of scalars. That said, I include it here since many people use it and I also have found it useful in past calculations. If nothing else at least it helps you learn what a determinant is. That is a calculation which is worthwhile since determinants have application far beyond mere cross products. We can also use the basic relations:

$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$

and the properties of cross products to work out cross products algebraically:

Example 1.6.7. *Let $\vec{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$ and $\vec{B} = 4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}$. Calculate $\vec{A}\times\vec{B}$ as follows:*

$$\begin{aligned}\vec{A}\times\vec{B} &= \hat{\mathbf{x}}\times(4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}) + 2\hat{\mathbf{y}}\times(4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}) + 3\hat{\mathbf{z}}\times(4\hat{\mathbf{x}} + 5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}) \\ &= \hat{\mathbf{x}}\times(5\hat{\mathbf{y}} + 6\hat{\mathbf{z}}) + 2\hat{\mathbf{y}}\times(4\hat{\mathbf{x}} + 6\hat{\mathbf{z}}) + 3\hat{\mathbf{z}}\times(4\hat{\mathbf{x}} + 5\hat{\mathbf{y}}) \\ &= 5\hat{\mathbf{x}}\times\hat{\mathbf{y}} + 6\hat{\mathbf{x}}\times\hat{\mathbf{z}} + 8\hat{\mathbf{y}}\times\hat{\mathbf{x}} + 12\hat{\mathbf{y}}\times\hat{\mathbf{z}} + 12\hat{\mathbf{z}}\times\hat{\mathbf{x}} + 15\hat{\mathbf{z}}\times\hat{\mathbf{y}} \\ &= 5\hat{\mathbf{z}} + 6(-\hat{\mathbf{y}}) + 8(-\hat{\mathbf{z}}) + 12\hat{\mathbf{x}} + 12\hat{\mathbf{y}} + 15(-\hat{\mathbf{x}}) \\ &= -3\hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 3\hat{\mathbf{z}}.\end{aligned}$$

This agrees with the conclusion of the previous pair of examples.

The calculation above is probably not the quickest for the example at hand here, but it is faster for other computations. For example:

Example 1.6.8. *Suppose $\vec{A} = \langle 1,2,3\rangle$ and $\vec{B} = \hat{\mathbf{x}}$ then*

$$\begin{aligned}\vec{A}\times\vec{B} &= (\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}})\times\hat{\mathbf{x}} \\ &= 2\hat{\mathbf{y}}\times\hat{\mathbf{x}} + 3\hat{\mathbf{z}}\times\hat{\mathbf{x}} \\ &= -2\hat{\mathbf{z}} + 3\hat{\mathbf{y}}.\end{aligned}$$

Example 1.6.9. *Let $\vec{A} = \langle 3,2,4\rangle$ and $\vec{B} = \langle 1,-2,-3\rangle$. We calculate,*

$$\begin{aligned}\vec{A}\times\vec{B} &= det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{bmatrix} \\ &= \hat{\mathbf{x}}(-6+8) - \hat{\mathbf{y}}(-9-4) + \hat{\mathbf{z}}(-6-2) \\ &= 2\hat{\mathbf{x}} + 13\hat{\mathbf{y}} - 8\hat{\mathbf{z}}.\end{aligned}$$

As a check on our computation, note that $\vec{A}\bullet(\vec{A}\times\vec{B}) = 0$ and $\vec{B}\bullet(\vec{A}\times\vec{B}) = 0$.

There are a number of identities which connect the dot and cross products. These formulas require considerable effort if you choose to use brute-force proof methods.

Proposition 1.6.10. *nontrivial properties of the cross product.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$

(1.) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$

(2.) Jacobi Identity: $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0,$

(3.) cyclicity of triple product: $\vec{A} \bullet (\vec{B} \times \vec{C}) = \vec{B} \bullet (\vec{C} \times \vec{A}) = \vec{C} \bullet (\vec{A} \times \vec{B})$

(4.) Lagrange's identity: $||\vec{A} \times \vec{B}||^2 = ||\vec{A}||^2 \, ||\vec{B}||^2 - [\ \vec{A} \bullet \vec{B}\ ]^2$

Use Lagrange's identity together with $\vec{A} \bullet \vec{B} = AB\cos(\theta)$,

$$||\vec{A} \times \vec{B}||^2 = A^2B^2 - [AB\cos(\theta)]^2 = A^2B^2(1 - \cos^2(\theta)) = A^2B^2\sin^2(\theta)$$

It follows there exists some unit-vector $\hat{\mathbf{n}}$ such that

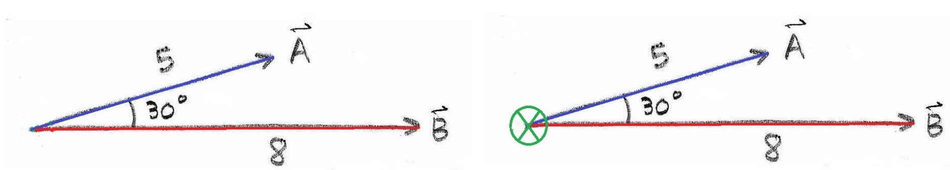
$$\boxed{\vec{A} \times \vec{B} = AB\sin(\theta)\hat{\mathbf{n}}}$$

The direction of the unit-vector $\hat{\mathbf{n}}$ is conveniently indicated by the **right-hand-rule**. I typically perform the rule as follows:

1. point fingers of **right hand** in direction $\vec{A}$
2. cross the fingers into the direction of $\vec{B}$
3. the direction your thumb points is the approximate direction of $\hat{\mathbf{n}}$

I say *approximate* because $\vec{A} \times \vec{B}$ is strictly perpendicular to both $\vec{A}$ and $\vec{B}$ whereas your thumb's direction is a little ambiguous. But, it does pick one side of the plane in which the vectors $\vec{A}$ and $\vec{B}$ reside.

Example 1.6.11. . *Consider $\vec{A}$ and $\vec{B}$ pictured below. Find the magnitude of $\vec{A} \times \vec{B}$ and describe its direction. We produce the right picture by the right hand rule:*



Note $||\vec{A} \times \vec{B}|| = AB\sin\theta = 40\sin 30^o = 20$. By the right hand rule, we find the direction of $\vec{A} \times \vec{B}$ is into the page. The $\otimes$ symbol intends we visualize the vector as an arrow pointing into the page.

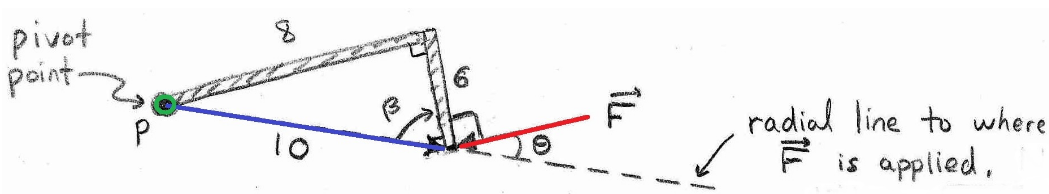
Example 1.6.12. *Let $\vec{u}$ and $\vec{v}$ be as pictured below with $u = 5$ and $v = 4\sqrt{3}$. Find the magnitude and direction vector of $\vec{v} \times \vec{u}$: we use the right hand rule to produce the diagram on the right:*

Note $||\vec{v} \times \vec{u}|| = vu \sin\theta = 20\sqrt{3}\sin 60^o = 30$. *By the right hand rule, we find the direction of* $\vec{v} \times \vec{u}$ *is out of the page. The* $\odot$ *symbol indicates a vector pointing out of the page.*

We will study torque towards the end of this course, and the Lorentz force is properly included in Physics 232. That said, I include these examples here without their full context.

Example 1.6.13. *In rotational physics the direction of a rotation is taken to be the axis of the rotation where a counter-clockwise-rotation (CCW) is taken to be positive. To decide which direction is CCW we grip the rotation axis and point our right-hand's thumb in the direction of the positive axis. Once that grip is made the fingers on the right hand encircle the axis in the CCW-rotational sense. A torque on a body allowed to rotate around some axis makes it rotate. In particular, if* $\vec{r}$ *is the* **moment arm** *and* $\vec{F}$ *is the force applied then* $\vec{\tau} = \vec{r} \times \vec{F}$ *is the torque produced by* $\vec{F}$ *relative to the given axis.*

Problem: *Find the torque due to the force* $\vec{F}$ *pictured below. Describe the rotation produced as CCW or CW given the axis of rotation points out of the page*



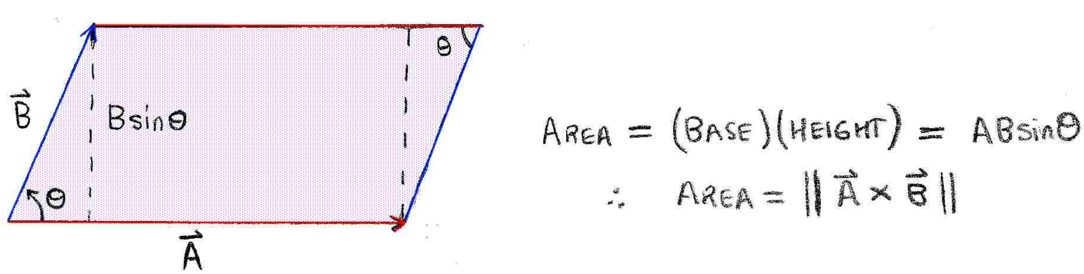
Solution: *Imagine moving* $\vec{F}$ *to* P *while maintaining its direction. This is called* **parallel transport***. We calculate* $\vec{r} \times \vec{F}$ *as if they are both attached to* P*. The right hand rule reveals the direction is into the page (*$\otimes$*) and we can determine* θ *from trigonometry and the given geometric data. Observe* θ *is also interior to the triangle at* P *hence* $\sin\theta = \frac{6}{10}$ *. Also, by pythagorean theorem,* $r = \sqrt{8^2 + 6^2} = 10$. *Therefore,* $\tau = rF \sin\theta = 6F$. *The direction of the torque is* $\otimes$ *which indicates a CW-rotation relative to the outward pointing axis through* P.

Example 1.6.14. *Another important application of the cross product to physics is the Lorentz force law. If a charge* q *has velocity* $\vec{v}$ *and travels through a magnetic field* $\vec{B}$ *then the force due to the electromagnetic interaction between* q *and the field is* $\vec{F} = q\vec{v} \times \vec{B}$.

Finally, we should investigate how the dot and cross product give nice formulas for the area of a parallelogram or the volume of a parallel piped. Suppose $\vec{A}, \vec{B}$ give the sides of a parallelogram.

$$Area = ||\ \vec{A} \times \vec{B}\ ||$$

The picture below shows why the formula above is true:



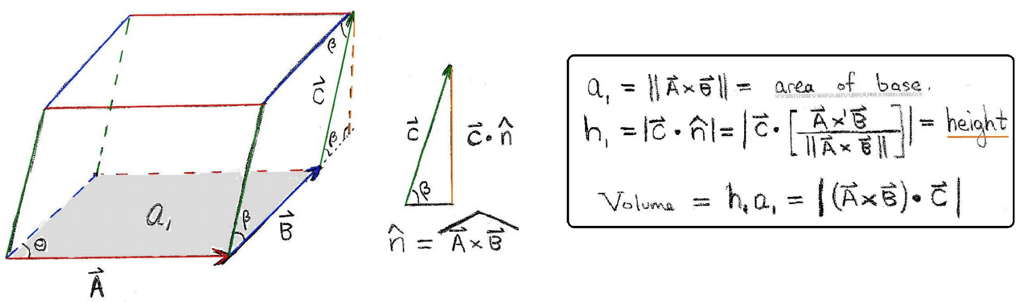
On the other hand, if $\vec{A}, \vec{B}, \vec{C}$ give the corner-edges of a parallelogram then[11]

$$Volume = \mid \vec{A} \bullet (\vec{B} \times \vec{C}) \mid$$

These formulas are connected by the following thought: the volume subtended by $\vec{A}, \vec{B}$ and the unit-vector $\hat{\mathbf{n}}$ from $\vec{A} \times \vec{B} = AB\sin(\theta)\hat{\mathbf{n}}$ is equal to the area of the parallelogram with sides $\vec{A}, \vec{B}$. Algebraically:

$$\Big| \ \hat{\mathbf{n}} \bullet (\vec{A} \times \vec{B}) \ \Big| = \Big| \ \hat{\mathbf{n}} \bullet (AB\sin(\theta)\hat{\mathbf{n}}) \Big| = |AB\sin(\theta)| = ||\vec{A} \times \vec{B}||.$$

The picture below shows why the triple product formula is valid.



Example Problem 1.6.15. *Find the volume of a parallel-piped with edge-vectors* $\vec{A} = \langle 0, 1, 1 \rangle$ *and* $\vec{B} = \langle 1, 0, 0 \rangle$ *and* $\vec{C} = \langle 0, 1, 0 \rangle$.

Solution: *We calculate* $\vec{B} \times \vec{C} = \hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$. *Therefore, the volume of the solid is* $V = \vec{A} \bullet (\vec{B} \times \vec{C}) = \langle 0, 1, 1 \rangle \bullet \hat{\mathbf{z}} = 1$.

Moreover, given this geometric interpretation we find a new proof (up to a sign) for the cyclic property. By the symmetry of the edges it follows that $\mid \vec{A} \bullet (\vec{B} \times \vec{C}) \mid = \mid \vec{B} \bullet (\vec{C} \times \vec{A}) \mid = \mid \vec{C} \bullet (\vec{A} \times \vec{B}) \mid$. We should find the same volume no matter how we label width, depth and height.

Example 1.6.16. *Suppose* $\vec{U} \neq 0$ *and* $\vec{A} \times \vec{U} = \vec{B} \times \vec{U}$ *and* $\vec{A} \bullet \vec{U} = \vec{B} \bullet \vec{U}$. *Choose coordinates for which* $\vec{U}$ *points in the* $\hat{\mathbf{x}}$ *direction and denote* $\vec{A} = \langle A_x, A_y, A_z \rangle$ *and* $\vec{B} = \langle B_x, B_y, B_z \rangle$. *Note,*

$$\vec{A} \bullet \hat{\mathbf{x}} = \vec{B} \bullet \hat{\mathbf{x}} \quad \Rightarrow \quad A_x = B_x.$$

and

$$(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times \hat{\mathbf{x}} = (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \times \hat{\mathbf{x}} \ \Rightarrow \ -A_y\hat{\mathbf{z}} + A_z\hat{\mathbf{y}} = -B_y\hat{\mathbf{z}} + B_z\hat{\mathbf{y}} \ \Rightarrow \ A_y = B_y, \ A_z = B_z.$$

Thus $\vec{A} = \vec{B}$. *If we only had equality* $\vec{A} \times \vec{U} = \vec{B} \times \vec{U}$ *or* $\vec{A} \bullet \vec{U} = \vec{B} \bullet \vec{U}$ *then we could not be sure of the equality of* $\vec{A}$ *and* $\vec{B}$.

Intuitively, the cross-product with $\vec{U}$ gives data about components which are perpendicular to $\vec{U}$ whereas the dot-product gives data about the component in the $\vec{U}$-direction. We will see that both the dot and the cross product play an essential role in describing the laws of physics.

[11] we could also show that $\det[\vec{A}|\vec{B}|\vec{C}] = \vec{A} \bullet (\vec{B} \times \vec{C})$ thus the determinant of the three edge vectors of a parallel piped yields its signed-volume. We can define the sign of the volume to be positive if the edges are ordered to respect the right hand rule. Respecting the right hand rule means the angle between $\vec{A} \times \vec{B}$ and $\vec{C}$ is less than 90^o.

Chapter 2

Motion

2.1 position and displacement

Given an object we can consider it as a point mass at its center of mass. I won't belabor this point, but this is our default understanding of the position of an object.

Definition 2.1.1. *The* **position** *of an object at time t is denoted $\vec{r}(t)$. If the object is found in the xy-plane then $\vec{r} = \langle x, y \rangle$. If the object is found in three dimensional space then $\vec{r} = \langle x, y, z \rangle$. If the object undergoes one dimensional motion then the position may be denoted by a single variable like x. The* **displacement** *of an object over the* **duration** $\triangle t = t_2 - t_1$ *from time $t = t_1$ to time $t = t_2$ is defined to be $\triangle \vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$, or in one-dimensional motion in the x-direction $\triangle x = x(t_2) - x(t_1)$. An* **event** *is a time and a position. Given two events we naturally define the displacement between the events to be the difference of their position vectors.*

Measurement of position implicitly assumes a choice of reference frame. It is understood that we use a fixed frame of reference for any given problem. We often choose the location of the origin for the coordinate system as that may simplify our calculations. We defer the larger discussion of how to understand coordinate change to a later section.

Example 2.1.2. *Consider Minato has position given by $x(t) = 0$ for $0 \leq t \leq 10\,s$ and $x(t) = 100\,m$ for $10\,s < t \leq 20\,s$ and $x(t) = 200\,m$ for $t > 20\,s$. The motion of Minato is physically unreasonable. Notice $\triangle x = x(20.01\,s) - x(20\,s) = 200m - 100m = 100m$ over a duration $\triangle t = 0.01\,s$.*

Example 2.1.3. *Suppose $\vec{r}(t) = \langle (10\,m/s)t, 10\ m \rangle$ be the position of a squirrel running along crest of a rooftop. When $t = 0$ the squirrel is at $\vec{r}(0) = \langle 0, 10\,m \rangle$. When $t = 2\,s$ then the squirrel is at $\vec{r}(2s) = \langle 20m, 10m \rangle$. We find $\triangle \vec{r} = \langle 20m, 10m \rangle - \langle 0, 10m \rangle = \langle 10m, 0 \rangle$ over the duration $\triangle t = 2\,s$.*

Example Problem 2.1.4. *A sailboat begins at Island of Cats then makes a displacement of 20 miles at 20^o South of East. Then the boat changes course to travel 30 miles at 30^o South of West. Finally, the boat travels 10 miles due North where it reaches to Island of Dogs. In what direction should the boat set sail in order to return to dreadful Island of Cats ? How far is the Island of Dogs from the Island of Cats ? Please express the direction in terms of the standard angle.*

Solution: *we calculate the net-displacement by writing each displacement in vector form. Notice 20^o South of East means $\theta_1 = -20^o$ hence*

$$\triangle \vec{r}_1 = (20mi)\langle \cos(-20^o), \sin(-20^o) \rangle = \langle 18.79\,mi, -6.84\,mi \rangle.$$

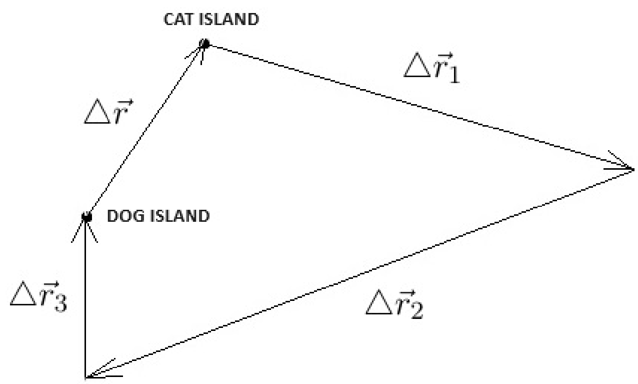
Next, we have a displacement at $\theta_2 = 210^o$ *thus*

$$\triangle\vec{r}_2 = (30mi)\langle\cos(210^o), \sin(210^o)\rangle = \langle -25.98\,mi, -15\,mi\rangle.$$

Finally, $\triangle\vec{r}_3 = \langle 0, 10\,mi\rangle$*. If* $\triangle\vec{r}$ *denotes the displacement necessary for the boat to return to the Island of Cats then we need:* $\triangle\vec{r}_1 + \triangle\vec{r}_2 + \triangle\vec{r}_3 + \triangle\vec{r} = 0$*. Therefore,*

$$\begin{aligned}\triangle\vec{r} &= -\langle 18.79\,mi, -6.84\,mi\rangle - \langle -25.98\,mi, -15\,mi\rangle - \langle 0, 10\,mi\rangle \\ &= \langle -18.79 + 25.98, 6.84 + 15 - 10\rangle\,mi \\ &= \langle 7.19, 11.84\rangle\,mi.\end{aligned}$$

Calculate $\|\triangle\vec{r}\| = \sqrt{7.19^2 + 11.84^2}\,mi = 13.85\,mi$ *and* $\tan^{-1}(11.84/7.19) = 58.73^o$*. We find the boat should set sail in the direction with standard angle* 58.73^o *and the distance between the Islands is* 13.85 *miles.*



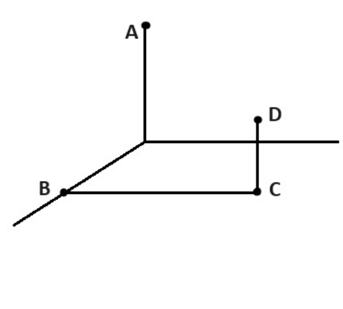
Example Problem 2.1.5. *Consider a room where a spy mouse enters the room through a small hole in the corner. Then the mouse sneaks along the wall a distance of* 3.2 *m to point B as pictured. Then the mouse darts* 2.5 *m over to lamp whose base is at point C where he then climbs* 35 *cm up the lamp. His mission is to shine a laser in the corner at A which is* 4.0 *m above the floor. What angle of inclination should the mouse set on his laser ?*

Solution: *let us use coordinates which base the origin at the corner where the mouse entered. We have* $A = (0, 0, 4)m$ *and* $B = (3.2, 0, 0)m$ *and* $C = (3.2, 2.5, 0)m$ *and* $D = (3, 2, 2.5, 0.35)m$ *where I have observed that* $35cm = 35cm\left(\frac{1m}{100cm}\right) = 0.35m$*. The displacement of the laser beam is*

$$\triangle\vec{r} = A - D = \langle 3.2, 2.5, 3.65\rangle m$$

The angle of inclination is complementary to the angle which the displacement of the laser makes with the $\hat{\mathbf{z}}$*-direction. Calculate* $\|\triangle\vec{r}\| = \sqrt{3.2^2 + 2.5^2 + 3.65^2}\,m = 5.46\,m$ *and*

$$\triangle\vec{r} \bullet \hat{\mathbf{z}} = (\langle 3.2, 2.5, 3.65\rangle m) \bullet \langle 0, 0, 1\rangle = 3.65\,m$$



Thus $90 - \alpha = \cos^{-1}\left(\frac{3.65\,m}{5.46\,m}\right) = 48.05^o$ *hence* $\alpha = 41.95^o$.

2.2 velocity, speed and distance traveled

Average velocity can be defined without calculus.

Definition 2.2.1. *The average velocity of an object with position $\vec{r}$ over the time interval $[t_1, t_2]$ is given by the ratio of the displacement by the duration:*

$$\vec{v}_{avg} = \frac{\triangle \vec{r}}{\triangle t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}.$$

In the one-dimensional context, $v_{avg} = \frac{\triangle x}{\triangle t} = \frac{x(t_2)-x(t_1)}{t_2-t_1}$.

Example 2.2.2. *Suppose I commute to work 40 miles from my house and it takes me 33 minutes to get home from work. My average velocity is $v_{avg} = \frac{40\ mi}{33\ min}$. To convert to miles per hour we multiply by an appropriate unit-conversion factor. Since $60\ min = 1\ hr$ we have $1 = \frac{60\ min}{1\ hr}$ thus*

$$v_{avg} = \frac{40\ mi}{33\ min}\frac{60\ min}{1\ hr} = 72.72\frac{mi}{hr}$$

or in the common notation, $v_{avg} = 72.72\,mph$.

Example Problem 2.2.3. *If the average velocity $\vec{v}_{avg} = \langle 2, -3, 10\rangle m/s$ over a duration $\triangle t = 2.0\,s$. If the object is initially at position $\langle 1, 2, 3\rangle m$ then find the final position.*

Solution: *we know $\langle 2, -3, 10\rangle m/s = \dfrac{\vec{r}_2 - \vec{r}_1}{2.0\,s}$ where $\vec{r}_1 = \langle 1, 2, 3\rangle m$ and $\vec{r}_2$ we wish to determine. Multiplying by $2.0\,s$ we find*

$$\langle 4, -6, 20\rangle m = \vec{r}_2 - \vec{r}_1 \quad \Rightarrow \quad \vec{r}_2 = \langle 4, -6, 20\rangle m + \langle 1, 2, 3\rangle m \quad \Rightarrow \quad \boxed{\vec{r}_2 = \langle 5, -4, 23\rangle m}.$$

Definition 2.2.4. *The* **velocity** *of an object with position $\vec{r}$ at time t is defined by*

$$\vec{v} = \lim_{\triangle t \to 0} \frac{\triangle \vec{r}}{\triangle t} = \lim_{\triangle t \to 0} \frac{\vec{r}(t + \triangle t) - \vec{r}(t)}{\triangle t} = \frac{d\vec{r}}{dt}.$$

Notice, if $\vec{r} = \langle x, y, z\rangle$ then $\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$. If the motion is one-dimensional with position x then the velocity in the x-direction is given by $v = \frac{dx}{dt}$.

The velocity is a vector. However, to be clear, in one dimension a vector is represented by a number whose sign indicated the direction. Velocity is the so-called *instantaneous velocity*. You can think of it as the average velocity over an exceedingly small duration. How small ? Thankfully we don't have to answer that since the concept of a limit makes the process of examining arbitrarily small increments of time a mathematically rigorous process. I leave the proof and further discussion of the existence of derivatives to your Calculus course work. Here we assume you know Calculus and we simply do the math.

Example 2.2.5. *Suppose a ninja is at $x = a + bt^3 + c\cos(\alpha t) + \gamma \sinh(\beta t)$ where $a, b, c, \alpha, \beta, \gamma$ are constants then the ninja has velocity is $v = \frac{dx}{dt} = 3bt^2 - c\alpha \sin(\alpha t) + \gamma\beta \cosh(\beta t)$.*

Example 2.2.6. *If the position of an evil cat is given by $\vec{r} = \vec{c} + \langle R\cos\omega t, R\sin\omega t\rangle$ where $\vec{c}$ and ω, R are contants. Then the velocity of the cat is given by*

$$\vec{v} = \frac{d\vec{r}}{dt} = \left\langle \frac{d}{dt}\, R\cos\omega t, \frac{d}{dt}\, R\sin\omega t \right\rangle = \langle -R\omega \sin\omega t, R\omega \cos\omega t\rangle.$$

Example Problem 2.2.7. *Suppose the velocity of a flying donkey is given by* $\vec{v} = \langle 2m/s, 3m/s, 4m/s - (9.8m/s^2)t \rangle$ *and suppose the donkey is initiall observed to have position* $\vec{r}(0) = \langle 0, 1, 3 \rangle m$*. Find the position of the donkey as a function of time* t*.*

Solution: *since* $\vec{v} = \frac{d\vec{r}}{dt}$ *we see* $\vec{r} = \int \vec{v} dt$*. We calculate, set* $b = m/s$ *for convenience and* $g = 9.8m/s^2$*,*

$$\vec{r}(t) = \left\langle \int 2b dt, \int 3b dt, \int (4b - gt) dt \right\rangle$$
$$= \left\langle 2bt + c_1, 3bt + c_2, 4bt - \frac{1}{2} g t^2 + c_3 \right\rangle.$$

Then $\vec{r}(0) = \langle c_1, c_2, c_3 \rangle = \langle 0, 1, 3 \rangle m$*. Thus,*

$$\boxed{\vec{r}(t) = \left\langle 2bt, 3bt + 1m, 4bt - \frac{1}{2} g t^2 + 3m \right\rangle}.$$

Example 2.2.8. *Suppose* $\vec{C}, \vec{B}$ *are constant vectors* α *is a constant. If a friendly righteous dog has position* $\vec{r} = e^{\alpha t}\vec{C} + \cos^2(\alpha t)\vec{B}$ *then the dog has velocity*

$$\vec{v} = \frac{d}{dt}[e^{\alpha t}]\vec{C} + \frac{d}{dt}[\cos^2(\alpha t)]\vec{B} = \alpha e^{\alpha t}\vec{C} - 2\alpha \sin(\alpha t)\cos(\alpha t)\vec{B}.$$

Definition 2.2.9. *The* **speed** *of an object is the magnitude of its velocity.*

In one-dimension, there is no special symbol for speed. For two or three dimensions, if the velocity is $\vec{v}$ then the speed is denoted v. In two-dimensions or three-dimensions,

$$v = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} \qquad \text{or} \qquad v = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}$$

To calculate the distance traveled we integrate the speed:

Definition 2.2.10. *If an object travels a path with position* $\vec{r} = \langle x, y, z \rangle$ *for* $t_1 \leq t$ *then the distance traveled from time* $t = t_1$ *to time* t *is given by:*

$$s = \int_{t_1}^{t} \sqrt{\frac{dx}{d\tau}^2 + \frac{dy}{d\tau}^2 + \frac{dz}{d\tau}^2}\, d\tau.$$

We've used τ as an integration variable to avoid confusion with t. Apply FTC I from Calculus,

$$\frac{ds}{dt} = \frac{d}{dt}\int_{t_1}^{t} \sqrt{\frac{dx}{d\tau}^2 + \frac{dy}{d\tau}^2 + \frac{dz}{d\tau}^2}\, d\tau = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2} = v.$$

Therefore, the speed is given by $\frac{ds}{dt}$. The speed is rate at which distance is traveled for a given object. Speed is necessarily non-negative.

Example 2.2.11. *Supppose* R, m, ω *are positive constants. Let* $\vec{r} = \langle R\cos\omega t, R\sin\omega t, mt \rangle$ *for* $t \geq 0$ *denote the position of an object travelling in a spiral. The velocity is given by*

$$\vec{v} = \langle -R\omega \sin\omega t, R\omega\cos\omega t, m \rangle$$

Calculate $\vec{v} \bullet \vec{v} = R^2\omega^2 + m^2$*. We find speed* $v = \sqrt{R^2\omega^2 + m^2}$ *and the distance traveled is given by* $s = \int_0^t \sqrt{R^2\omega^2 + m^2}\, d\tau = t\sqrt{R^2\omega^2 + m^2}$*.*

Example 2.2.12. *If $x = \alpha t$ and $y = \beta t^2$ for $t \geq 0$ then $\vec{v} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \langle \alpha, 2\beta t \rangle$ is the velocity at time t. The speed*

$$v = \sqrt{\alpha^2 + 4\beta^2 t^2}$$

The distance traveled up to time t requires we calculate a somewhat challenging integral:

$$s = \int_0^t \sqrt{\alpha^2 + 4\beta^2 \tau^2}\, d\tau$$

We will content ourselves with an answer in **integral form**[1].

Example Problem 2.2.13. *Suppose the position of a cat attached to a spring undergoing simple harmonic motion with amplitude $A > 0$ and angular frequency $\omega > 0$ is given by $x(t) = A\sin(\omega t)$ for $0 \leq t \leq 2T$ where $T = 2\pi/\omega$. Find the distance traveled by the cat under the motion.*

Solution: *calculate the velocity $v = \frac{dx}{dt} = A\omega \cos \omega t$. In this context, the speed is given by the absolute value of v. The distance traveled is:*

$$s = \int_0^{2T} |A\omega \cos \omega t| dt$$

Notice $0 \leq t \leq 2T = 4\pi/\omega$ gives $0 \leq \omega t \leq 4\pi$ so if we let $u = \omega t$ then as $du = \omega dt$ we calculate[2].

$$s = A \int_0^{4\pi} |\cos(u)| du = 8A \int_0^{\pi/2} \cos(u) du = 8A \sin(u)\Big|_0^{\pi/2} = 8A.$$

Example 2.2.14. *The vertical position of an object under the acceleration of gravity near the surface of earth is given by $y = y_o + v_o t - \frac{1}{2} g t^2$ where $y_o, v_o, g > 0$ are all constants. The vertical velocity is $v = \frac{dy}{dt} = v_o - gt$ and the speed is given by $|v_o - gt|$. The distance traveled is a bit complicated since we have to break into cases. For $0 \leq t \leq t_1$ where $v_o - gt_1 = 0$ then*

$$s = \int_0^t |v_o - g\tau| d\tau = \int_0^t (v_o - g\tau) d\tau = v_o t - \frac{1}{2} g t^2.$$

Let s_1 be the distance traveled up to time $t = t_1 = v_o/g$ then

$$s_1 = \frac{v_o^2}{g} - \frac{1}{2} g \frac{v_o}{g}^2 = \frac{v_o^2}{2g}.$$

Then for $t > t_1$,

$$s = \int_0^t |v_o - g\tau| d\tau = \int_0^{t_1} |v_o - g\tau| d\tau + \int_{t_1}^t |v_o - g\tau| d\tau = s_1 + \int_{t_1}^t |v_o - g\tau| d\tau$$

Notice $v = v_o - g\tau < 0$ when $\tau > v_o/g$ thus $|v_o - g\tau| = g\tau - v_o$ for $t_1 \leq \tau < t$ and

$$s = s_1 + \int_{t_1}^t (g\tau - v_o) d\tau = s_1. + \frac{1}{2} g (t^2 - t_1^2) - v_o (t - t_1).$$

[1] you ought to be able to calculate this integral by the conclusion of your second term in Calculus
[2] Think about the graph of the absolute value of cosine

Notice when $t_2 = 2t_1$ we have $y = y_o$ once more and the formula gives the total distance

$$s_2 = s_1 + \frac{3}{2}gt_1^2 - v_o t_1 = s_1 + \frac{3}{2}\frac{v_o^2}{g} - \frac{v_o^2}{g} = 2s_1.$$

This makes perfect sense. We throw a ball up, it reaches height s_1 from our hand, then it fall back into our hand and the total distance traveled by the ball is twice the distance to the apex of the flight.

The previous example is part of an important general class of motion known as *projectile motion*. We discuss general properties of such motion in Section 2.4

2.3 acceleration

Acceleration is a vector quantity which decribe the change in the velocity of the object.

Definition 2.3.1. *The* **acceleration** *of an object with velocity $\vec{v}$ is given by $\vec{a} = \frac{d\vec{v}}{dt}$. If the position is $\vec{r}$ then the acceleration is also given by $\vec{a} = \frac{d^2\vec{r}}{dt^2}$. For one-dimnensional motion in x we have acceleration defined by $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.*

Given the position we can twice differentiate to find the acceleration. On the other hand, we need to know both an initial velocity and an initial position in order to calculate position from a given acceleration.

Example 2.3.2. *If $\vec{r} = \langle \alpha t, \beta t^2 \rangle$ then $\vec{v} = \frac{d\vec{r}}{dt} = \langle \alpha, 2\beta t \rangle$ and $\vec{a} = \frac{d\vec{v}}{dt} = \langle 0, 2\beta \rangle$.*

The example above has constant acceleration.

Example Problem 2.3.3. *If $\vec{a} = \left\langle 2m/s^2, (3m/s^4)t^2 \right\rangle$ and you are given initial velocity $\vec{v}(1s) = \langle 3\,m/s, 0 \rangle$ and initial position $\vec{r}(1s) = \langle 3.7\,m, 1.4\,m \rangle$ then find the position, velocity of this object at time t.*

Solution: *integration from $\tau = 1s$ to $\tau = t$ naturally builds initial data into our solutions:*

$$\frac{d\vec{v}}{dt} = \left\langle 2m/s^2, (3m/s^4)t^2 \right\rangle \quad \Rightarrow \quad \int_{1s}^{t} \frac{d\vec{v}}{d\tau} d\tau = \int_{1s}^{t} \left\langle 2m/s^2, (3m/s^4)\tau^2 \right\rangle d\tau$$

thus $\vec{v}(t) - \vec{v}(1s) = \left\langle (2m/s^2)(t - 1s), (m/s^4)(t^3 - 1s^3) \right\rangle$ and we find

$$\boxed{\vec{v}(t) = \left\langle 3m/s + (2m/s^2)(t - 1s), (m/s^4)(t^3 - 1s^3) \right\rangle}.$$

Next, I'll use the dummy-variable integration technique once more, (i'll omit units for brevity)

$$\frac{d\vec{r}}{dt} = \left\langle 3 + 2(t-1), t^3 - 1 \right\rangle \quad \Rightarrow \quad \int_{1}^{t} \frac{d\vec{r}}{d\tau} d\tau = \int_{1}^{t} \left\langle 2\tau + 1, \tau^3 - 1 \right\rangle d\tau$$

Thus $\vec{r}(t) - \vec{r}(1) = \langle t^2 + t - 2, t^4/4 - t - 1/4 + 1 \rangle$. But $\vec{r}(1) = \langle 3.7, 1.4 \rangle$ thus solve for

$$\boxed{\vec{r}(t) = \langle 1.7m + (m/s^2)t^2 + (m/s)t, 2.15m + (0.25m/s^4)t^4 - (m/s)t \rangle}$$

2.3.1 calculus of paths and geometry of curves

We should take a few moments to appreciate the calculus of paths. I will not prove these assertions, the proofs are simple applications of first semester calculus to the formulas for scalar multiplication, the dot-product or the cross products. If $\vec{A}, \vec{B}$ are functions of time t and f is a scalar function of time t then there are natural product rules:

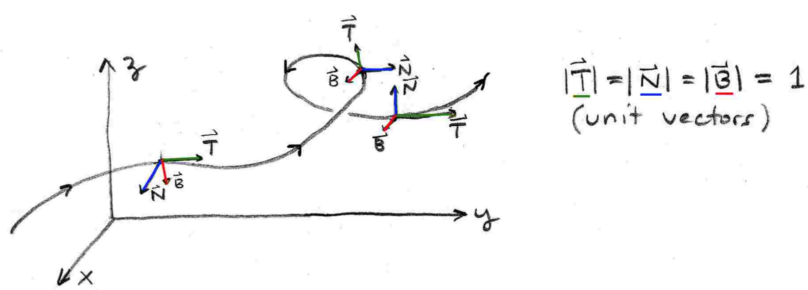
$$\frac{d}{dt} f\vec{A} = \frac{df}{dt}\vec{A} + f\frac{d\vec{A}}{dt}, \qquad \frac{d}{dt}\vec{A} \bullet \vec{B} = \frac{d\vec{A}}{dt} \bullet \vec{B} + \vec{A} \bullet \frac{d\vec{B}}{dt}, \qquad \frac{d}{dt}\vec{A} \times \vec{B} = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}.$$

One simple application of the above calculus is to see that a unit-vector field is alway perpendicular to its derivative:

$$\vec{B} \bullet \vec{B} = 1 \quad \Rightarrow \quad \frac{d\vec{B}}{dt} \bullet \vec{B} + \vec{B} \bullet \frac{d\vec{B}}{dt} = 0 \quad \Rightarrow \quad \vec{B} \bullet \frac{d\vec{B}}{dt} = 0.$$

Given a non-stop path, if we define $T = \frac{\vec{v}}{v}$ then clearly $T \bullet T = 1$ thus $T \perp dT/dt$. In fact, if the path with position $\vec{r}$ is non-stop and non-linear then we can describe the shape of the path using the Frenet-frame T, N, B which are the unit **tangent**, **normal** and **binormal** vector fields of the path. In particular, we define:

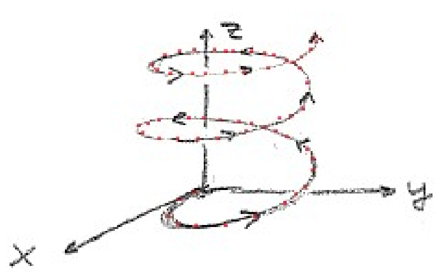
$$T = \frac{\vec{v}}{v}, \quad N = \frac{dT/dt}{||dT/dt||}, \quad B = T \times N$$

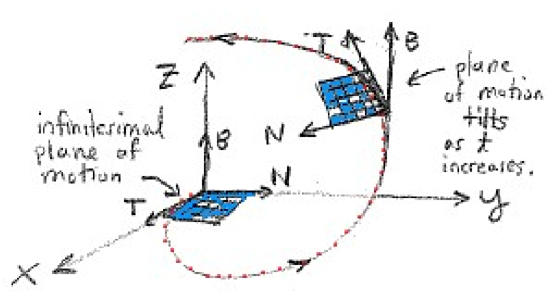


In Calculus III it is sometimes shown that the Frenet-frame satisfies the Frenet-Serret equations:

$$\tfrac{dT}{dt} = v\kappa N \qquad \tfrac{dN}{dt} = -v\kappa T + v\tau B \qquad \tfrac{dB}{dt} = -v\tau N.$$

where the **curvature** $\kappa = \frac{1}{v}||dT/dt||$ and **torsion** $\tau = -\frac{1}{v}\frac{dB}{dt} \bullet N$ describe the shape of the trajectory. Curvature of a unit-speed circle is simply the reciprocal of the radius of the circle; $\kappa = 1/R$. The torsion is zero if and only if the path lies in a particular plane, nonzero torsion indicates the motion of the path is lifting up of its plane of motion.





We can study motion in view of the Frenet-frame. To begin, the velocity is given by

$$\vec{v} = vT$$

the velocity is the product of the speed and its direction, the unit-tangent vector T. Next, differentiate the equation above to derive the formula for acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}T + v\frac{dT}{dt} = \frac{dv}{dt}T + v^2\kappa N \quad \Rightarrow \quad \vec{a} = \frac{dv}{dt}T + \frac{v^2}{R}N$$

where I have introduced $R = 1/\kappa$ do denote the radius of the **osculating circle** to the motion. The acceleration can be broken into a tangential and normal component. Since $T \perp N$ we have

$$a = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{R^2}}.$$

Example 2.3.4. *Suppose R and ω are positive constants and the motion of an object is observed to follow the path $\vec{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle = R\langle \cos(\omega t), \sin(\omega t)\rangle$. We wish to calculate the velocity and acceleration as functions of time.*

Differentiate to obtain the velocity

$$\boxed{\vec{v}(t) = R\omega\langle -\sin(\omega t), \cos(\omega t)\rangle.}$$

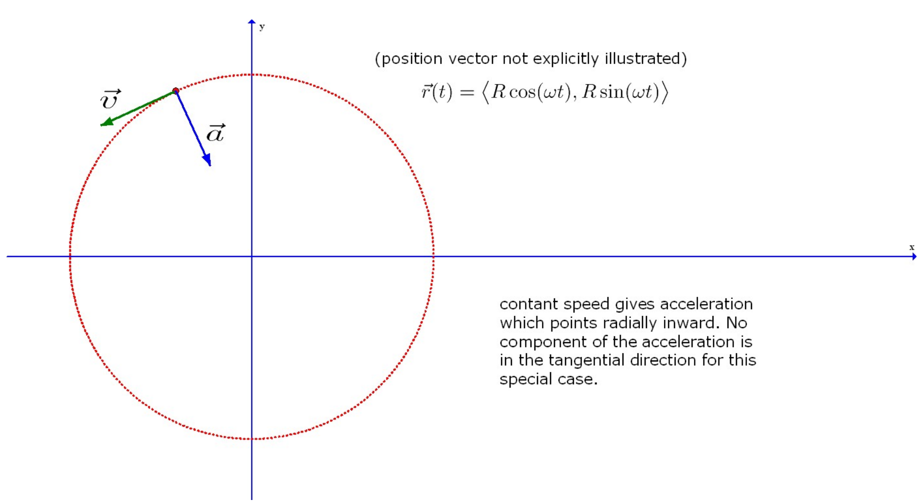
Differentiate once more to obtain the acceleration:

$$\vec{a}(t) = R\omega\langle -\omega\cos(\omega t), -\omega\sin(\omega t)\rangle = \boxed{-R\omega^2\langle \cos(\omega t), \sin(\omega t)\rangle.}$$

Notice we can write that $\vec{a}(t) = -\omega^2\vec{r}(t) = R\omega^2\vec{N}$ in this very special example. This means the acceleration is opposite the direction of the position and it is purely normal. Furthermore, we can calculate

$$r = R, \qquad v = R\omega, \qquad a = R\omega^2$$

Thus the magnitudes of the position, velocity and acceleration are all constant. However, their directions are always changing. The acceleration in this example is completely **centripetal** *or center-seeking acceleration since it points towards the center. Here we imagine attaching the acceleration vector to the object which is traveling in the circle.*



Example 2.3.5. *Suppose* $\vec{r}(t) = \langle R\cos(\theta), R\sin(\theta)\rangle$ *for* $t \geq 0$ *where* $\theta_o, \omega_o, \alpha$ *are constants and* $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$. *To calculate the distance travelled it helps to first calculate the velocity:*

$$\frac{d\vec{r}}{dt} = \langle -R(\omega_o + \alpha t)\sin(\theta), R(\omega_o + \alpha t)\cos(\theta)\rangle$$

Next, the speed is the length of the velocity vector,

$$v = \sqrt{[-R(\omega_o + \alpha t)\sin(\theta)]^2 + [R(\omega_o + \alpha t)\cos(\theta)]^2} = R\sqrt{(\omega_o + \alpha t)^2} = R|\omega_o + \alpha t|.$$

Therefore, the distance travelled is given by the integral below:

$$s(t) = \int_0^t R|\omega_o + \alpha\tau|d\tau$$

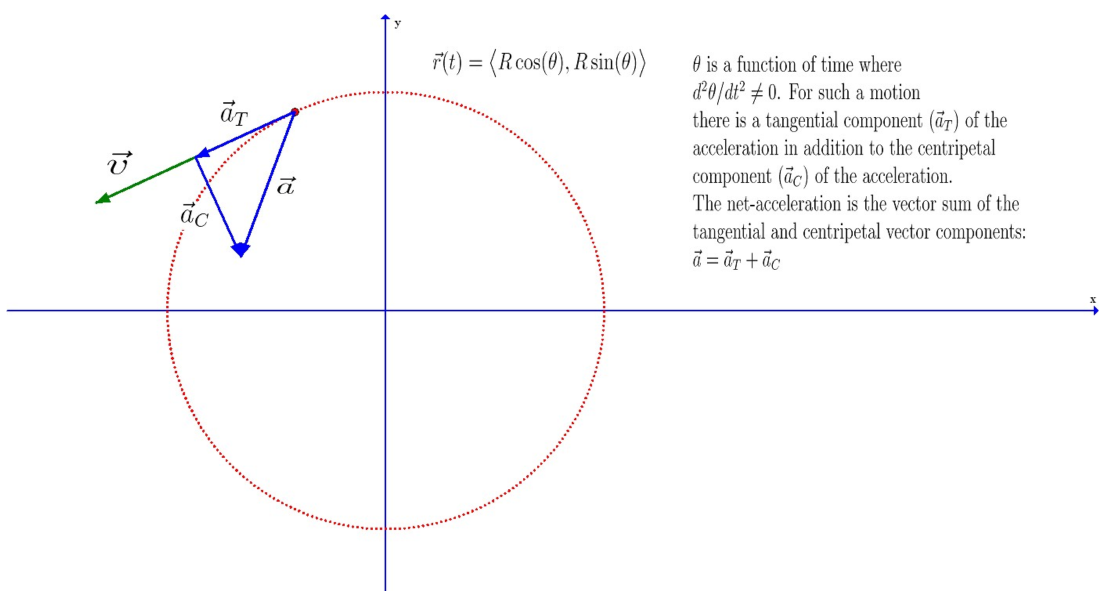
To keep things simple, let's suppose that ω_o, α *are given such that* $\omega_o + \alpha t \geq 0$ *hence* $v = R\omega_o + R\alpha t$. *To suppose otherwise would indicate the motion came to a stopping point and reversed direction, which is interesting, just not to us here.*

$$s(t) = R\int_0^t (\omega_o + \alpha\tau)d\tau = R\left(\omega_o t + \frac{1}{2}\alpha t^2\right).$$

Twice differentiate the position twice and show that

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = -\underbrace{R\omega^2\Big\langle \cos(\theta(t)), \sin(\theta(t))\Big\rangle}_{centripetal} + \underbrace{R\alpha\Big\langle -\sin(\theta(t)), \cos(\theta(t))\Big\rangle}_{tangential}$$

where $\omega = \omega_o + \alpha t$. *Notice, the term* **centripetal** *could be replaced with* **normal** *in the sense of the Frenet-Serret frames. Recall the normal pointed towards the center of the osculating circle thus the center-seeking acceleration is precisely the normal acceleration.*



Notice the magnitude of the acceleration in the above example is given by $a = \sqrt{R^2\omega^4 + R^2\alpha^2}$. If $\alpha = 0$ then the formula reduces to $a = R\omega^2$ in good agreement with the previous example.

2.4 Constant Acceleration and Projectile Motion

There are many special formulas which apply only in the case of constant acceleration. Let us derive all such formulas and record them in this section. I'll conclude the section with many typical applications to the study of projectile motion on Earth.

2.4.1 one dimensional constant acceleration formulas

Suppose $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt} = A$ where A is a constant. Then

$$\int \frac{dv}{dt} dt = \int A dt \quad \Rightarrow \quad v(t) = At + c_1.$$

If $v(0) = v_o$ then $A(0) + c_1 = v_o$ hence $c_1 = v_o$ and $\boxed{v(t) = v_o + At}$. Integrating once more,

$$\int \frac{dx}{dt} dt = \int (v_o + At) dt \quad \Rightarrow \quad x(t) = v_o t + \frac{t^2}{2} A + c_2.$$

If $x(0) = x_o$ then observe $x(0) = c_2$ hence $\boxed{x(t) = x_o + v_o t + \frac{t^2}{2} A}$. Notice if we wrote these **kinematic formulas** in terms of $v(t_1) = v_1$ and $x(t_1) = x_1$ then we would derive

$$\boxed{v(t) = v_1 + A(t - t_1) \qquad \& \qquad x(t) = x_1 + v_1(t - t_1) + \frac{1}{2} A (t - t_1)^2}.$$

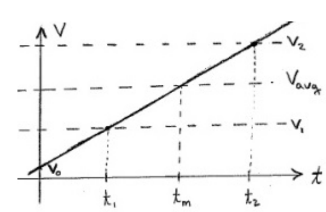
Next, we derive the **timeless equation**. In particular, we wish to calculate the velocity as a function of position. Notice

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} = A \quad \Rightarrow \quad \int_{v_o}^{v_f} v\, dv = \int_{x_o}^{x_f} A dx \quad \Rightarrow \quad \boxed{v_f^2 = v_o^2 + 2A(x_f - x_o)}.$$

The equation above is great for constant acceleration problems where we don't care about time, but we have information about initial and final velocities and positions. If we use $x_f - x_o = \triangle x$ then the timeless equation reads $\boxed{v_f^2 = v_o^2 + 2A\triangle x}$.

Next we derive the seemingly vacuous statement that the average of the velocities is the average velocity in the case of constant acceleration motion. Let $v(t_1) = v_1$ and $v(t_2) = v_2$. Recall,

$$\begin{aligned} v_{avg} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} &= \frac{1}{t_2 - t_1}\left(x_o + v_o t_2 + \frac{t_2^2}{2} A - x_o - v_o t_1 - \frac{t_1^2}{2} A\right) \\ &= \frac{1}{t_2 - t_1}\left(v_o(t_2 - t_1) + \frac{1}{2} A (t_2 - t_1)(t_2 + t_1)\right) \\ &= \frac{1}{2}(v_o + t_1 A + v_o + t_2 A) \\ &= \frac{1}{2}(v_1 + v_2). \quad \Rightarrow \quad \boxed{\frac{\triangle x}{\triangle t} = \frac{v_o + v_f}{2}}. \end{aligned}$$



The average velocity as given by the ratio of displacement to duration is the same as the average of the initial and final instantaneous velocities. This is a result which is special to the constant acceleration problem. It is not generally true.

2.4.2 three dimensional constant acceleration formulas

Next, suppose $\vec{A}$ is a constant vector. Suppose the acceleration of a body is given by

$$\vec{a}(t) = \vec{A}$$

for all t. Since $\vec{a} = \frac{d\vec{v}}{dt}$ integration $\int \frac{d\vec{v}}{dt}dt = \int \vec{A}dt$ yields $\vec{v}(t) = t\vec{A} + \vec{C}_1$. If $\vec{v}(0) = \vec{v}_o$ then we find $\vec{C}_1 = \vec{v}_o$ hence

$$\boxed{\vec{v}(t) = \vec{v}_o + t\vec{A}}$$

Then, as $\vec{v} = \frac{d\vec{r}}{dt}$ integration $\int \frac{d\vec{r}}{dt}dt = \int (\vec{v}_o + t\vec{A})dt$ yields $\vec{r}(t) = t\vec{v}_o + \frac{t^2}{2}\vec{A} + \vec{C}_2$. If $\vec{r}(0) = \vec{r}_o$ then note $\vec{r}_o = \vec{C}_2$ hence

$$\boxed{\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \frac{t^2}{2}\vec{A}}$$

Notice that the equations boxed above imply the form of the equations is the same for each Cartesian component of the motion. If $\vec{A} = \langle A_x, A_y, A_z \rangle$ then

$$\boxed{v_x(t) = v_{ox} + tA_x, \quad v_y(t) = v_{oy} + tA_y, \quad v_z(t) = v_{oz} + tA_z.}$$

where $\vec{v} = \langle v_x, v_y, v_z \rangle$ and $\vec{v}_o = \langle v_{ox}, v_{oy}, v_{oz} \rangle$. Likewise, if $\vec{r}_o = \langle x_o, y_o, z_o \rangle$ and $\vec{r} = \langle x, y, z \rangle$ then

$$\boxed{x(t) = x_o + tv_{ox} + \frac{t^2}{2}A_x, \quad y(t) = y_o + tv_{oy} + \frac{t^2}{2}A_y, \quad z(t) = z_o + tv_{oz} + \frac{t^2}{2}A_z.}$$

If we were given initial position and velocities at time $t = t_1$ then the formulas derived are the same as those above, except we replace t with $t - t_1$ and $\vec{v}_o$ with $\vec{v}_1 = \vec{v}(t_1)$ and $\vec{r}_o$ with $\vec{r}_1 = \vec{r}(t_1)$ etc. In particular,

$$\boxed{\vec{v}(t) = \vec{v}_1 + (t - t_1)\vec{A} \qquad \& \qquad \vec{r}(t) = \vec{r}_1 + (t - t_1)\vec{v}_o + \frac{1}{2}(t - t_1)^2\vec{A}}.$$

and

$$\boxed{v_x(t) = v_{1x} + (t - t_1)A_x, \quad v_y(t) = v_{1y} + (t - t_1)A_y, \quad v_z(t) = v_{1z} + (t - t_1)A_z.}$$

where $\vec{v}(t_1) = \vec{v}_1 = \langle v_{1x}, v_{1y}, v_{1z} \rangle$. Likewise, if $\vec{r}(t_1) = \vec{r}_1 = \langle x_1, y_1, z_1 \rangle$ then

$$\boxed{\begin{array}{l} x(t) = x_1 + (t - t_1)v_{1x} + \frac{(t-t_1)^2}{2}A_x, \\ y(t) = y_1 + (t - t_1)v_{1y} + \frac{(t-t_1)^2}{2}A_y, \\ z(t) = z_1 + (t - t_1)v_{1z} + \frac{(t-t_1)^2}{2}A_z. \end{array}}.$$

Finally both the timeless equation and the average velocity formula we found in the one-dimensional context have natural extensions here: for $\triangle\vec{r} = \langle \triangle x, \triangle y, \triangle z \rangle$,

$$\boxed{\frac{\triangle\vec{r}}{\triangle t} = \frac{\vec{v}_1 + \vec{v}_2}{2}} \qquad \& \qquad \boxed{v_f^2 = v_o^2 + 2\vec{A} \bullet \triangle\vec{r}.}$$

It may be that the derivation of the formulas above is part of the homework. We also note that similar two-dimensional formulas are similarly derived.

2.4.3 examples of constant acceleration

Example Problem 2.4.1. *Suppose a cat is thrown height h into the air above the initial point from which it is thrown. With what speed was the cat thrown ?*

Solution: *let the initial position of the cat be given by $y = 0$. Assume the cat was thrown vertically. When the cat reaches $y = h$ it must be that $v_f = 0$ hence the timeless equation gives*

$$0 = v_o^2 - 2gh \quad \Rightarrow \quad \boxed{v_o = \sqrt{2gh}}.$$

Example Problem 2.4.2. *suppose a car has brakes which apply a constant deceleration of $a = -2.0m/s^2$. If the car has an initial velocity of $20m/s$ then what is the stopping distance of the car and how much time is required to stop ?*

Solution: *the timeless equation with $v_f = 0$ gives*

$$0 = v_o^2 + 2a\triangle x \quad \Rightarrow \quad \triangle x = \frac{-v_o^2}{2a} = \frac{-(20m/s)^2}{2(-2m/s^2)} = \boxed{100\,m}.$$

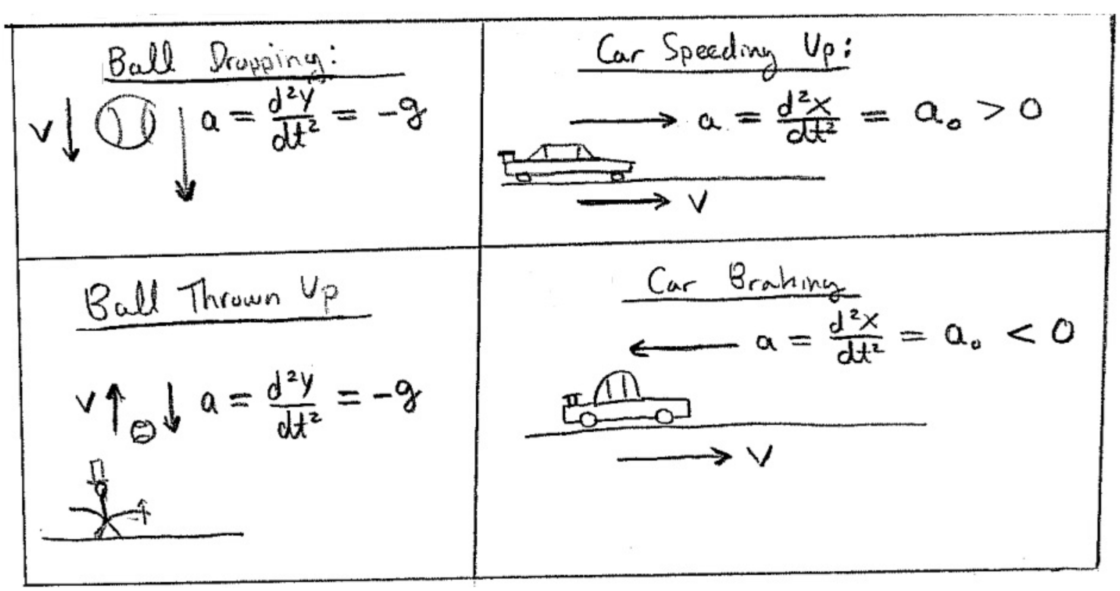
Also $v_f = v_o + at$ gives

$$t = \frac{v_f - v_o}{a} = \frac{-20m/s}{-2.0m/s^2} = \boxed{10\,s}.$$

Example Problem 2.4.3. *Does a car going $50\,mph$ have a positive or negative acceleration ?*

Solution: *we cannot say. If the velocity of the car points in the positive direction then the car could be increasing velocity in which case $a > 0$. Or, the car could be braking in which case $a < 0$.*

Let me illustrate a few common cases:



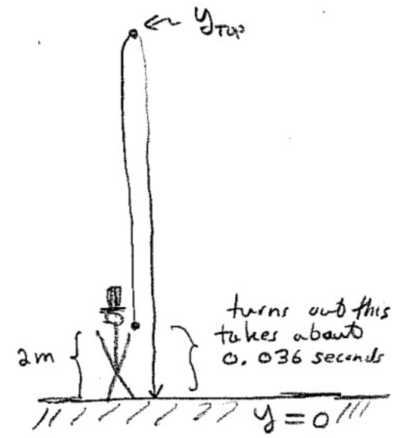
Example Problem 2.4.4. *If we throw a ball vertically with an initial height of* $2m$ *and an initial speed of* $50m/s$ *then what is the maximum height of the ball and how much time is it in flight ? Assume the ball falls back to the ground where* $y = 0$ *and assume the motion has constant acceleration of* $g = 9.8m/s^2$ *directed downward.*

Solution: *the equation of motion is given by (omitting units of meters and seconds)*

$$y = 2 + 50t - 4.9t^2$$

notice $\frac{dy}{dt} = 50 - 9.8t = 0$ *gives critical time* $t_{top} = \frac{50m/s}{9.8m/s^2} = 5.097\,s$. *Thus*

$$y_{top} = 2 + 50(5.097) - 4.9(5.097)^2 = \boxed{129.6\,m}.$$



The ball hits the ground when $y = 0$ *hence solve the quadratic equation* $0 = 2 + 50t - 4.9t^2$ *to obtain*

$$t = \frac{-50 \pm \sqrt{50^2 - 4(-4.9)(2)}}{2(-4.9)} = 10.24,\ 0.03984$$

Consequently, we find the ball is in flight for $\boxed{10.24s}$.

Example Problem 2.4.5. *Two stones are dropped from a cliff. The second stone is dropped* $1.6s$ *after the first. How far below the top of the cliff is the second stone when the separation between the stones is* $36m$ *?*

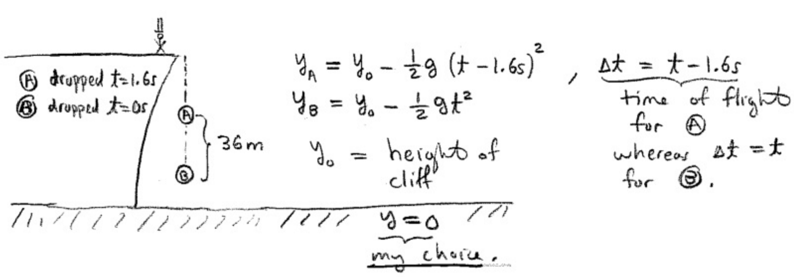
Solution: *both stones are dropped with* $v_o = 0m/s$ *and both stones are falling at* $g = 9.8m/s^2$. *Let* y_A *denote the position of the second stone and* y_B *denote the position of the first stone. We have the following equations of motion:*

$$y_A = y_o - \tfrac{1}{2}g(t - 1.6s)^2$$
$$y_B = y_o - \tfrac{1}{2}gt^2$$

To find the time at which the stones are $36m$ *apart we should solve*

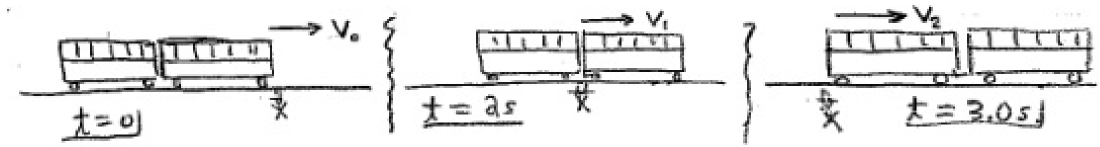
$$y_A - y_B = -\frac{1}{2}g(t - 1.6s)^2 + \frac{1}{2}gt^2 = 36m \quad \Rightarrow \quad -(t - 1.6s)^2 + t^2 = \frac{72m}{9.8m/s^2}$$

thus $3.2t - 2.56 = 7.347$ *and* $t = 3.096s$. *Then* $y_A - y_o = -\frac{g}{2}(3.096s)^2 = -10.96m$. *The second stone is* $\boxed{10.96\,m}$ *below the top of the cliff at the time of interest.*



Example Problem 2.4.6. *A subway car accelerates as it leaves the station. As it passes a certain person waiting for another train it takes* $2.0s$ *for one car, and* $1.0s$ *for the next subway car to pass. If the cars have length* $10m$ *then how fast are they traveling as they begin to pass the person ?*

Solution: *our goal is to find* v_o *as pictured:*



We assume the train accelerates at a constant rate hence the average velocity formula holds over (a.) $[0, 1s], (b.)[1s, 2s]$ *and (c.)* $[0, 2s]$. *Thus,*

$$\begin{aligned} v_{avg(a)} &= 10m/2s = 5\tfrac{m}{s} = \tfrac{1}{2}(v_o + v_1) \\ v_{avg(b)} &= 10m/1s = 10\tfrac{m}{s} = \tfrac{1}{2}(v_1 + v_2) \\ v_{avg(c)} &= 20m/3s = \tfrac{20}{3}\tfrac{m}{s} = \tfrac{1}{2}(v_o + v_2) \end{aligned}$$

Algebra shows $v_o = \frac{5}{3}\frac{m}{s}$ *that is* $\boxed{1.667m/s}$ *is the initial speed of the train passing the person.*

Example Problem 2.4.7. *Suppose a rocket car accelerates from rest at* $a = 2g$ *over a distance* L *then it releases an air-brake which decelerates the car at* $a = -g$ *until it comes to rest. How far did the car travel ?*

Solution: *let* L_2 *be the distance the car travels while it slows to a stop. If* v_1 *is the maximum velocity the car reaches right as it begins to brake then we can relate the velocity to the distances involved via the timeless equation for the accelerating and braking phases of the motion:*

$$v_1^2 = 2gL \qquad \& \qquad 0 = v_1^2 - gL_2$$

Thus $v_1^2 = 2gL = gL_2$ *and so* $L_2 = 2L$ *and we find the car traveled a distance of* $L + L_2 = \boxed{3L}$.

Example Problem 2.4.8. *Suppose an object undergoes constant acceleration motion as it moves from* $(-1, -2, 3)m$ *to* $(4, 3, 2)m$ *under* $\vec{a} = \langle 1, 0, 2\rangle m/s^2$. *Given that the object started at rest, what is the final speed of the object ?*

Observe $\triangle\vec{r} = (4, 3, 2)m - (-1, -2, 3)m = \langle 5, 5, -1\rangle m$ *thus*

$$\triangle\vec{r} \bullet \vec{a} = \langle 5, 5, -1\rangle \bullet \langle 1, 0, 2\rangle m^2/s^2 = 3m^2/s^2.$$

Recall, the timeless equation in vector form is given by $v_f^2 = v_o^2 + 2\vec{a} \bullet \triangle\vec{r}$. *Since* $v_o = 0$ *is given we find that* $v_f = \sqrt{6}m/s$. *That is,* $\boxed{v_f = 2.449m/s}$.

Example Problem 2.4.9. *Suppose an object undergoes constant acceleration motion as it moves from* $(-1, -2, 3)m$ *to* $(4, 3, 2)m$ *under* $\vec{a} = \langle 1, 0, 14\rangle m/s^2$. *Given that the object started at rest, what can you say about the motion ?*

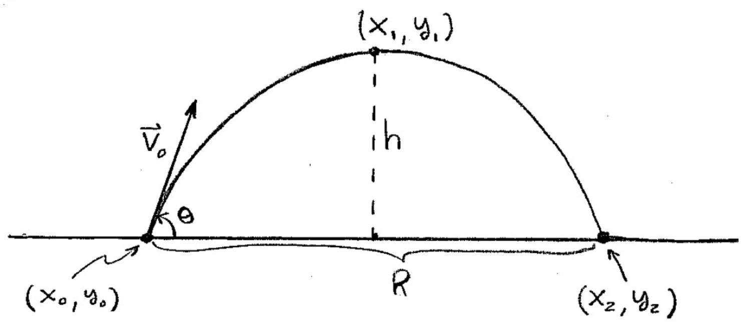
Observe $\triangle\vec{r} = (4, 3, 2)m - (-1, -2, 3)m = \langle 5, 5, -1\rangle m$ *thus*

$$\triangle\vec{r} \bullet \vec{a} = \langle 5, 5, -1\rangle \bullet \langle 1, 0, 14\rangle m^2/s^2 = -9m^2/s^2.$$

Recall, the timeless equation in vector form is given by $v_f^2 = v_o^2 + 2\vec{a} \bullet \triangle\vec{r}$. *Since* $v_o = 0$ *is given we find that* $v_f^2 = -18m^2/s^2$. *But, this is impossible since* v_f *is a real value. It is impossible for such a motion to occur. Wait, that can happen in a physics example ? Yes. It can.*

2.4.4 projectile motion

We say an object in flight which is accelerated by gravity alone is under projectile motion. We assume friction can be ignored along with the variability of gravity and the rotation of the earth etc. Typically we use coordinates x, y where we think of x as horizontal to the earth and y as vertical. In this set-up we face the constant acceleration of $\vec{a} = \langle 0, -g \rangle$ where $g = 9.8\,m/s^2$. To be honest, this is an approximation since the acceleration due to gravity varies by about $0.1m/s^2$ as we change position on the earth. In fact, the frame-effect of the earth rotating at the equator effectively decreases the gravitational acceleration by about $0.1m/s^2$. More on that later, we can do the calculation to explain the magnitude[3]. Of course, air-friction cannot actually be ignored in a variety of actual physical motions. From Nerf guns, to badminton, to ping pong, to bullets from a sniper rifle, all these motions are significantly impacted by the effect of friction. That said, we can view actual motion of physical bodies as a modification of the ideal case of projectile motion. One reason many engineers are required to take Differential Equations is that the mathematics to study friction is largely found in that course. We must start with the basics. Let's study the case that the projectile is fired on a flat field with initial speed v_o at an **angle of inclination** θ as pictured below:



We seek to find the formulas for the **maximum height** h and the **range** R as pictured. Notice

$$\vec{v}_o = \langle v_o \cos\theta, v_o \sin\theta \rangle$$

thus assuming the motion begins at $t = 0$,

$$v_x = v_o \cos\theta \qquad \& \qquad v_y = v_o \sin\theta - gt$$

The motion features constant velocity in x and decreasing velocity in y. Then

$$x = x_o + tv_o \cos\theta \qquad \& \qquad y = y_o + tv_o \sin\theta - \frac{1}{2}gt^2.$$

To find (x_1, y_1) seek t_1 for which $v_y(t_1) = v_o \sin\theta - gt_1 = 0$ thus $t_1 = v_o \sin\theta / g$ is the time to the **apex** of the flight. Observe

$$h = y(t_1) - y_o = \frac{v_o \sin\theta}{g} v_o \sin\theta - \frac{1}{2} g \left(\frac{v_o \sin\theta}{g} \right)^2 \quad \Rightarrow \quad \boxed{h = \frac{v_o^2 \sin^2\theta}{2g}} \quad \text{(max. height)}$$

Then, by the symmetry, the time t_2 to reach (x_2, y_2) where $y_2 = y_o$ is simply $t_2 = 2t_1$ hence

$$R = x_2 - x_o = 2\left(\frac{v_o \sin\theta}{g} \right) v_o \cos\theta \quad \Rightarrow \quad \boxed{R = \frac{v_o^2 \sin(2\theta)}{g}} \quad \text{(range)}$$

where I've used the trigonometric identity $2\sin\theta\cos\theta = \sin(2\theta)$.

[3]this explains why we don't go flying off the earth due to the rotational motion, it turns out the gravitational attraction much larger

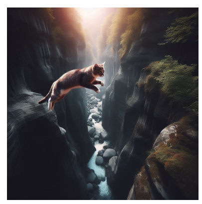
Example 2.4.10. *If we fire a projectile with an initial speed of* $500m/s$ *then since*

$$v_o^2/g = (500m/s)^2/9.8m/s^2 = 25.51km$$

we find a maximum height $h = (12.76km)\sin\theta$ *which is at most* $h = 12.76km$ *if we fire at* $\theta = 90^o$ *(I don't advise this). On the other hand, the range* $R = \frac{v_o^2}{g}\sin(2\theta) = (25.51km)\sin(2\theta)$ *is maximized when we select* $\theta = 45^o$*. This is clear from trigonometry alone, we'll need calculus for less obvious problems.*

There exist naval guns which could bombard targets 20 miles away ($32km$). The range equation and the max height equation are useful for quick fact finding checks in more complicated problems. Let me illustrate the use with a couple examples.

Example Problem 2.4.11. *A cat can jump vertically* $2.5m$*. If the cat wishes to jump across a ravine in such a way that it begins and ends at the same height then what is the largest distance the cat can jump ?*



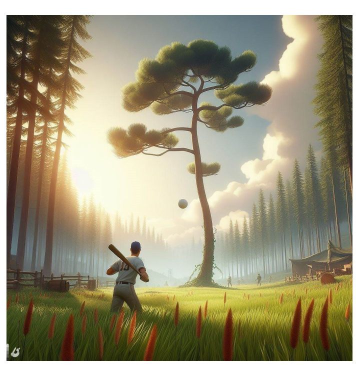
Solution: *we assume the cat can jump the same initial speed independent of the angle at which the cat launches itself. If* $\theta = 90^o$ *the max height equation gives* $h = v_o^2/2g$ *thus* $v_o = \sqrt{2gh} = \sqrt{2(9.8m/s^2)(2.5m)} = 7m/s$*. Then the max range is given by* $\theta = 45^o$ *in this context hence* $R = v_o^2/g = (7m/s)^2/(9.8m/s^2) = 5m$*. We find the maximum distance such a cat can jump is* $5m$*.*

Example Problem 2.4.12. *A baseballer can throw 100mph. Can he throw the ball over a tree which is* 70 *m tall a distance of* $200\,m$ *away ? What if the tree was just* $20\,m$ *away ?*

Solution: *let us convert* $100mph$ *using the fact that* $mi = 1609m$ *and* $hr = 3600s$ *hence*

$$v_o = 100\frac{mi}{hr}\frac{1609m}{mi}\frac{hr}{3600s} = 44.70\frac{m}{s}.$$

Calculate $v_o^2/g = (44.70m/s)^2/(9.8m/s^2) = 203.8m$ *so the maximum height reached by such a throw would be* $h = (101.9\,m)\sin\theta$ *with a range of* $R = (203.8m)\sin(2\theta)$*. We see that reaching the tree's base requires an angle of about* 45^o *in which case the maximum height attained is* $h = 72.1\,m$*. Unfortunately, that height occurs in the middle of the flight and it is clear that it is not possible to clear the tree in question if it is* $200\,m$ *away. On the other hand, if the tree was just* $20\,m$ *away then since a height of* $101.9m$ *can be attained it seems plausible that he could clear the tree.*



If $R = 40m$ then $40 = (203.8m)\sin(2\theta)$ hence $2\theta = \sin^{-1}(40/203.8) = 11.32^o$ or $\theta = 5.66^o$. But, trigonometry implies $90^o - 5.66^o = 84.24^o$ will also yield the same $40m$ range since

$$\sin(2(90^o - \theta)) = \sin(180^o - 2\theta) = \sin(2\theta).$$

You doubt me ? Check it out, $\sin(2(84.24^o)) = 0.2 = \sin(2(5.66^o))$. Ok, so using the larger angle obviously gives the larger max height; $h = (101.9m)\sin(84.24^o) = 101.4m$. So, yeah, the tree is cleared by quite a bit if he throws the ball at 84.24^o.

Clearly the range and maximum height equations have many uses, but not all problems permit their use. Whenever the begining and ending height is not the same the range equation can be misleading. Whenever the height in question is not necessarily at the apex of the flight, the maximum height equation can be misleading. Many problems require us to go back to basics and work out the kinematic equations from scratch. Let's examine a few typical problems.

Example Problem 2.4.13. *Suppose a cat is thrown by batman over a river with an initial velocity of $v_o = 20m/s$ at an angle of 30^o. The opposite bank of the river happens to rise vertically $5m$ above where batman is standing and batman released the cat a height of $1m$ above the ground. If the river is $30m$ wide then will the cat fall into the river full of alligators ?*

Solution: *Notice $\vec{v}_o = \langle (20m/s)\cos(30^o), (20m/s)\sin(30^o)\rangle = \langle 17.32m/s, 10m/s\rangle$. Since $\vec{a} = \langle 0, -g\rangle$ we have*

$$v_x = 17.32m/s, \qquad \& \qquad v_y = 10m/s - (9.8m/s^2)t.$$

I'll put the origin at batman's feet. Thus,

$$x = (17.32m/s)t, \qquad \& \qquad y = 1m + (10m/s)t - (4.9m/s^2)t^2.$$

Since the river is $30m$ wide we can find the time the cat would reach the other side by solving the x-equation for t:

$$t = \frac{30m}{17.32m/s} = 1.732s.$$

Notice, the value of y at this time is given by plugging $t = 1.732s$ into the equation of motion for y,

$$y = 1m + (10m/s)(1.732s) - (4.9m/s^2)(1.732s)^2 = 3.621m.$$

Good news, the cat did not make it since it is not above the $5m$ mark.

Example Problem 2.4.14. *The helicopter travels horizontally with speed $10m/s$ at the moment that batman drops off it. If batman lands on the ground a distance $20m$ from where he dropped off then how far off the ground was he when he let go ?*

Solution: *the equations of motion are given by*

$$x = (10m/s)t, \qquad \& \qquad y = y_o - \frac{1}{2}gt^2$$

where we've defined Batman's initial position as $(0, y_o)$. Let $(x_o, 0)$ be the point where Batman lands. Since $x(t_f) = x_o$ and $y(t_f) = 0$ we find $y_o = \frac{1}{2}gt_f^2$ or $t_f = \sqrt{2y_o/g}$ thus $x_o = (10m/s)\sqrt{2y_o/g}$ which gives $x_o^2 = \frac{(200m^2/s^2)y_o}{g}$ and as we are given $d((0, y_o), (x_o, 0)) = 20m$ thus $x_o^2 + y_o^2 = (20m)^2$. Hence

$$\frac{(200m^2/s^2)y_o}{g} + y_o^2 = (20m)^2 \quad \Rightarrow \quad y_o^2 + (20.41m)y_o - 400m^2 = 0$$

which has solutions $y_o = 12.25m$ or $y_o = -32.66m$. Therefore, Batman was $\boxed{12.25m}$ above the ground when he dropped.

Example Problem 2.4.15. *A cat is in a hotair balloon when it throws a mouse with velocity $10m/s$ at an angle of 40^o relative to the horizontal floor of the balloon's bucket. At the time the balloon is rising vertically at a speed of $2m/s$. If the mouse is initially $300m$ above the ground when it is thrown then how far horizontally does the mouse travel before it hits the ground ?*

Solution: *the velocity of the mouse relative to the balloon as it is thrown is given by $(10m/s)\cos(40^o) = 7.660m/s$ horizontally and $(10m/s)\sin(40^o) = 6.428m/s$ vertically, but since the balloon is rising at $2m/s$ we need to add that amount to the initial vertical velocity of the mouse relative to the earth:*

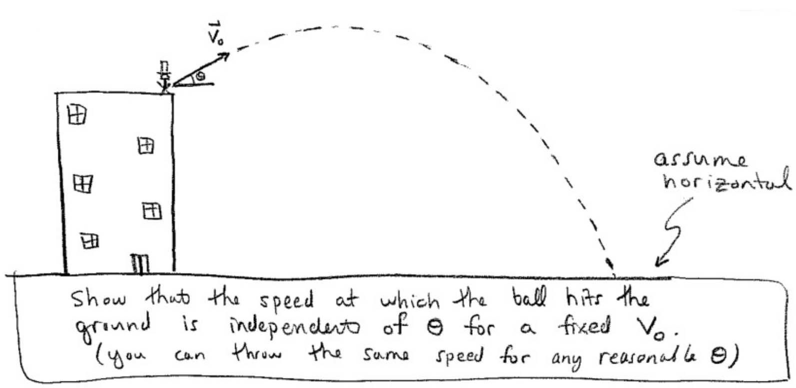
$$v_{ox} = 7.660m/s \qquad \& \qquad v_{oy} = 8.428m/s$$

Then, projectile motion with $\vec{a} = \langle 0, -9.8m/s^2 \rangle$ yields

$$x = (7.660m/s)t, \qquad \& \qquad y = 300m + (8.428m/s)t - (4.9m/s^2)t^2$$

The mouse hits the ground when $y = 0$ thus solve $300 + 8.428t - 4.9t^2 = 0$ to obtain $t = 8.732s$ or $t = -7.012s$. We choose the physically reasonable time and plug it into the equation of motion for x to find the horizontal distance traveled by the mouse; $x = (7.660m/s)(8.732s) = \boxed{66.89m}$.

Example 2.4.16. *Consider the following projectile motion problem:*



Let y_1 be the height from which the ball is thrown and let $y = 0$ be the ground. Notice $v_{ox} = v_o \cos\theta$ and $v_{oy} = v_o \sin\theta$. Timeless equation gives

$$v_{fy}^2 = (v_o \sin\theta)^2 - 2gy_1$$

and $a_x = 0$ gives $v_{fx} = v_o \cos\theta$. Thus

$$v_f^2 = v_{fx}^2 + v_{fy}^2 = (v_o \cos\theta)^2 + (v_o \sin\theta)^2 - 2gy_1 = v_o^2 - 2gy_1.$$

Thus $v_f = -\sqrt{v_o^2 - 2gy_1}$ and it is clear that this result is independent from θ.

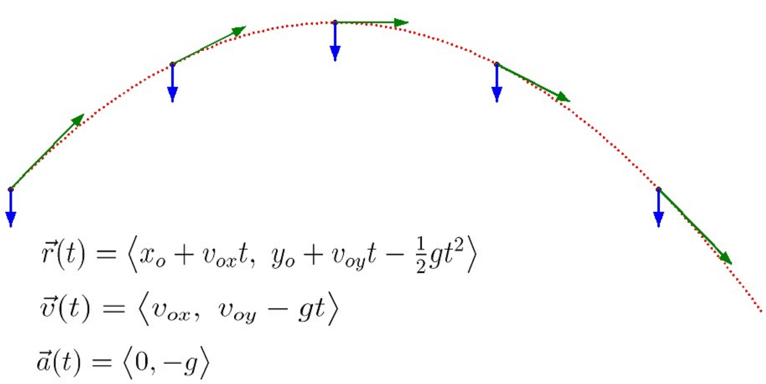
Example 2.4.17. *Suppose that the acceleration of an object is known to be $\vec{a} = \langle 0, -g\rangle$ where g is a positive constant. Furthermore, suppose that initially the object is at $\vec{r}_o$ and has velocity $\vec{v}_o$. We wish to calculate the position and velocity as functions of time.*

Integrate the acceleration from 0 to t,

$$\int_0^t \frac{d\vec{v}}{d\tau} d\tau = \int_0^t a(\tau) d\tau \quad \Rightarrow \quad \vec{v}(t) - \vec{v}(0) = \int_0^t \langle 0, -g\rangle d\tau \quad \Rightarrow \quad \boxed{\vec{v}(t) = \vec{v}_o + \langle 0, -gt\rangle}$$

Integrate the velocity from 0 to t,

$$\int_0^t \frac{d\vec{r}}{d\tau} d\tau = \int_0^t v(\tau) d\tau \quad \Rightarrow \quad \vec{r}(t) - \vec{r}(0) = \int_0^t \big(\vec{v}_o + \langle 0, -gt\rangle\big) d\tau \quad \Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \langle 0, -\frac{1}{2}gt^2\rangle}$$



The acceleration is constant for this parabolic trajectory.
The velocity is changing in the vertical direction, but is constant in the x-direction.

Distance travelled is not always something we can calculate in closed form. Sometimes we need to relegate the calculation of the arclength integral to a numerical method. However, the example that follows is still calculable without numerical assistance. It did require some thought.

Example 2.4.18. *We found that* $\vec{a} = \langle 0, -g\rangle$ *twice integrated yields a position of* $\vec{r}(t) = \vec{r}_o + t\vec{v}_o + \langle 0, -\frac{1}{2}gt^2\rangle$ *for some constant vectors* $\vec{r}_o = \langle x_o, y_o\rangle$ *and* $\vec{v}_o = \langle v_{ox}, v_{oy}\rangle$*. Thus,*

$$\vec{r}(t) = \big\langle x_o + v_{ox}t, y_o + v_{oy}t - \frac{1}{2}gt^2\big\rangle$$

From which we can differentiate to derive the velocity,

$$\vec{v}(t) = \big\langle v_{ox}, v_{oy} - gt\big\rangle.$$

Observe that the zero-acceleration in the x-direction gives rise to constant-velocity motion in the x-direction whereas the gravitational acceleration in the y-direction makes the object fall back down as a consequence of gravity. If you think about $v_{oy} - gt$ *it will be negative for some* $t > 0$ *whatever the initial velocity* v_{oy} *happens to be, this point where* $v_{oy} - gt = 0$ *is the turning point in the flight of the object and it gives the top of the parabolic*[4] *trajectory which is parametrized by* $t \to \vec{r}(t)$*. Suppose* $x_o = y_o = 0$ *and calculate the distance travelled from time* $t = 0$ *to time* $t_1 = v_{oy}/g$*. Additionally, let us assume* $v_{ox}, v_{oy} \geq 0$*.*

$$\begin{aligned} s = \int_0^{t_1} v(t)dt &= \int_0^{t_1} \sqrt{(v_{ox})^2 + (v_{oy} - gt)^2}dt \\ &= \int_{v_{oy}}^{0} \sqrt{(v_{ox})^2 + (u)^2}\left(\frac{du}{-g}\right) \qquad u = v_{oy} - gt \\ &= \frac{1}{g}\int_0^{v_{oy}} \sqrt{(v_{ox})^2 + (u)^2}du \end{aligned}$$

Recall that a nice substitution for an integral such as this is provided by the $\sinh(z)$ *since* $1 + \sinh^2(z) = \cosh^2(z)$ *hence a* $u = v_{ox}\sinh(z)$ *subsitution will give*

$$(v_{ox})^2 + (u)^2 = (v_{ox})^2 + (v_{ox}\sinh(z))^2 = v_{ox}^2\cosh^2(z)$$

and $du = v_{ox}\cosh(z)dz$ *thus,* $\int \sqrt{(v_{ox})^2 + (u)^2}du = \int \sqrt{v_{ox}^2\cosh^2(z)}v_{ox}\cosh(z)dz = \int v_{ox}^2\cosh^2(z)dz$*. Furthermore,* $\cosh^2(z) = \frac{1}{2}(1 + \cosh(2z))$ *hence*

$$\int \sqrt{(v_{ox})^2 + (u)^2}du = \frac{v_{ox}^2}{2}\big[z + \frac{1}{2}\sinh(2z)\big] + c = \frac{v_{ox}^2}{2}\big[z + \sinh(z)\cosh(z)\big] + c$$

Note $u = v_{ox}\sinh(z)$ *and* $v_{ox}\cosh(z) = \sqrt{(v_{ox})^2 + (u)^2}$ *hence substituting,*

$$\int \sqrt{(v_{ox})^2 + (u)^2}du = \frac{1}{2}\left[v_{ox}^2\sinh^{-1}\left(\frac{u}{v_{ox}}\right) + u\sqrt{v_{ox}^2 + u^2}\right] + c$$

Returning to the definite integral to calculate s we can use the antiderivative just calculated together with FTC II to conclude: (provided $v_{ox} \neq 0$*)*

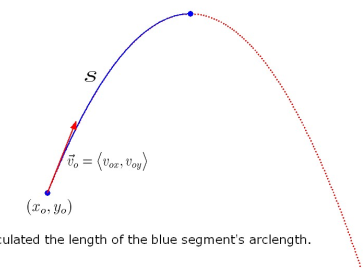
$$\boxed{s = \frac{1}{2g}\left[v_{ox}^2\sinh^{-1}\left(\frac{v_{oy}}{v_{ox}}\right) + v_{oy}\sqrt{v_{ox}^2 + v_{oy}^2}\right]}$$

[4] no, we have not shown this is a parabola, I invite the reader to verify this claim. That is find A, B, C such that the graph $y = Ax^2 + Bx + C$ is the same set of points as $\vec{r}(\mathbb{R})$.

If $v_{ox} = 0$ then the problem is easier since $v(t) = |v_{oy} - gt| = v_{oy} - gt$ for $0 \leq t \leq t_1 = v_{oy}/g$ hence

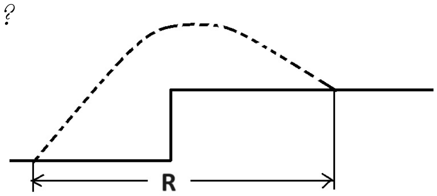
$$s = \int_0^{t_1} v(t)dt = \int_0^{t_1} (v_{oy} - gt)dt = \left[v_{oy}t - \frac{1}{2}gt^2 \right]\Bigg|_0^{v_{oy}/g} = \boxed{\frac{v_{oy}^2}{2g}}$$

Interestingly, this is the formula for the height of the parabola even if $v_{ox} \neq 0$. The initial x-velocity simply determines the horizontal displacement as the object is accelerated vertically by gravity.



We calculated the length of the blue segment's arclength.

Example Problem 2.4.19. *Suppose you have a water balloon launcher in a valley $10\,m$ below a level field and your launcher is attached to a tree which is $20\,m$ from the vertical cliff where the field begins. The launcher can send balloons with $v_o = 15\,m/s$. What angle should we lauch balloons in order to maximize the range R ?*



Solution: *let v_o be the speed at which the balloons are launched. Then*

$$x = tv_o \cos\theta \qquad \& \qquad y = tv_o \sin\theta - \frac{g}{2}t^2$$

Balloon hits the field at time t for which $y = 10m$

$$tv_o \sin\theta - \frac{g}{2}t^2 = 10m \quad \Rightarrow \quad t^2 - \frac{2v_o \sin\theta}{g}t + \frac{20m}{g} = 0 \quad \Rightarrow \quad \left(t - \frac{v_o \sin\theta}{g}\right)^2 = \frac{v_o^2 \sin^2\theta - 20mg}{g^2}.$$

We find positive time solution of

$$t = \frac{v_o \sin\theta + \sqrt{v_o^2 \sin^2\theta - 20mg}}{g}$$

Thus $R = tv_o \cos\theta = \frac{v_o^2}{g}\left(\cos\theta \sin\theta + \cos\theta\sqrt{\sin^2\theta - 20mg/v_o^2}\right)$ thus

$$R = \frac{v_o^2}{g}\left(\cos\theta \sin\theta + \cos\theta\sqrt{\sin^2\theta - 0.871}\right)$$

To maximize R we seek the critical angle θ for which $\frac{dR}{d\theta} = 0$. Differentiating,

$$\cos^2(\theta) - \sin^2(\theta) + \frac{\cos^2(\theta)\sin(\theta)}{\sqrt{-0.871 + \sin^2(\theta)}} - \sin(\theta)\sqrt{-0.871 + \sin^2(\theta)} = 0$$

which has its first real zero at $\theta = 1.23$ (this is in radians since we assumed $\frac{d}{d\theta}\sin\theta = \cos\theta$ etc.) hence $\theta = 1.23\left(\frac{180^o}{\pi}\right) = \boxed{70.5^o}$. When the initial and final height are not the same there is no reason for 45^o to be the angle for maximum range. I used a numerical method to solve this problem since the algebraic solution of the critical equation looked rather tricky.

Chapter 3

Force and Motion

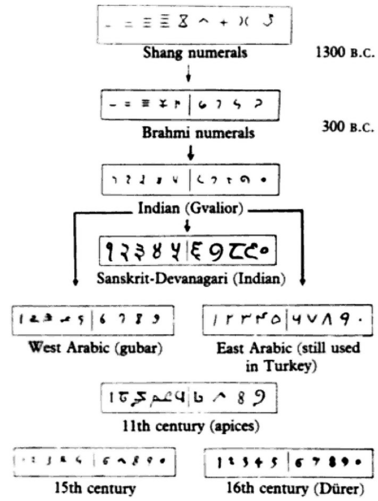
3.1 history

The term *Physics* is attributed to Aristotle, the famous Greek philosopher who lived 384-322 BC. Aristotle saw all matter and its motion in terms of aether and the four elements fire, air, water and earth. The elements could be transformed into one another and each had a natural tendency. Rocks fall because they belong on the ground. Heavier things fall faster, lighter things fall slower. This was the dominant view of Physics in much of the known world from the time of Aristotle to about 1600 AD. What changed ?

- Nicolaus Copernicus (1473–1543) put forth a heliocentric model of the Solar System which was at odds with the popular and generally accepted Earth-centered model accepted by Aristotle. Ptolemy (100-170 AD) made charts and predictions which explained many things and it's easy to see why so many people found it convincing for over a thousand years. It's very understandable that Copernicus was reluctant to publish his idea when he found it around 1515, only near his death was a publication of his theory finally authorized.

- Galileo Galilei (1564–1642) championed experiments as a means to test existing Physics, found rates of falling were independent of mass. Championed Copernicus helocentric and he is rightly remembered as the father of modern physics. Galileo also more or less put forth Newton's First Law of motion in his book *Dialogue Concerning the Two Chief World Systems* in 1632. That book upset the inquisition so much so that it was banned. In the book, Galileo describes how we would not be able to tell the difference between a boat at rest on a smooth sea and a boat in constant motion. That concept is more or less equivalent to Newton's First Law. Galileo also is famous for the quote: *The laws of nature are written by the hand of God in the language of mathematics.*. Certainly that captures a large part of what separates Physics in the last 500 years from Physics of antiquity; Physics is written in terms of math. Use of math and verification by experiment are two pillars of Physics as we know it today.

- Johannes Kepler (1571-1630) found the motion of planets could be described by ellipses and the period of their orbit was related to the radius of the orbit in the same way for all planets. This work further verified the heliocentric view, but challenged other assumptions such as the need for perfectly circular orbits. Overall, the mathematics of Kepler was far simpler than competing theories of epicycles.



- Simon Stevin (1548–1620) in 1585 published *De Thiende (The Tenth* which made decimal calculation accessible to many people. Decimal representations of numbers were know in other forms by various ancient cultures, but this was a turning point for the advancement of math which laid the foundation for Newton's work. John Napier (1550- 1617) also played an important role in turning the notation of a real number in decimal form closer to the modern notation, he introduced the *decimal point*. Napier's work in logarithms also have lasting value. What is most interesting about Napier is his various shennigans, like getting pigeons drunk, or tricking people into thinking his rooster had special powers. Of course, all of this is built over a vast prehistory of the development of number systems by mathematicians of various ancient cultures. See the picture.

- René Descartes (1596–1650) had a theory that motion could all be explained by an invisible sea of "corpuscles". Apparently, only human thought and God were outside his theory of everything. Far more importantly, Descartes was the inventor of coordinate geometry. The so-called Cartesian Plane is named in honor of this natural philosopher. We take this for granted, but it is a huge part of what makes Newtonian Mechanics a plausible theory of nature.

All of the events above and many other we don't have space for here set the stage for Newton to discover what we know as *Newtonian Mechanics.* Sir Isaac Newton (1642-1727) showed that motion could be derived from three basic laws. Newton published these laws[1] in his 1687 work entitled *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy). It was written in Latin, so what follows is a translation:

- **First Law:** Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

- **Second Law:** The change of motion of an object is proportional to the force impressed; and is made in the direction of the straight line in which the force is impressed.

- **Third Law:** To every action, there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

He also explained the motion of planets and derived Kepler's Laws. Newton's Universal Law of Gravitation explains almost every aspect of gravity which is familar to our experience. Newton invented Calculus in order to make his physical theory mathematically precise. It should be noted that Leibniz (1646-1716), a French contemporary of Newton, also found and popularized Calculus in the 17th century.

There were other ancient natural philosophers with a multitude of ideas about how to describe motion and the structure of matter. Indian and Chinese and Arabic scholars all pioneered various bits and pieces of Newtonian Physics. The concept of inertia can be seen in multiple ancient sources. However, ultimately, it is Newton who set the stage for the modern mechanics as we know it. Of course, I just speak of introductory mechanics, there are better methods invented by Euler and Lagrange, expanded by Hamilton and others which allow elegant solutions of problems

[1]The First Law was also known to Christiaan Huygens (1629-1695), Galileo and Descartes and others.

which would be exceedingly difficult to set-up in the Newtonian framework. Also, to be fair, we do think the Physics of this chapter are only accurate for motion which is not *relativistic*. When speeds approach the speed of light we need to describe motion with relativistic mechanics. Furthermore, Newton's Universal Law of Gravitation (which we discuss later in this course) has been replaced with Einstein's General Theory of Relativity which describes the physics of black holes and gravitational waves, there is even an aspect of GPS technology which requires a correction from General Relativity. On the other hand, when things are very small, Quantum Mechanics seems to govern the motion of such systems. All of this said, we will focus our attention on Newtonian Mechanics, so, let's get to it.

3.2 Newton's Laws

Let me restate Newton's Laws in a form which is explicitly tied to the physical variables we use to study motion in nature:

- **First Law:** A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.
- **Second Law:** If $\vec{F}$ is the net-force which acts on a body with mass m then $\vec{F} = m\vec{a}$ where $\vec{a}$ is the acceleration of the body.
- **Third Law:** If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

Notice, we can derive the First Law from the Second Law. Let's see how this is done. If the net-force on a mass m is $\vec{F} = 0$ then $m\vec{a} = 0$ which gives $\vec{a} = 0$ hence

$$\vec{a} = \frac{d\vec{v}}{dt} = 0 \quad \Rightarrow \quad \vec{v} = \vec{v}_o = \frac{d\vec{r}}{dt} \quad \Rightarrow \quad \vec{r} = \vec{r}_o + t\vec{v}_o$$

where $\vec{r}_o$ and $\vec{v}_o$ are respectively the position and velocity of the object at time $t = 0$. The formula $\vec{r} = \vec{r}_o + t\vec{v}_o$ describes a line with direction-vector $\vec{v}_o$ and base-point $\vec{r}_o$. In short, if the net-force is zero then by twice integrating Newton's Second Law we derive the motion follows a straight line with constant velocity $\vec{v}_o$. Recall speed $v_o = \|\vec{v}_o\| = \sqrt{\vec{v}_o \bullet \vec{v}_o}$ so clearly constant velocity implies constant speed.

Definition 3.2.1. *The unit of force is known as the* **Newton**. *We define* $N = kg \cdot m/s^2$.

What is the force ? Well, there are many various examples:

- Force of gravity near surface of earth on mass m has magnitude mg pointed towards the center of the earth.
- Spring with spring constant k is stretched ($x > 0$) or compressed ($x < 0$) from its equilbrium position ($x = 0$) then $F = -kx$ where we assume the motion of the spring is along the x-coordinate axis. Here the units of k might be N/m or N/cm as is sometimes provided to catch careless students[2]
- Pulling or pushing of one object on another.

[2]not that I'd do such a thing, well at least not without warning you, but, you're reading this so...

- Electric or Magnetic Force (next semester, relax for now)
- Friction forces

This list is by no means exhaustive.

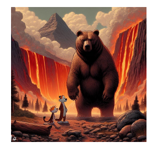
3.2.1 examples and problems involving Newton's Second Law

We now look at a variety of examples which illustrate how Newton's Second Law pairs with our previous work on vectors and calculus to describe the motion of physical objects in three dimensions. Ok, but first, let's be boring and look at a one-dimensional examples. In one-dimensional problems we can omit the vector notation.

Example 3.2.2. *If a mass $m = 3\,kg$ is observed to accelerate at $5\,m/s^2$ then the net-force on the mass is given by*

$$F = ma = (3\,kg)\left(5\frac{m}{s^2}\right) = 15\,\frac{kg\,m}{s^2} = 15\,N.$$

Example 3.2.3. *The* **weight** *of an object on earth is given by $F = mg$ where $g = 9.8m/s^2$. If we have a bear which weighs $250,000\,N$ then the mass of the bear is given by $m = \frac{250,000\,N}{9.8\,m/s^2} \cong 25510\,kg$. That's a big bear.*



Remark 3.2.4. *A given object has the same mass on any planet, but the weight depends on the gravitational acceleration of the planet where we find the mass. All weights given in this course are assumed to be Earth weights unless explicitly indicated otherwise.*

Example Problem 3.2.5. *A cat is thrown vertically with an initial velocity of $10\,m/s$ on the surface of a planet where it takes $2.0\,s$ for the cat to reach the apex of the flight. Find the force of gravity on the cat if the cat has mass $4\,kg$.*

Solution: *We need to find the acceleration of the cat. Let us assume the mass of the cat is constant and the force of gravity is constant over the motion which is a reasonable assumption for a large planet. Since $F_{gravity} = ma$ we see a is constant. Apply our handy constant acceleration formulas, note that $v_f = 0$ at the apex of the flight hence*

$$v_f = v_o + at \quad \Rightarrow \quad a = \frac{v_f - v_o}{t} = \frac{-10\,m/s}{2s} = -5\,m/s^2$$

Thus the force of gravity on the cat is $F_{gravity} = (4\,kg)(-5\,m/s^2) = \boxed{-20\,N}$.

Example Problem 3.2.6. *Suppose the net-force on an evil cat of mass m is given by $F = \alpha + 2\beta t$ as it is strapped onto a rocket cart to begin its journey into a live volcano. Find the velocity of the cat as a function of time given that it is intially at rest with $x = 0$ when $t = 0$.*

Solution: *Since we are not given values for m, α or β we expect the answer will involve these symbols. Apply Newton's Second Law, divide by m, and integrate with respect to time,*

$$ma = \alpha + 2\beta t \quad \Rightarrow \quad \frac{dv}{dt} = \frac{1}{m}(\alpha + 2\beta t) \quad \Rightarrow \quad \boxed{v = \frac{1}{m}\left(\alpha t + \beta t^2\right)}.$$

Of course, in this case it is NOT *true that* $v = v_o + at$ *since* $a = \frac{1}{m}(\alpha + 2\beta t)$ *is not constant. Please keep this in mind, we can only use the constant acceleration formulas when the acceleration is constant.*

Remark 3.2.7. *I am not missing units in the answer of the example above. The units are within the variables* α *and* β *which have units of* N *and* N/s *respectively. If I was to write* $v = \frac{1}{m}\left(\alpha t + \beta t^2\right) m/s$ *this would be dimensionally inconsistent nonsense. I take off points when you put incorrect units on answers. Consider this fair warning.*

Example 3.2.8. *Suppose the net force on a mass* m *is given by the velocity dependent friction force* $F = -\beta v$ *then we can solve Newton's Second Law via calculus. Let* v_f *and* v_o *denote the velocities at time* t_f *and* t_o *respective in what follows. Consider:*

$$m\frac{dv}{dt} = -\beta v \quad \Rightarrow \quad \int_{v_o}^{v_f} \frac{dv}{v} = \int_{t_o}^{t_f} \frac{-\beta dt}{m} \quad \Rightarrow \quad \ln|v_f| - \ln|v_o| = -\frac{\beta}{m}(t_f - t_o)$$

Setting $t_f = t$, $v_f = v$ *and* $t_o = 0$ *we derive* $\ln|v/v_o| = -\beta t/m$. *Exponentiating,*

$$|v/v_o| = e^{-\beta t/m} \quad \Rightarrow \quad v = \pm v_o e^{-\beta t/m} \quad \Rightarrow \quad v = v_o e^{-\beta t/m}$$

where in the last step we notice $m\frac{dv}{dt} = -\beta v$ *implies* v *is a differentiable function*[3] *of time* t *hence* v *is continuous at* $t = 0$ *and we must have* $\lim_{t\to 0} v(t) = v(0) = v_o$. *We also note that* $v(t) = v_o e^{-\beta t/m} \to 0$ *as* $t \to \infty$. *Then to find the position as a function of time, if* $x(0) = x_o$ *then noting* $v = \frac{dx}{dt}$ *we integrate the velocity function and derive*

$$\int_0^t \frac{dx}{d\tau} d\tau = \int_0^t v_o e^{-\beta\tau/m} d\tau \quad \Rightarrow \quad x - x_o = \frac{-mv_o}{\beta}(e^{-\beta t/m} - 1) \quad \Rightarrow \quad x = x_o + \frac{mv_o}{\beta}(1 - e^{-\beta t/m}).$$

Notice that the displacement $x - x_o \to \frac{mv_o}{\beta}$ *as* $t \to \infty$. *Finally, one last calculation we can study with this example is the problem of finding velocity as a function of position. Of course, one approach would be to solve the formula for* x *for* t *then just plug it into the velocity function. But, we prefer a calculus-based approach. We will use the same technique as when we derived the timeless equation for the constant acceleration problem. Notice*

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

hence as $F = ma = -\beta v$ *we face:*

$$mv\frac{dv}{dx} = -\beta v \quad \Rightarrow \quad \int_{v_o}^{v_f} dv = -\int_{x_o}^{x_f} \frac{\beta dx}{m} \quad \Rightarrow \quad v_f - v_o = -\frac{\beta}{m}(x_f - x_o)$$

Thus, setting $v_f = v$ *and* $x_f = x$ *we find* $v = v_o - \frac{\beta}{m}(x - x_o)$. *The velocity as a function of position is rather simple. Notice as* $x - x_o \to \frac{mv_o}{\beta}$ *we see* $v \to v_o - \frac{\beta}{m}\frac{mv_o}{\beta} = 0$. *The limit* $x \to \infty$ *is unphysical for this problem.*

In two or three dimensional problems we must include proper vector notation.

[3]notice, it is possible for velocity to be discontinuous if the force tends to infinity at a point, like with a hammer striking a nail, but that requires the mathematics of distributions which is properly part of Math 334 under the banner of Laplace Transforms

Example 3.2.9. *If a mass $m = 10\,kg$ is observed to accelerate at $\vec{a} = \langle 3,4\rangle m/s^2$ then the net-force on the mass is given by*

$$\vec{F} = m\vec{a} = (10\,kg)\langle 3,4\rangle m/s^2 = (10\,N)\langle 3,4\rangle = \langle 30,40\rangle N.$$

The magnitude of this net force is $F = \sqrt{30^2+40^2}\,N = 50\,N$ which is directed at the standard angle $\theta = \tan^{-1}(4/3) \approx 53.13^o$

Example Problem 3.2.10. *If a mass $m = 3\,kg$ has a $15\,N$ force placed on it in the $\langle 1,2,2\rangle$-direction then what is the acceleration of this mass if no other forces act on the mass ?*

Solution: *To find the net-force we need to find the unit-vector in the given direction. Since $\|\langle 1,2,2\rangle\| = \sqrt{1+4+4} = 3$ we find $\vec{F} = 15\,N\left(\frac{1}{3}\langle 1,2,2\rangle\right) = 5\,N\langle 1,2,2\rangle$. If $\vec{F} = m\vec{a}$ then $\vec{a} = \frac{1}{m}\vec{F}$ hence*

$$\vec{a} = \frac{5\,N}{3\;kg}\langle 1,2,2\rangle \quad\Rightarrow\quad \boxed{\vec{a} = \langle 5/3, 10/3, 10/3\rangle m/s^2}.$$

The examples given thus far were fairly tame. It is more fun when the net-force actually involves multiple forces.

Example Problem 3.2.11. *A unicorn is being pulled on by Naruto, Goku and Mickey Mouse. If Naruto pulls North with $1100\,N$ and Goku pulls East with $1000\,N$ and Mickey Mouse pulls South with $100\,N$ the find the acceleration of the unicorn given it has a mass of $500\,kg$. Also find the magnitude and direction in terms of standard angle for the acceleration.*

Solution: *Let us begin by finding the net force. Let $F_o = 100\,N$ for convenience*

$$\vec{F}_{net} = \underbrace{11F_o\langle 0,1\rangle}_{Naruto} + \underbrace{10F_o\langle 1,0\rangle}_{Goku} + \underbrace{F_o\langle 0,-1\rangle}_{Mickey\ Mouse} = \langle 10F_o, 10F_o\rangle.$$

Hence, by Newton's Second Law,

$$(500\,kg)\vec{a} = \langle 10F_o, 10F_o\rangle \quad\Rightarrow\quad \vec{a} = \left\langle \frac{10F_o}{500\,kg}, \frac{10F_o}{500\,kg}\right\rangle \quad\Rightarrow\quad \boxed{\vec{a} = \langle 20,20\rangle\,m/s^2}$$

The magnitude $a = \sqrt{20^2+20^2}\,m/s^2 \approx \boxed{28.28\,m/s^2}$ at $\boxed{\theta = 45^o}$.

Remark 3.2.12. *Sadly my art does not yet allow so many characters acting at once.*

Example Problem 3.2.13. *Suppose four forces act on an object which remains at rest. If $F_1 = 50\,N$ is applied in the direction 30^o West of North and $F_2 = 30\,N$ is applied 20^o South of East and $F_3 = 180\,N$ is applied 30^o North of East then find the magnitude and direction of the fourth force.*

Solution: *let us begin by converting the given angles into the standard angles for each given force; $\theta_1 = 120^o$ and $\theta_2 = -20^o$ and $\theta_3 = 30^o$. Thus,*

$$\begin{aligned}\vec{F}_1 &= 50\,N\langle\cos 120^o, \sin 120^o\rangle \cong \langle -25, 43.30\rangle N\\ \vec{F}_2 &= 30\,N\langle\cos(-20^o), \sin(-20^o)\rangle \cong \langle 28.19, -10.26\rangle N\\ \vec{F}_3 &= 180\,N\langle\cos(30^o), \sin(30^o)\rangle \cong \langle 155.88, 90\rangle N\end{aligned}$$

The net force must be zero since the object it at rest; $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 0$. *Solve for* $\vec{F}_4$,

$$\begin{aligned} \vec{F}_4 &= -(\vec{F}_1 + \vec{F}_2 + \vec{F}_3) \\ &= -(\langle -25 + 28.19 + 155.88, 43.30 - 10.26 + 90\rangle N) \\ &= \langle -159.07, -123.04\rangle N. \end{aligned}$$

Hence the magnitude of $\vec{F}_4$ *is given by* $F_4 = \sqrt{159.07^2 + 123.04^2}N \cong \boxed{201.1\,N}$. *Since* $\vec{F}_4$ *points in Quadrant III we know it has a standard angle which is between* 180^o *and* 270^o. *By geometry,*

$$\theta = 180^o + \tan^{-1}\left(\frac{123.04}{159.07}\right) \cong \boxed{217.7^o}$$

Example Problem 3.2.14. *Suppose Ron has a pair of rocket boots with thrust vectoring. The boots create a constant thrust of* 3 *times Ron's weight during operation. If Ron begins at rest and boosts vertically for time* $3.0\,s$ *then boosts at an angle of* 45^o *for another* $4.0\,s$. *If Ron then falls under projectile motion then find the horizontal distance traveled by Ron.*

Solution: *we divide the motion into three stages. Stage I, the net-force is given by*

$$\vec{F}_I = \underbrace{\langle 0, -mg\rangle}_{gravity} + \underbrace{\langle 0, 3mg\rangle}_{boots} = \langle 0, 2mg\rangle$$

hence $\vec{a}_I = \langle 0, 2g\rangle = \langle 0, 19.6m/s^2\rangle$. *Since Ron begins at rest we find the velocity after* $3.0\,s$ *of constant acceleration is given by* $\vec{v}_1 = \triangle t_1 \vec{a}_I = 3.0s\langle 0, 19.6m/s^2\rangle = \langle 0, 58.8m/s\rangle$. *The position at time* $t = 3.0\,s$ *is given by* $\vec{r}_I = \frac{1}{2}(\triangle t_1)^2\vec{a}_I = \langle 0, 88.2\,m\rangle$. *In stage II we'll assume the thrust from the boots points in the* $\langle 1, 1\rangle$ *direction so* $\vec{F}_{boots} = 3mg\langle 0.7071, 0.7071\rangle$ *and thus*

$$\vec{F}_{II} = m\vec{a}_{II} = \langle 0, -mg\rangle + 3mg\langle 0.7071, 0.7071\rangle = m\langle 20.79\,m/s^2, 10.99\,m/s^2\rangle$$

Thus, $\vec{a}_{II} = \langle 20.79\,m/s^2, 10.99\,m/s^2\rangle$ *from time* $t_1 = 3.0\,s$ *to time* $t_2 = 7.0\,s$ *where* $\triangle t_{II} = 4.0\,s$. *Once more we face constant acceleration motion so the formulas for finding the velocity and position at* $t = 7.0\,s$ *are simply:*

$$\vec{v}_{II} = \vec{v}_I + \triangle t_{II}\vec{a}_{II} \qquad \& \qquad \vec{r}_{II} = \vec{r}_I + \triangle t_{II}\vec{v}_I + \frac{1}{2}(\triangle t_{II})^2\vec{a}_{II}$$

We calculate,

$$\vec{v}_{II} = \langle 0,\ 58.8m/s\rangle + (4.0s)\langle 20.79\,m/s^2,\ 10.99\,m/s^2\rangle = \langle 83.16m/s,\ 102.76m/s\rangle$$

and

$$\vec{r}_{II} = \langle 0, 88.2\,m\rangle + (4.0s)\langle 0,\ 58.8m/s\rangle + \frac{1}{2}(4.0s)^2\langle 20.79\,m/s^2,\ 10.99\,m/s^2\rangle = \langle 166.32\,m,\ 411.32\,m\rangle$$

Finally, in stage III we face $\vec{a}_{III} = \langle 0, -9.8m/s^2\rangle$. *Thus*

$$\vec{r}_{III}(t) = \langle 166.32m + (t - 7.0s)83.16m/s, 411.32m + (t - 7.0s)102.76m/s - 4.9m/s^2(t - 7.0s)^2\rangle$$

In other words, setting $\triangle t = t - 7.0s$ *we face*

$$x = 166.32m + (83.16m/s)\triangle t \qquad \& \qquad y = 411.32m + 102.76m/s\triangle t - 4.9m/s^2\triangle t^2$$

Of course $y = 0$ *when Ron comes back to the ground hence we must solve the quadratic equation* $411.32m + 102.76m/s\triangle t - 4.9m/s^2\triangle t^2 = 0$ *which gives* $\triangle t \cong -3.44\,s,\ 24.41\,s$. *Clearly* $\triangle t = 24.41\,s$ *is the physically relevant solution hence we calculate the horizontal distance travelled by Ron is:*

$$x = 166.32m + (83.16m/s)(24.41\,s) = \boxed{2196\,m}.$$

3.3 necessity of inertial coordinate frames for Newton's Laws

The concept which is implicit, and absolutely necessary, to make proper application of Newton's Laws is the assumption that both the force $\vec{F}$ and the acceleration $\vec{a}$ along with the other kinematic variables like position $\vec{r}$ and velocity $\vec{v}$ must all be written with respect to an **inertial coordinate system**. In other words, Newton's Laws presuppose the existence of an inertial frame of reference with which we can judge speed, rest and directions of various vectors.

Definition 3.3.1. *Two coordinate systems $\vec{r}_1, \vec{r}_2$ to be* **inertially related** *if there exists a constant rotation matrix R and constant vectors $\vec{c}, \vec{b}$ for which*

$$\vec{r}_2 = R\,\vec{r}_1 + t\vec{b} + \vec{c}.$$

This equation means that the two coordinate systems under discussion are in constant velocity motion with respect to one another. We typically focus our attention on the case that our coordinate systems are not rotated with respect to one another, this means the matrix R drops from the consideration and we just face $\vec{r}_2 = \vec{r}_1 + t\vec{b} + \vec{c}$. That said, I'm including the rotation matrix as it is worthy of mention and is ultimately required if one wishes to understand interesting phenomenon like the Coriolis effect[4]. If we differentiate the equation above with respect to time t then we find

$$\frac{d\vec{r}_2}{dt} = R\frac{d\vec{r}_1}{dt} + \vec{b} \quad \Rightarrow \quad \boxed{\vec{v}_2 = R\,\vec{v}_1 + \vec{b}}$$

where we have denoted the velocity with respect to frame 1 as $\vec{v}_1 = \frac{d\vec{r}_1}{dt}$ and the velocity with respect to frame 2 as $\vec{v}_2 = \frac{d\vec{r}_2}{dt}$. Finally, notice that if we differentiate once more and denote $\vec{a}_2 = \frac{d\vec{v}_2}{dt}$ and $\vec{a}_1 = \frac{d\vec{v}_1}{dt}$ then

$$\boxed{\vec{a}_2 = R\,\vec{a}_1}.$$

Suppose the net-force on a mass m in frame 1 is $\vec{F}_1$ then by Newton's Second Law

$$m\vec{a}_1 = \vec{F}_1 \quad \Rightarrow \quad mR\,\vec{a}_1 = R\,\vec{F}_1 \quad \Rightarrow \quad m\vec{a}_2 = R\,\vec{F}_1$$

Since Newton's Second Law in frame 2 yields $m\vec{a}_2 = \vec{F}_2$ where $\vec{F}_2$ is the net-force measured in frame 2. Therefore, we find $\vec{F}_2 = R\vec{F}_1$. If the net-force is zero in a frame of reference then it will be zero in every other frame which inertially related to the given frame. Furthermore, if Newton's Laws hold in one inertial frame of reference then they will likewise hold in other frames of reference provided we use the transformation rule $\vec{F}_2 = R\vec{F}_1$ to rotate the force if need be.

3.3.1 accelerated frames of reference

Remark 3.3.2. *This section can be skipped in a first reading of the subject*

Let $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ denote the Cartesian coordinate frame of a fixed inertial frame of reference we label S. Then, suppose is the coordinate frame $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ of a moving frame of reference

$$\vec{r}_S = \vec{r}_o + \bar{x}\hat{\mathbf{u}} + \bar{y}\hat{\mathbf{v}} + \bar{z}\hat{\mathbf{w}} = \vec{r}_o + \vec{r}_{\bar{s}}$$

[4]see http://www.supermath.info/CoriolisEffect.pdf

here $\vec{r}_o$ is the position of the moving frame with respect to the fixed frame whereas $\vec{r}_{\bar{S}} = \bar{x}\hat{\mathbf{u}}+\bar{y}\hat{\mathbf{v}}+\bar{z}\hat{\mathbf{w}}$ is the position with respect to the moving frame $\bar{S}$. We define

$$\vec{v}_{\bar{s}} = \frac{d\bar{x}}{dt}\hat{\mathbf{u}} + \frac{d\bar{y}}{dt}\hat{\mathbf{v}} + \frac{d\bar{z}}{dt}\hat{\mathbf{w}} \neq \frac{d\vec{r}_{\bar{s}}}{dt} \qquad \& \qquad \vec{a}_{\bar{s}} = \frac{d^2\bar{x}}{dt^2}\hat{\mathbf{u}} + \frac{d^2\bar{y}}{dt^2}\hat{\mathbf{v}} + \frac{d^2\bar{z}}{dt^2}\hat{\mathbf{w}} \neq \frac{d^2\vec{r}_{\bar{s}}}{dt^2}.$$

The derivatives involve terms arising from the change in the moving frame $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$. Notice

$$\frac{d\vec{r}_S}{dt} = \frac{d\vec{r}_o}{dt} + \frac{d\bar{x}}{dt}\hat{\mathbf{u}} + \frac{d\bar{y}}{dt}\hat{\mathbf{v}} + \frac{d\bar{z}}{dt}\hat{\mathbf{w}} + \bar{x}\frac{d\hat{\mathbf{u}}}{dt} + \bar{y}\frac{d\hat{\mathbf{v}}}{dt} + \bar{z}\frac{d\hat{\mathbf{w}}}{dt}$$

thus

$$\boxed{\vec{v}_S = \vec{v}_o + \vec{v}_{\bar{S}} + \bar{x}\frac{d\hat{\mathbf{u}}}{dt} + \bar{y}\frac{d\hat{\mathbf{v}}}{dt} + \bar{z}\frac{d\hat{\mathbf{w}}}{dt}}$$

Next, differentiating the expression above yields

$$\begin{aligned}
\vec{a}_S = \frac{d\vec{v}_S}{dt} &= \frac{d\vec{v}_o}{dt} + \frac{d}{dt}\left[\frac{d\bar{x}}{dt}\hat{\mathbf{u}} + \frac{d\bar{y}}{dt}\hat{\mathbf{v}} + \frac{d\bar{z}}{dt}\hat{\mathbf{w}} + \bar{x}\frac{d\hat{\mathbf{u}}}{dt} + \bar{y}\frac{d\hat{\mathbf{v}}}{dt} + \bar{z}\frac{d\hat{\mathbf{w}}}{dt}\right] \\
&= \frac{d\vec{v}_o}{dt} + \frac{d^2\bar{x}}{dt^2}\hat{\mathbf{u}} + \frac{d^2\bar{y}}{dt^2}\hat{\mathbf{v}} + \frac{d^2\bar{z}}{dt^2}\hat{\mathbf{w}} + 2\frac{d\bar{x}}{dt}\frac{d\hat{\mathbf{u}}}{dt} + 2\frac{d\bar{y}}{dt}\frac{d\hat{\mathbf{v}}}{dt} + 2\frac{d\bar{z}}{dt}\frac{d\hat{\mathbf{w}}}{dt} + \bar{x}\frac{d^2\hat{\mathbf{u}}}{dt^2} + \bar{y}\frac{d^2\hat{\mathbf{v}}}{dt^2} + \bar{z}\frac{d^2\hat{\mathbf{w}}}{dt^2} \\
&= \vec{a}_o + \vec{a}_{\bar{S}} + 2\frac{d\bar{x}}{dt}\frac{d\hat{\mathbf{u}}}{dt} + 2\frac{d\bar{y}}{dt}\frac{d\hat{\mathbf{v}}}{dt} + 2\frac{d\bar{z}}{dt}\frac{d\hat{\mathbf{w}}}{dt} + \bar{x}\frac{d^2\hat{\mathbf{u}}}{dt^2} + \bar{y}\frac{d^2\hat{\mathbf{v}}}{dt^2} + \bar{z}\frac{d^2\hat{\mathbf{w}}}{dt^2}
\end{aligned}$$

thus

$$\boxed{\vec{a}_S = \vec{a}_o + \vec{a}_{\bar{S}} + 2\frac{d\bar{x}}{dt}\frac{d\hat{\mathbf{u}}}{dt} + 2\frac{d\bar{y}}{dt}\frac{d\hat{\mathbf{v}}}{dt} + 2\frac{d\bar{z}}{dt}\frac{d\hat{\mathbf{w}}}{dt} + \bar{x}\frac{d^2\hat{\mathbf{u}}}{dt^2} + \bar{y}\frac{d^2\hat{\mathbf{v}}}{dt^2} + \bar{z}\frac{d^2\hat{\mathbf{w}}}{dt^2}.}$$

Example 3.3.3. *Let $\bar{S}$ be the rotating frame of reference where*

$$\vec{r}_o = \langle R\cos\omega t, R\sin\omega t, 0\rangle$$

Furthermore, $\hat{\mathbf{u}} = \langle\cos\omega t, \sin\omega t, 0\rangle$ and $\hat{\mathbf{v}} = \langle -\sin\omega t, \cos\omega t, 0\rangle$ and $\hat{\mathbf{w}} = \hat{\mathbf{z}}$. Then

$$\frac{d\hat{\mathbf{u}}}{dt} = \omega\langle -\sin\omega t, \cos\omega t\rangle = \omega\hat{\mathbf{v}} \qquad \& \qquad \frac{d\hat{\mathbf{v}}}{dt} = \omega\langle -\cos\omega t, -\sin\omega t\rangle = -\omega\hat{\mathbf{u}}$$

thus $\frac{d^2\hat{\mathbf{u}}}{dt^2} = -\omega^2\hat{\mathbf{u}}$ and $\frac{d^2\hat{\mathbf{v}}}{dt^2} = -\omega^2\hat{\mathbf{v}}$. Since $\hat{\mathbf{z}}$ is constant we also have $\frac{d\hat{\mathbf{z}}}{dt} = \frac{d^2\hat{\mathbf{z}}}{dt^2} = 0$. Using the formulas derived in this section, notice $\hat{\mathbf{w}} \times \hat{\mathbf{u}} = \hat{\mathbf{v}}$ and $\hat{\mathbf{w}} \times \hat{\mathbf{v}} = -\hat{\mathbf{u}}$,

$$\vec{v}_S = \vec{v}_o + \vec{v}_{\bar{S}} + \omega\bar{x}\hat{\mathbf{v}} - \omega\bar{y}\hat{\mathbf{u}} = \vec{v}_o + \vec{v}_{\bar{S}} + (\omega\hat{\mathbf{w}}) \times (\bar{x}\hat{\mathbf{u}} + \bar{y}\hat{\mathbf{v}} + \bar{z}\hat{\mathbf{w}}) = \vec{v}_o + \vec{v}_{\bar{S}} + \vec{\omega} \times \vec{r}_{\bar{S}}$$

where $\vec{\omega} = \omega\hat{\mathbf{w}}$. Continuing[5]*,*

$$\vec{a}_S = \vec{a}_o + \vec{a}_{\bar{S}} + 2\omega\frac{d\bar{x}}{dt}\hat{\mathbf{v}} - 2\omega\frac{d\bar{y}}{dt}\hat{\mathbf{u}} - \omega^2\bar{x}\hat{\mathbf{u}} - \omega^2\bar{y}\hat{\mathbf{v}} = \vec{a}_o + \vec{a}_{\bar{S}} + 2\vec{\omega} \times \vec{r}_{\bar{S}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{\bar{S}})$$

In summary, if S denotes the fixed frame of reference and $\bar{S}$ the rotating frame with axis $\vec{\omega}$ and angular rotation rate ω then

$$\boxed{\vec{v}_S = \vec{v}_o + \vec{v}_{\bar{S}} + \vec{\omega} \times \vec{r}_{\bar{S}}} \qquad \& \qquad \boxed{\vec{a}_S = \vec{a}_o + \vec{a}_{\bar{S}} + 2\vec{\omega} \times \vec{r}_{\bar{S}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{\bar{S}})}$$

If we solve for $\vec{a}_{\bar{S}}$ and use $m\vec{a}_S = \vec{F}_{net}$ then the analog of Newton's Second Law for the rotating frame is:

$$m\vec{a}_{\bar{S}} = \vec{F}_{net} - m\vec{a}_o - 2m\vec{\omega} \times \vec{r}_{\bar{S}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_{\bar{S}})$$

where we see three ficticious forces arising from the rotation of the frame.

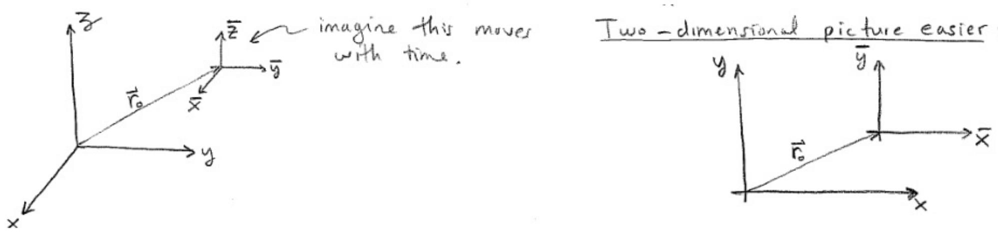
[5] I use $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{C} \bullet \vec{A})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$ identity with $\vec{A} = \vec{B} = \vec{\omega}$ and $\vec{C} = \vec{r}_{\bar{S}}$ hence $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{\bar{S}}) = (\vec{\omega} \bullet \vec{r}_{\bar{S}})\vec{\omega} - (\vec{\omega} \bullet \vec{\omega})\vec{r}_{\bar{S}} = \omega^2\bar{z}\hat{\mathbf{w}} - \omega^2(\bar{x}\hat{\mathbf{u}} + \bar{y}\hat{\mathbf{v}} + \bar{z}\hat{\mathbf{w}} = -\omega^2(\bar{x}\hat{\mathbf{u}} + \bar{y}\hat{\mathbf{v}})$.

3.4 relative motion

Let us consider two frames of reference which both take coordinate directions in the same sense. In particular,

$$\vec{r}_S = \vec{r}_o + \vec{r}_{\bar{S}}$$

where $\vec{r}_{\bar{S}} = \bar{x}\hat{\mathbf{x}} + \bar{y}\hat{\mathbf{y}} + \bar{z}\hat{\mathbf{z}} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ and $\vec{r}_S = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = \langle x, y, z \rangle$. We assume[6] $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are independent of time t. Furthermore, let us denote $\vec{r}_o = \langle x_o, y_o, z_o \rangle$



Differentiating once than twice gives transformation laws for velocity and acceleration:

$$\vec{v}_S = \vec{v}_o + \vec{v}_{\bar{S}} \qquad \& \qquad \vec{a}_S = \vec{a}_o + \vec{a}_{\bar{S}}.$$

Notice $\vec{a}_{\bar{S}} = \vec{a}_S - \vec{a}_o$ and since Newton's Second Law holds in S we find $m\vec{a}_S = \vec{F}_{net}$ thus

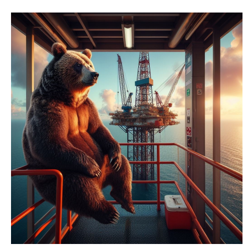
$$\boxed{m\vec{a}_{\bar{S}} = \vec{F}_{net} - m\vec{a}_o}$$

In the accelerated frame of reference $\bar{S}$ we find the acceleration of the origin of the frame appears as a *ficticious force*. The term $m\vec{a}_o$ appears to be a force, but in reality it is merely a frame-effect. Of course, we can feel frame effects, so they are certainly real in that regard. Most amusement park rides are one frame effect after another throwing you this way and that. The fictional aspect is that $m\vec{a}_o$ is not a force in the same way as $\vec{F}_{net}$ is a force. Notice this sets up a principle; forces which are proportional to mass are potentially frame effect forces (like $F = mg$). This is part of what made Einstein pursue General Relativity which takes this idea to its logical end and proposes gravity itself is a frame effect. The mathematics to explain that is a bit beyond this course, but you can begin to get the idea here.

Example 3.4.1. *Suppose you throw a baseball vertically with speed v_o on a train traveling v_T in the North East direction. Then the velocity of the baseball relative the earth is given by*

$$\vec{v} = v_T\langle 0.707, 0.707, 0 \rangle + v_o\langle 0, 0, 1 \rangle.$$

Example 3.4.2. *If a bear stands on a scale in an elevator which accelerates upward at $a = g$ then we can find the weight of the bear on the scale by considering the elevator as an accelerated reference frame with $a_o = g$. Since the bear is at rest in the elevator we have $a_{\bar{S}} = 0$. The forces on the scale include the weight mg of the bear and the normal force F_w from the scale pushing back. This normal force causes the weight reading on the scale.*



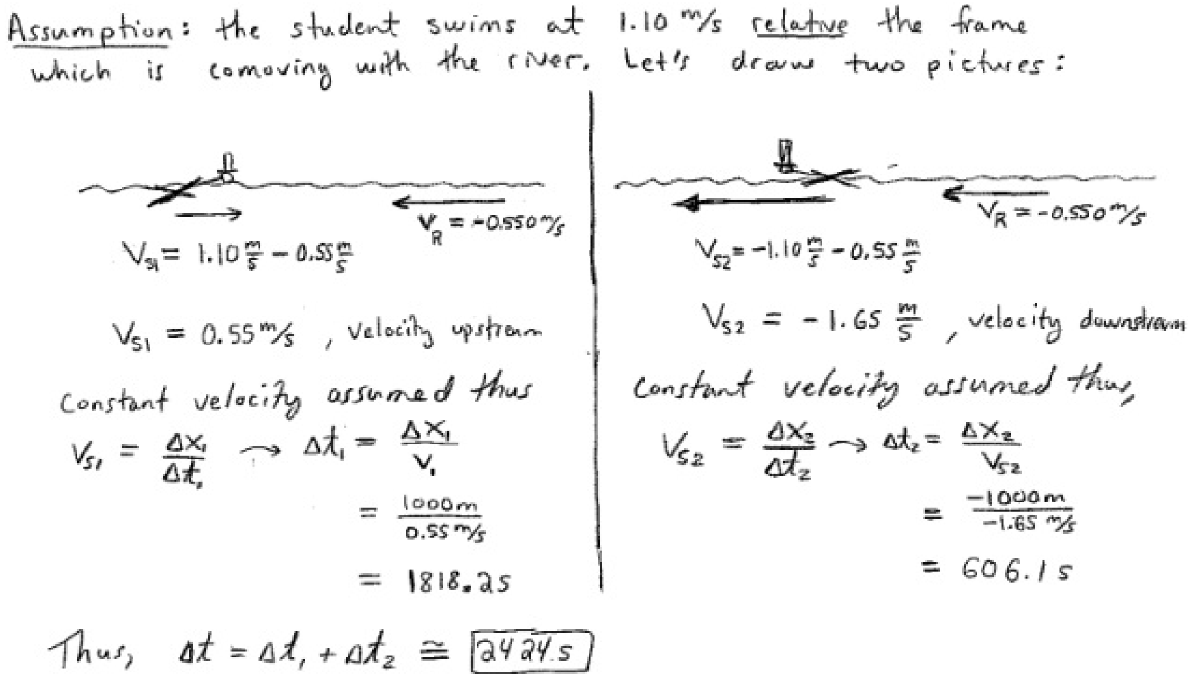
$$0 = F_{net} - ma_o = -mg + F_w - mg \quad \Rightarrow \quad \boxed{F_w = 2mg}$$

If the elevator accelerated downward at g then similar calculation shows the bear is weightless.

[6] feel free to peruse the previous section if you wish to see what the calculational impact of allowing a time-variate basis entails

Example Problem 3.4.3. *A river has a speed of* $0.550m/s$*. Suppose a student swims up the river (against the current) a distance of* $1.00km$ *and then swims back to where he began. If the student can swim* $1.10m/s$ *in still water then how long did his swim in the river take ?*

Solution:



3.5 circular motion

Motion in a circle is a common problem. We develop some tools to deal with such problems in this section. Our analysis rests both on calculus and vectors. Let us begin with the equation of a circle of radius R and center $\vec{r}_o$. The typical point $\vec{r}$ is distance R from $\vec{r}_o$ hence

$$(\vec{r}-\vec{r}_o)\bullet(\vec{r}-\vec{r}_o)=R^2$$

Differentiate with respect to time, use product rule:

$$\frac{d\vec{r}}{dt}\bullet(\vec{r}-\vec{r}_o)+(\vec{r}-\vec{r}_o)\bullet\frac{d\vec{r}}{dt}=0 \quad\Rightarrow\quad \vec{v}\bullet(\vec{r}-\vec{r}_o)=0$$

Differentiate once more, to obtain

$$\frac{d^2\vec{r}}{dt^2}\bullet(\vec{r}-\vec{r}_o)+\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}=0 \quad\Rightarrow\quad \vec{a}\bullet(\vec{r}-\vec{r}_o)+\vec{v}\bullet\vec{v}=0$$

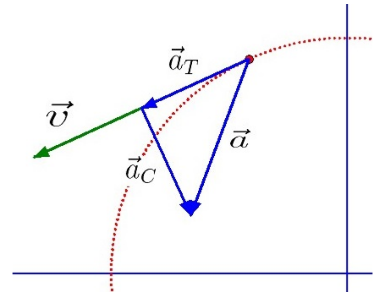
If we use $\hat{r}$ for the unit-vector pointing in the $\vec{r}-\vec{r}_o$ then $\vec{r}-\vec{r}_o=R\hat{r}$. Thus, as $\vec{v}\bullet\vec{v}=v^2$,

$$\vec{a}\bullet R\hat{r}=-v^2 \quad\Rightarrow\quad \vec{a}\bullet\hat{r}=-\frac{v^2}{R}$$

Recall that $\vec{v}=vT$ hence $\vec{a}=\frac{dv}{dt}T+v\frac{dT}{dt}=\frac{dv}{dt}T+\kappa v^2N$ by the Frenet Serret equation $\frac{dT}{dt}=\kappa vN$ where N is the center-pointing unit-normal. Identify $N=-\hat{r}$ so $\vec{a}\bullet N=-(\vec{a}\bullet(-\hat{r})$ thus $v^2\kappa=$

v^2/R. It follows we have shown that

$$\vec{a} = \frac{dv}{dt}T - \frac{v^2}{R}\hat{r}.$$



If the circular motion is constant speed then $\frac{dv}{dt} = 0$ so $\vec{a} = -\frac{v^2}{R}\hat{r}$. In general for circular motion we have

$$a = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{R^2}}$$

Here $\vec{a}_c = -\frac{v^2}{R}\hat{r}$ whereas $\vec{a}_T = \frac{dv}{dt}T$. We can have $\vec{a}_T = 0$ however the same is not true for the centripetal part. If the motion is circular then there has to be a center-seeking component of the acceleration with magnitude given by v^2/R.

Example 3.5.1. *Suppose a $1500kg$ car goes around a circular track at constant speed. If the track has a radius of $R = 200\,m$ and the speed is $v = 50\,m/s$ then the acceleration of the car is center-seeking with magnitude*

$$a = \frac{v^2}{R} = \frac{(50m/s)^2}{200\,m} = 12.5m/s^2$$

Newton's Second Law from the Earth frame gives $ma = F_{net,radial}$. The force responsible for this motion is the friction force[7] *of the tires against the track $F_{net,radial} = F_f$. We find $F_f = ma = (1500kg)(12.5m/s^2) = 18.75kN$.*

Example 3.5.2. *The Earth spins around its axis once a day $(1day = 86,400s)$. The radius of the Earth is approximately $R = 6371km$. Thus,*

$$v = \frac{2\pi R}{86,400s} = 463.3m/s$$

which is about $1036.6mph$. Wow, it seems amazing the folks on the equator don't go flying off into space... until you calculate the centripetal acceleration:

$$a_c = \frac{v^2}{R} = \frac{(463.3m/s)^2}{6371km} = 0.034m/s^2$$

For a person standing on the equator the net-acceleration in the center-seeking is a mere $0.034m/s^2$. This acceleration is the result of gravity mg balancing against the normal force F_N of the ground on your feet.

$$-m(0.034m/s^2) = -mg + F_N$$

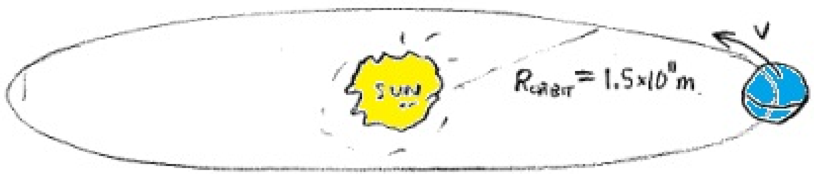
In other words, $F_N = mg - ma_c = m(g - 0.034m/s^2)$ which means you'd weigh approximately $\frac{9.8-0.034}{9.8} \times 100\% = 99.7\%$ of your weight at the North Pole. Notice, at the North Pole, your acceleration towards the center of the Earth would simply be zero and there Newton's Second Law gives $0 = -mg + F_N$ so the normal force $F_N = mg$. In most cases we neglect the rotation of the Earth in our examples.

[7] more on this next chapter

Example 3.5.3. *The Earth rotates around the Sun each year in a roughly circular path. We can show the acceleration from the yearly orbit amounts to a mere* $a = 0.00595m/s^2$.

$$V_{orb.} = \frac{2\pi R_{orbit}}{(365)(86400s)} = \frac{(2\pi)(1.5\times 10^{11}m)}{(365)(86400)} = 29885.8\,\frac{m}{s}$$

$$a_c = \frac{(29885.8\,m/s)^2}{1.5\times 10^{11}m} = \underline{0.00595\,m/s^2}.$$



Chapter 4

Application of Newton's Laws

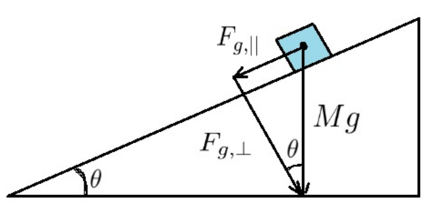
Newton's Second Law is a vector law. We study examples in this Chapter where typically more than one direction come into play. To organize the work we use a standard book-keeping device: the **free body diagram**. When multiple masses are studied at once we often draw a free-body-diagram for each mass. Newton's Third Law is also seen in this Chapter as we balance forces at interfaces. We also introduce the **tension** force of an unstretchable rope which requires some common sense in its application. Both **static** and **kinetic friction** are introduced. Static friction is especially subtle since it is characterized by an inequality rather than a simple equality. We also introduce agents (usually Mr. Tophat) which push or pull on a system with a given force in some direction. Finally, as if all this was not enough, we put systems on inclined planes where gravity is split between both the parallel and perpendicular directions.

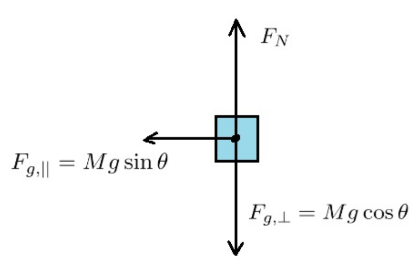
4.1 free body diagrams and friction

Example 4.1.1. *Suppose a box with mass M is placed on an inclined plane with angle of inclination θ. Then notice the force of gravity on the box has magnitude $F_g = Mg$, but this force partly aligns with both the direction parallel to the plane and to the direction perpendicular to the plane. Observe*

$$F_{g,||} = Mg\sin\theta \qquad \& \qquad F_{g,\perp} = Mg\cos\theta$$

When $\theta = 90^o$ the formulas above give $F_{g,||} = Mg$ and $F_{g,\perp} = 0$ which makes sense in this limiting case. Checking limiting cases is a good habit to cultivate in Physics. The plane pushes against the box with the normal force F_N which is directed in the perpendicular direction.





The free body diagram has gravity pictured as two components. Notice the acceleration is nontrivial only in the parallel direction. We expect $a_\perp = 0$ since the box is on the plane. Notice:

$$Ma = Mg\sin\theta \qquad \& \qquad 0 = F_N - Mg\cos\theta$$

Thus $a = g\sin\theta$ and $F_N = Mg\cos\theta$. Once more notice the edge case $\theta = 0$ makes good sense as it gives $a = 0$ and $F_N = Mg$ as we expect.

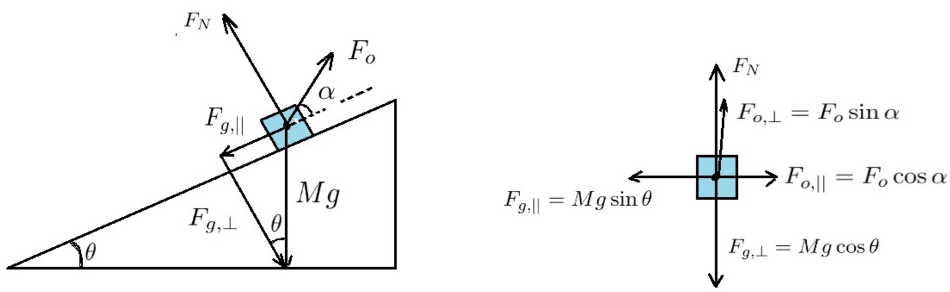
Example 4.1.2. *Suppose a box with mass M is placed on an inclined plane with angle of inclination θ. In addition a force F_o pulls on the mass at an angle α as pictured. Notice*

$$F_{o,||} = F_o\cos\alpha \qquad \& \qquad F_{o,\perp} = F_o\sin\alpha$$

and as in our previous example,

$$F_{g,||} = Mg\sin\theta \qquad \& \qquad F_{g,\perp} = Mg\cos\theta$$

The free-body-diagram plots the parallel and perpendicular vector components of each force which acts on M (in this example, there are three forces, gravity $\vec{F}_g$, the normal force $\vec{F}_N$ and the pulling force $\vec{F}_o$)



The free body diagram has gravity pictured as two components. Notice the acceleration is nontrivial only in the parallel direction. We expect $a_\perp = 0$ since the box is on the plane. Notice:

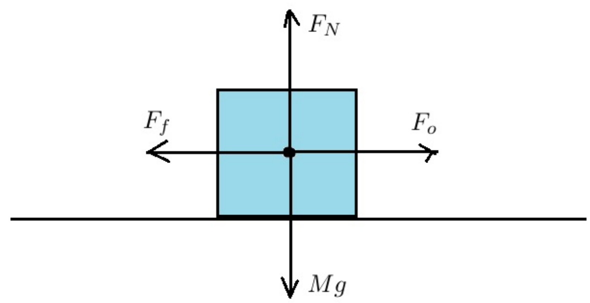
$$Ma = Mg\sin\theta - F_o\cos\alpha \qquad \& \qquad 0 = F_N - Mg\cos\theta + F_o\sin\alpha$$

Thus $a = g\sin\theta - (F_o/M)\cos\alpha$ and $F_N = Mg\cos\theta - F_o\sin\alpha$. The external force $\vec{F}_o$ changes both the acceleration and the normal force.

Definition 4.1.3. *Let F_N be the magnitude of the normal force acting on an object placed on a surface. The force of* **static friction** *is a force directed opposite the direction of potential motion and it has a magnitude of $F_f \leq \mu_S F_N$. The constant μ_S is called the* **coefficient of static friction** *and it is characteristic of both the material of the object as well as the surface on which the object rests. Likewise, the force of* **kinetic friction** *is a force directed opposite the direction of the velocity with a magnitude $F_f = \mu_k F_N$. The constant μ_k is characteristic of the types of material forming both the object and the surface.*

Notice that $0 \leq F_f \leq \mu_S F_N$ in the static case. This is a bit tricky since the available force is variable. We call $\mu_S F_N$ the **maximum force of static friction**. It should also be mentioned these rules are not like other laws of physics. In practice these are approximations of an incredibly complicated process of one microscopic landscape rubbing against another. Pressure, humidity and speed in the kinetic case all play a role in real physical motion. In short, the simplistic model of friction defined above is a toy macroscopic description. Short of more complicated models and a wealth of additional data it is the best we can do. We do study some simplistic models of friction in gases or liquids as well from time to time in this course, however those mostly appear as exercises in the methods and application of calculus to kinematics and dynamics.

Example 4.1.4. *A box with mass M rests on a horizontal plane and we push against it with force of magnitude F_o directed horizontally rightward. In this case, the friction force acts leftward. Vertically, Newton's Second Law gives $Ma_y = F_N - Mg = 0$ hence $F_N = Mg$. Horizontally, since $F_f \leq \mu_S F_N = \mu_S Mg$, we find in the case of maximum static friction:*



$$Ma_x = F_o - \mu_S Mg.$$

However, in the static case, $a_x = 0$ thus $F_o = \mu_S Mg$.
If we push with a force with magnitude larger than $\mu_S Mg$ then the analysis above breaks down since we cannot offer sufficient frictional force to oppose the potential motion. If $F_o > \mu_S Mg$ then $F_f = \mu_k Mg$ where $\mu_k < \mu_s$ and

$$Ma = F_o - \mu_k Mg \quad \Rightarrow \quad a = F_o/M - \mu_k g.$$

Example Problem 4.1.5. *Consider masses M_1, M_2 connected by a very light unstretchable rope over an essentially massless pulley. Find the tension in the rope and the friction force in the static case and the acceleration of the system in the kinetic case. Use μ_s and μ_k for the respective static and kinetic coefficients of friction.*

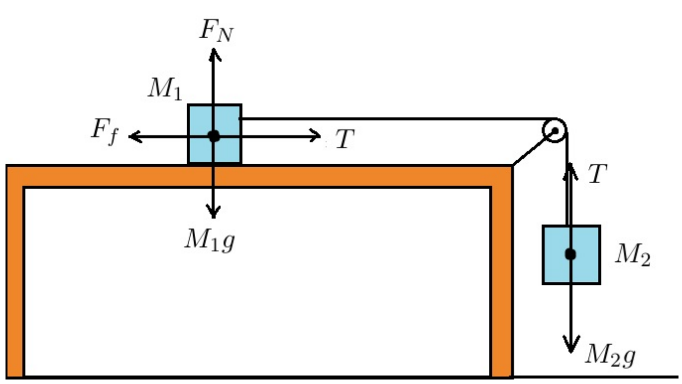
Solution: *Notice that $M_1 a_y = F_N - M_1 g$ by Newton's Second Law and since $a_y = 0$ we have $F_N = M_1 g$ thus $F_f \leq \mu_s M_1 g$ if the system is at rest or $F_f = \mu_k M_1 g$ if the mass is sliding. If we take rightward motion of M_1 as positive then Newton's Second Law for M_1 and M_2 are given by*

$$M_1 a = T - \mu_k M_1 g \qquad \& \qquad M_2 a = M_2 g - T$$

Hence, adding the equations above and solving for a yields

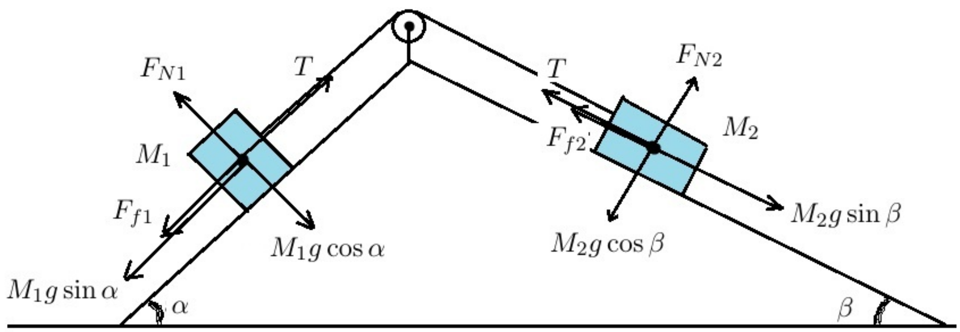
$$\boxed{a = \frac{M_2 g - \mu_k M_1 g}{M_1 + M_2}}.$$

If the system is at rest then $\boxed{T = M_2 g}$ from Newton's Second Law applied to M_2. Likewise, as $0 = T - F_f$ we find $\boxed{F_f = M_2 g}$ in the case the box is not sliding. Notice the static case presumes that $M_2 g \leq \mu_s M_1 g$ as F_f cannot exceed a magnitude of $\mu_s M_1 g$ in this context.



Example Problem 4.1.6. *Consider mass M_1 and M_2 connected by a rope of very small mass over a pulley with neglible mass. Let μ_k be the coefficient of kinetic friction between the inclined plane pictured below and the masses which are put in motion to the right. Find the acceleration of the system taking rightward overall motion as positive.*

Solution: *the perpendicular acceleration of both masses is zero since both M_1 and M_2 are on the planes in question. Thus $F_{N1} - M_1 g\cos\alpha = 0$ and $F_{N2} - M_2 g\cos\beta = 0$ where we've used the usual trigonometry to break down $-M_1 g\hat{\mathbf{y}}$ and $-M_2 g\hat{\mathbf{y}}$ into their respective perpendicular components. Then we find the magnitude of the friction force on M_1 as $F_{f1} = \mu_k F_{N1} = \mu_k M_1 g\cos\alpha$ and that on M_2 as $F_{f2} = \mu_k F_{N2} = \mu_k M_2 g\cos\beta$.*



Let T be the tension force of the rope then we find Newton's Second Law for M_1 yields

$$M_1 a = T - \mu_k M_1 g\cos\alpha - M_1 g\sin\alpha$$

whereas for M_2 we find[1]

$$M_2 a = -T - \mu_k M_2 g\cos\beta + M_2 g\sin\beta$$

Now we wish to solve for a. Adding the equations gives

$$(M_1 + M_2)a = -\mu_k M_1 g\cos\alpha - M_1 g\sin\alpha - \mu_k M_2 g\cos\beta + M_2 g\sin\beta$$

Thus,

$$a = \frac{g}{M_1 + M_2}\left(M_2\sin\beta - M_1\sin\alpha - \mu_k(M_1\cos\alpha + M_2\cos\beta)\right)$$

Remark 4.1.7. *If the situation in the last example is duplicated, but the system is at rest then $a = 0$. In that case the analysis leading to Newton's Laws for masses M_1 and M_2 is mostly unchanged. However, rather than assuming F_{f1} and F_{f2} are at their static maximums of $\mu_s M_2 g\cos\beta$ and $\mu_s M_1 g\cos\alpha$ respective, we should leave the friction forces symbollically as F_{f1} and F_{f2} since all we know is that in the static case $|F_{f1}| \leq \mu_s M_1 g\cos\alpha$ and $|F_{f2}| \leq \mu_s M_2 g\cos\beta$. Then,*

$$0 = T - F_{f1} - M_1 g\sin\alpha \qquad \& \qquad 0 = -T - F_{f2} + M_2 g\sin\beta$$

Add the equations above to obtain $-F_{f1} - M_1 g\sin\alpha - F_{f2} + M_2 g\sin\beta = 0$. Hence:

$$F_{f1} + F_{f2} = M_2 g\sin\beta - M_1 g\sin\alpha$$

For a given set of masses and angles there are infinitely many solutions to the equation above which is constrained by the inequalities $-\mu_s M_1 g\cos\alpha \leq F_{f1} \leq \mu_s M_1 g\cos\alpha$ and $-\mu_s M_2 g\cos\beta \leq$

[1]notice the fact that the rope is massless and does not stretch forces us to conclude that both masses share the same acceleration and that the tension is constant in magnitude in the rope

$F_{f2} \leq \mu_s M_2 g \cos\beta$. *We can visualize the solution set by making a graph with axes representing the* F_{f1} *and* F_{f2}. *The solution allows many possible values for* F_{f1} *and* F_{f2} *subject to the inequalities and equation given in this remark. To make the answer unique, we'd need to specify additional information.*

Remark 4.1.8. *See http://www.supermath.info/physics231lecture9.pdf for several additional examples of inclined planes and forces.*

4.2 contact forces and Newton's 3rd Law

Example 4.2.1. *two boxes with horizontal pushing force*

Example 4.2.2. *three boxes with horizontal pushing force*

Example 4.2.3. *two boxes with pushing force at angle and friction*

Chapter 5

energy methods

I should mention, the first two sections in this chapter are more or less aligned with my handwritten crash course in multivariate calculus seen here: http://www.supermath.info/physics231lecture15.pdf. The order of topics here is a bit different in that I begin with partial differentiation and then later discuss integration. I'm trying something different in the Spring 2025 term to see how it lands. The later sections concern how we can use energy conservation and work done by forces to solve physical problems. Finally we study simple harmonic motion with a focus on springs. The simple pendulum is given as an example of a system which is locally behaving as a simple harmonic oscillator. Finally we introduce energy analysis as a means to analyze possible motions of one-dimensional system in view of the total energy and a given potential energy plot. This chapter corresponds to Chapters 6 and 7 of the 9th edition of the Young and Freedman Physics text.

5.1 multivariate differential calculus

If the function has more than one independent variable in its domain then when we study the change in the function we need to calculate **partial derivatives** for the function.

Definition 5.1.1. *Given a function of several variables say $x_1, \dots, x_n$ we define for $f = f(x_1, \dots, x_n)$,*

$$\frac{\partial f}{\partial x_i}(p) = \lim_{h \to 0} \frac{f(p + h\hat{x}_i) - f(p)}{h} = \frac{d}{dt}\left[f(p + t\hat{x}_i)\right]\bigg|_{t=0}$$

The formula above says we study the change in f at p as it changes along the $\hat{x}_i$-direction. I'll organize the rest of this section in terms of context. There are three I'll consider, (1.) $\mathbb{R}^2$ with x, y variables, (2.) $\mathbb{R}^3$ with x, y, z variables, and (3.) $\mathbb{R}^n$ with $x_1, \dots, x_n$ variables.

(1.) For $\mathbb{R}^2$ containing $p_0 = (x_0, y_0)$ the partial derivatives can be formulated as

$$\frac{\partial f}{\partial x}(p_0) = \frac{d}{dx} f(x, y_0)\bigg|_{x=x_0} \qquad \& \qquad \frac{\partial f}{\partial y}(p_0) = \frac{d}{dy} f(x_0, y)\bigg|_{y=y_0}.$$

In other words, to calculate a partial derivative we differentiate as usual while assuming the non-differentiated variables are held constant. For $\mathbb{R}^2$ we have

$$\frac{\partial x}{\partial x} = 1, \quad \& \quad \frac{\partial x}{\partial y} = 0, \quad \& \quad \frac{\partial y}{\partial x} = 0, \quad \& \quad \frac{\partial y}{\partial y} = 1.$$

Given $f = f(x, y)$ and $g = g(x, y)$ and a constant c the rules of calculus are natural

$$\frac{\partial}{\partial x}(f + g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \quad \& \quad \frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}, \quad \& \quad \frac{\partial}{\partial x}(cf) = c\frac{\partial f}{\partial x}.$$

$$\frac{\partial}{\partial y}(f+g) = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}, \quad \& \quad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}, \quad \& \quad \frac{\partial}{\partial y}(cf) = c\frac{\partial f}{\partial y}.$$

Example 5.1.2. *Calculate* $\frac{\partial}{\partial x}(x^3y+y^2) = \frac{\partial}{\partial x}(x^3)y + \frac{\partial}{\partial x}(y^2) = 3x^2y$. *since* $\frac{\partial}{\partial x}(y^2) = 0$ *as* y *is held constant in the* x*-partial differentiation. Likewise,*

$$\frac{\partial}{\partial y}(x^3y+y^2) = x^3\frac{\partial y}{\partial y} + \frac{\partial}{\partial y}(y^2) = x^3 + 2y.$$

Example 5.1.3. *Notice* $\frac{\partial}{\partial x}(x^y) = yx^{y-1}$ *whereas* $\frac{\partial}{\partial y}(x^y) = \ln(x)x^y$. *These claims follow from Calculus I where we learned* $\frac{d}{dt}t^n = nt^{n-1}$ *whereas* $\frac{d}{dt}a^t = \ln(a)a^t$.

If $h = h(t)$ and $f = f(x,y)$ so $(h\circ f)(x,y) = h(f(x,y))$ then we have chain rules:

$$\frac{\partial}{\partial x}(h\circ f) = \frac{dh}{dt}(f(x,y))\frac{\partial f}{\partial x} \quad \& \quad \frac{\partial}{\partial y}(h\circ f) = \frac{dh}{dt}(f(x,y))\frac{\partial f}{\partial y}.$$

Example 5.1.4. *Calculate* $\frac{\partial}{\partial x}\sin(x^2+xy) = \cos(x^2+xy)\frac{\partial}{\partial x}(x^2+xy) = (2x+y)\cos(x^2+xy)$. *Likewise,* $\frac{\partial}{\partial y}\sin(x^2+xy) = \cos(x^2+xy)\frac{\partial}{\partial y}(x^2+xy) = x\cos(x^2+xy)$.

Example 5.1.5. *Calculate* $\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{2\sqrt{x^2+y^2}}(2x+0) = \frac{x}{\sqrt{x^2+y^2}}$. *By symmetry we must also find* $\frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}}$.

Another chain-rule is very important to applications of multivariate calculus. If we have a path $\vec{r}(t) = (x(t), y(t))$ and $f = f(x,y)$ then $(f\circ\vec{r})(t) = f(x(t),y(t))$ and

$$\frac{d}{dt}(f\circ\vec{r}) = \frac{d}{dt}(f(x(t),y(t)) = \frac{\partial f}{\partial x}(x(t),y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t))\frac{dy}{dt}.$$

Or to be more terse, $\frac{d}{dt}(f\circ(x,y)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Notice we can rewrite this chain rule as a dot-product,

$$\frac{d}{dt}(f\circ\vec{r}) = \left\langle \frac{\partial f}{\partial x}(x(t),y(t)), \frac{\partial f}{\partial y}(x(t),y(t))\right\rangle \bullet \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle = \nabla f(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}.$$

where we have introduced ∇f as defined below:

Definition 5.1.6. *If* $f = f(x,y)$ *then the* **gradient** *of* f *is* $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$.

If we use the notation $\frac{\partial f}{\partial x} = \partial_x f$ and $\frac{\partial f}{\partial y} = \partial_y f$ then $\nabla f = \langle \partial_x f, \partial_y f\rangle$. In Physics and Engineering it popular to view ∇ as an **vector operator** given by

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$$

then the formula for ∇f follows since $\nabla f = \langle \partial_x f, \partial_y f\rangle = \hat{x}\partial_x f + \hat{y}\partial_y f$. What is ∇f ? It is a vector field which points in the direction in-which f increases most rapidly. There are nice rules for ∇: for $f = f(x,y)$ and $g = g(x,y)$ and $h = h(t)$,

$$\nabla(f+g) = \nabla f + \nabla g, \quad \& \quad , \nabla(fg) = g\nabla f + f\nabla g, \quad \& \quad \nabla(h \circ f) = \frac{dh}{df}\nabla f.$$

To be explicit, $\frac{dh}{df}(x,y) = \frac{dh}{dt}(f(x,y))$. Examples are probably more helpful at this stage in the game.

Example 5.1.7. $\nabla(x^2+y^2) = \langle \partial_x(x^2+y^2), \partial_y(x^2+y^2)\rangle = \langle 2x, 2y\rangle$.

Example 5.1.8. $\nabla\sqrt{x^2+y^2} = \frac{1}{2\sqrt{x^2+y^2}}\nabla(x^2+y^2) = \frac{1}{2\sqrt{x^2+y^2}}\langle 2x, 2y\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right\rangle$.

Example 5.1.9. $\nabla(e^{xy}) = \langle \partial_x e^{xy}, \partial_y e^{xy}\rangle = \langle ye^{xy}, xe^{xy}\rangle$.

A **vector field** is an assignment of a vector at each point in space. The gradient of a scalar function gives a vector field which describes how the scalar function changes through space. This is very important to physics because a vector field which conserves energy can be written as the gradient of a scalar function. This is the reason for our interest in ∇ here. You will learn much more about ∇ in the full course on multivariate calculus. My goal here is to give you just enough to intelligently express the fundamental laws of physics for mechanical energy.

(2.) For $f = f(x,y,z)$, and $p_0 = (x_0, y_0, z_0)$,

$$\frac{\partial f}{\partial x}(p_0) = \frac{d}{dx}f(x,y_0,z_0)\bigg|_{x=x_0} \quad \& \quad \frac{\partial f}{\partial y}(p_0) = \frac{d}{dy}f(x_0,y,z_0)\bigg|_{y=y_0} \quad \& \quad \frac{\partial f}{\partial z}(p_0) = \frac{d}{dz}f(x_0,y_0,z)\bigg|_{z=z_0}.$$

For $\mathbb{R}^3$ we have

$$\frac{\partial x}{\partial x} = 1, \ \frac{\partial x}{\partial y} = 0, \ \frac{\partial x}{\partial z} = 0, \ \frac{\partial y}{\partial x} = 0, \ \frac{\partial y}{\partial y} = 1, \ \frac{\partial y}{\partial z} = 0, \ \frac{\partial z}{\partial x} = 0, \ \frac{\partial z}{\partial y} = 0, \ \frac{\partial z}{\partial z} = 1.$$

Given $f = f(x,y,z)$ and $g = g(x,y,z)$ and a constant c the rules of calculus are natural

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \quad \& \quad \frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}, \quad \& \quad \frac{\partial}{\partial x}(cf) = c\frac{\partial f}{\partial x}.$$

$$\frac{\partial}{\partial y}(f+g) = \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x}, \quad \& \quad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}, \quad \& \quad \frac{\partial}{\partial y}(cf) = c\frac{\partial f}{\partial y}.$$

$$\frac{\partial}{\partial z}(f+g) = \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}, \quad \& \quad \frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}, \quad \& \quad \frac{\partial}{\partial z}(cf) = c\frac{\partial f}{\partial z}.$$

Example 5.1.10. *Calculate* $\frac{\partial}{\partial x}(x^3y+y^2+z^3) = \frac{\partial}{\partial x}(x^3)y + \frac{\partial}{\partial x}(y^2) = 3x^2y$. *since* $\frac{\partial}{\partial x}(y^2) = 0$ *and* $\frac{\partial}{\partial x}(z^3) = 0$ *since* y, z *are held constant in the* x*-partial differentiation. Likewise,*

$$\frac{\partial}{\partial y}(x^3y+y^2+z^3) = x^3\frac{\partial y}{\partial y} + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(z^3) = x^3 + 2y.$$

and $\frac{\partial}{\partial z}(x^3y+y^2+z^3) = 3z^2$.

If $h = h(t)$ and $f = f(x, y, z)$ so $(h \circ f)(x, y, z) = h(f(x, y, z))$ then we have chain rules:

$$\frac{\partial}{\partial x}(h \circ f) = \frac{dh}{dt}(f(x,y,z))\frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial y}(h \circ f) = \frac{dh}{dt}(f(x,y,z))\frac{\partial f}{\partial y}, \quad \frac{\partial}{\partial z}(h \circ f) = \frac{dh}{dt}(f(x,y,z))\frac{\partial f}{\partial z}.$$

Example 5.1.11. *Calculate* $\frac{\partial}{\partial x}\sin(xyz) = \cos(xyz)\frac{\partial}{\partial x}(xyz) = yz\cos(xyz)$. *Likewise,*

$$\frac{\partial}{\partial y}\sin(xyz) = xz\cos(xyz) \quad \& \quad \frac{\partial}{\partial z}\sin(xyz) = xy\cos(xyz).$$

Example 5.1.12. *Calculate* $\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{x}{\sqrt{x^2+y^2+z^2}}$. *By symmetry we must also find*

$$\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} = \frac{y}{\sqrt{x^2+y^2+z^2}} \quad \& \quad \frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

If we set $r = \sqrt{x^2+y^2+z^2}$ *we have shown that* $\frac{\partial r}{\partial x} = \frac{x}{r}$ *and* $\frac{\partial r}{\partial y} = \frac{y}{r}$ *and* $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Another chain-rule is very important to applications of multivariate calculus. If we have a path $\vec{r}(t) = (x(t), y(t), z(t))$ and $f = f(x, y, z)$ then $(f \circ \vec{r})(t) = f(x(t), y(t), z(t))$ and

$$\frac{d}{dt}(f \circ \vec{r}) = \frac{d}{dt}(f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

where we have omitted the dependence of the partial derivatives on $\vec{r}(t)$. We can rewrite this chain rule as a dot-product,

$$\frac{d}{dt}(f \circ \vec{r}) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \bullet \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \nabla f(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}.$$

where we have introduced ∇f as defined below:

Definition 5.1.13. *If* $f = f(x, y, z)$ *then the* **gradient** *of* f *is* $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

If we use the notation $\frac{\partial f}{\partial x} = \partial_x f$ etc. then $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$. In Physics and Engineering it popular to view ∇ as an **vector operator** given by

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

then the formula for ∇f follows since

$$\nabla f = \langle \partial_x f, \partial_y f, \partial_y f \rangle = \hat{x}\partial_x f + \hat{y}\partial_y f + \hat{z}\partial_z f.$$

What is ∇f ? It is a vector field which points in the direction in-which f increases most rapidly. There are nice rules for ∇: for $f = f(x, y, z)$ and $g = g(x, y, z)$ and $h = h(t)$,

$$\nabla(f+g) = \nabla f + \nabla g, \quad \& \quad , \nabla(fg) = g\nabla f + f\nabla g, \quad \& \quad \nabla(h \circ f) = \frac{dh}{df}\nabla f.$$

To be explicit, $\frac{dh}{df}(x, y, z) = \frac{dh}{dt}(f(x, y, z))$. Examples are probably more helpful at this stage in the game.

Example 5.1.14. $\nabla(x^2+y^2+z^2) = \langle 2x, 2y, 2z\rangle$.

Example 5.1.15. *Let* $r = \sqrt{x^2+y^2+z^2}$*. We saw* $\partial_x r = x/r$ *and* $\partial_y r = y/r$ *and* $\partial_z r = z/r$ *earlier in this section. Thus calculate,*

$$\nabla r = \langle \partial_x r, \partial_y r, \partial_z r\rangle = \langle x/r, y/r, z/r\rangle = \frac{1}{r}\langle x, y, z\rangle.$$

Notice $\|\nabla r\| = 1$ *since* $\|\langle x,y,z\rangle\| = r$*. In fact,* $\nabla r = \hat{r}$*. We should recognize this is the unit-vector which points in the direction of increasing* r*.*

The result in the example above matches nicely with similar calculations for the Cartesian coordinate frame in $\mathbb{R}^3$. Note $\nabla x = \langle 1,0,0\rangle = \hat{x}$ and $\nabla y = \langle 0,1,0\rangle = \hat{y}$ and $\nabla z = \langle 0,0,1\rangle = \hat{z}$.

(3.) In $\mathbb{R}^n$ we can write less for $f = f(x_1,\dots,x_n)$ and $g = g(x_1,\dots,x_n)$ let us use the notation $\frac{\partial}{\partial x_i} = \partial_i$ then $\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j\end{cases}$. For constant c,

$$\partial_i(f+g) = \partial_i f + \partial_i g, \quad \& \quad , \partial_i(fg) = g\partial_i f + f\partial_i g, \quad \& \quad \partial_i(cf) = cf.$$

Given $h = h(t)$ and $f = f(x_1,\dots,x_n)$ we have $\partial_i(h\circ f) = h'(f)\partial_i f$. Also, if $\vec{r}(t) = \langle x_1(t),\dots,x_n(t)\rangle$ then

$$\frac{d}{dt}(f\circ\vec{r}) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{r}(t))\frac{dx_i}{dt} = \nabla f(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}.$$

Definition 5.1.16. *If* $f = f(x_1,\dots,x_n)$ *then the* **gradient** *of* f *is* $\nabla f = \langle \partial_1 f,\dots,\partial_n f\rangle$*.*

Alternatively, we can define ∇ as the vector operator $\nabla = \sum_{i=1}^n \hat{x}_i \frac{\partial}{\partial x_i}$ and write

$$\nabla f = \sum_{i=1}^{n}\hat{x}_i\frac{\partial f}{\partial x_i}.$$

Example 5.1.17. *If* $r = \sqrt{\vec{r}\bullet\vec{r}}$ *where* $\vec{r} = \langle x_1,\dots,x_n\rangle$ *then* $r^2 = \vec{r}\bullet\vec{r}$ *and since* $\partial_i\vec{r} = \hat{x}_i$ *(to partial differentiate a vector we partial differentiate each component)*

$$\partial_i(r^2) = 2r\partial_i r = \partial_i(\vec{r}\bullet\vec{r}) = (\partial_i\vec{r})\bullet\vec{r} + \vec{r}\bullet(\partial_i\vec{r}) = 2\hat{x}_i\bullet\vec{r} = 2x_i \quad \Rightarrow \quad \partial_i r = \frac{x_i}{r}.$$

Of course, we could also have calculated $\partial_i r = \frac{x_i}{r}$ *directly, but it's more fun to see a bit of the sort of calculus we can do in* $\mathbb{R}^n$*.*

Enough of this, you can safely ignore (3.) for most of our purposes. We're primarily working in $\mathbb{R}^2$ or $\mathbb{R}^3$ for this course.

5.2 line integral of a vector field

In this section we begin by discussing the concept of a force field and we offer some common examples. Then we define the line integral which serves to calculate the work done by a force. We detail common special cases such as the work done by a constant force, or work done in one-dimensional motion. Then we turn to analyze deeper questions about the structure of vector fields such as path-independence and the corresponding existence of potential energy functions. Finally, we circle back to explain the work-energy theorem and the conservation of total mechanical energy in the context of a conservative net-force. The penultimate subsection lays the groundwork for our application of energy methods to solve physics problems. Our typical approach is to use conservation of energy as a means to relate physical variables like position and speed. Finally, we detail common strategies for calculation of potential energy functions for a given force field.

5.2.1 what is a vector field ?

Before I define the line integral we should pause to appreciate the concept of a vector field. A **vector field** in a space S is an assignment of a vector $\vec{F}$ at p for each $p \in S$. In other words, $p \mapsto \vec{F}(p)$ is a vector field.

Example 5.2.1. *For example, $\vec{F}(x, y) = \langle 0, -mg \rangle$ is the constant vector field which represents the force of gravity on a mass m in the plane where we take y to be the vertical direction near the surface of the Earth.*

Example 5.2.2. *Or, $\vec{F}(x, y, z) = \langle 0, 0, -mg \rangle$ denotes the force of gravity taking z to be the vertical direction. These force fields are* **constant** *since the force does not depend on the position.*

Example 5.2.3. *Another example is friction for a moving object with mass m on a plane. If the plane is horizontal then $F_n = mg$ and $F_f = \mu_k mg$ directed opposite the direction of motion. A formula for that would be:*

$$\vec{F}_f = -\mu_k mgT$$

where T is the unit-tangent to the trajectory. If you prefer, $T = \hat{v}$, it is the unit-vector in the direction of the velocity. In this case, the friction force field is only assigned along the trajectory of the object. So, $\vec{F}_f$ is probably not a constant vector because T is changing for most motions. On the other hand, the magnitude of the friction force is simply $\mu_k mg$.

Our goal is to consider a general force field which is possibly non-constant and it may change direction from point to point. I will ignore dimensional analysis in many of the examples which follow since I wish to reduce notational clutter. Rather than writing $\vec{F}(x, y, z) = \langle \alpha x^2, \beta yz, \gamma(z^3 + x) \rangle$ with constants α, β, γ, I'll opt for the less burdensome $\vec{F}(x, y, z) = \langle x^2, yz, z^3 + x \rangle$. Please understand, technically the simplified expression does beg questions of dimensional analysis if we wish to be picky on that point.

Example 5.2.4. *Let $\vec{F}(x, y) = \langle x, y \rangle$. This vector fields points radially outward from the origin with a magnitude that is given by the distance from the origin.*

Example 5.2.5. *Let $\vec{G}(x, y) = \langle -y, x \rangle$. This vector fields wraps around the origin with a magnitude that is given by the distance from the origin. Notice $\vec{F}(x, y) \bullet \vec{G}(x, y) = \langle x, y \rangle \bullet \langle -y, x \rangle = -xy + yx = 0$. The vector fields $\vec{F}$ and $\vec{G}$ are everywhere orthogonal.*

Example 5.2.6. *Newton's Universal Law of Gravitation states the force of gravity from a mass M on a mass m is an attractive force which is inversely proportional to the squared distance between the centers of mass of M and m. If we assume M is at the origin and $\vec{r}$ is the position vector of m then we can express Newton's Law as*

$$\vec{F}(\vec{r}) = \frac{-GmM}{r^2}\hat{r} = \frac{-GmM}{r^3}\vec{r}$$

Or, in Cartesian variables $\vec{r} = \langle x, y, z\rangle$ and $r = \sqrt{x^2+y^2+z^2}$ hence

$$\vec{F}(x,y,z) = \frac{-GmM\langle x,y,z\rangle}{(x^2+y^2+z^2)^{3/2}} = \left\langle \frac{-GmMx}{(x^2+y^2+z^2)^{3/2}}, \frac{-GmMy}{(x^2+y^2+z^2)^{3/2}}, \frac{-GmMz}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

There are infinitely many different vector fields we can consider. Sometimes their formula can be written in terms of x, y, z independent of time, othertimes the definition of the vector field requires some time dependence. Other times the vector field may only reasonably be defined where the material object in question resides. The concept of a vector field has a lot of different possibilities from a physical perspective. That said, when we study line integrals in Calculus III we almost always just think about **static vector fields** which have no time dependence. Fortunately, the definition of line integral for static vector fields is not modified in its form when we consider **dynamic** fields. Let us continue to the main event:

5.2.2 definition of the line integral and examples

Here is the definition:

Definition 5.2.7. *Let $\vec{F}$ be a vector field and suppose C is a curve in the domain in which $\vec{F}$ is defined then if C is an oriented curve which is parameterized by $\vec{r}(t)$ for $t_1 \leq t \leq t_2$ then we define the* **line integral** *of $\vec{F}$ over C by*

$$\int_C \vec{F} \bullet d\vec{r} = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} dt.$$

We say $\int_C \vec{F} \bullet d\vec{r}$ gives the **work done by $\vec{F}$ along** *C.*

It is also useful to reformulate the line integral in terms of the tangent vector field $T = \frac{1}{\|\frac{d\vec{r}}{dt}\|}\frac{d\vec{r}}{dt}$ since $\frac{d\vec{r}}{dt} = \|\frac{d\vec{r}}{dt}\|T = vT$ where v is the **speed** we find

$$\int_C \vec{F} \bullet d\vec{r} = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \bullet T v dt.$$

However, $v = \frac{ds}{dt}$ hence we can rewrite the integral above as an arclength integral[1]

$$\int_C \vec{F} \bullet d\vec{r} = \int_C \vec{F} \bullet T ds.$$

The formula above helps explain why $\int_C \vec{F} \bullet d\vec{r}$ is at times called the **circulation** of $\vec{F}$ along C since it calculates how $\vec{F}$ aligns with the curve C. It is funny that we call it a **line integral** because

[1] the definition of the integral with respect to arclenth of a function f along a curve C parametrized by $\vec{r}(t)$ for $t_1 \leq t \leq t_2$ is $\int_C f ds = \int_{t_1}^{t_2} f(\vec{r}(t))\|d\vec{r}/dt\|dt$.

there is no assumption that C is a line. Maybe we should call it the *aligns integral* because it's really more about how $\vec{F}$ aligns with C. In particular, if the vector field at $\vec{r}(t)$ is perpendcular to the direction of the curve then $\vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} = 0$ hence such points give no contribution to the work done by $\vec{F}$ along C. If $\vec{F} \bullet T$ along all of C then $\int_C \vec{F} \bullet d\vec{r} = 0$.

Example 5.2.8. *Suppose a yo-yo is spun in a circle by a string. The tension force of the string is directed radially inward. If $\vec{F}_T$ is this tension force and T is the unit-tangent field to the trajectory of the yo-yo then $\vec{F}_T \bullet T = 0$ around the entire trajectory. Thus if C is some part of this trajectory then $\int_C \vec{F}_T \bullet d\vec{r} = 0$.*

Another interesting example where I can't give a nice formula for the force field in question is friction.

Example 5.2.9. *Let a box with mass M be pushed along a path C in the plane. In this case the magnitude of the force of friction is simple $F_f = \mu_k M g$ since the normal force and weight are the only forces normal to the plane and as such they must balance. The direction of the force of friction is naturally described by*

$$\vec{F}_f = -\mu_k M g T$$

where T is the unit-tangent vector to the path travelled by the box. We calculate the work done by F_f along C by

$$W_f = \int_C \vec{F}_f \bullet d\vec{r} = \int_C \vec{F}_f \bullet T ds = \int_C -\mu_k M g T \bullet T ds = -\mu_k M g \int_C ds = -\mu_k M g L(C)$$

where $L(C)$ denotes the arclength of C. In short, the magnitude of the work done by friction is simply the magnitude of the friction force times the distance travelled. The work done by friction is negative since the friction force is always directed opposite the direction of motion in this example.

Example 5.2.10. *The force of a spring on a mass in one-dimensional motion along x is given by* **Hooke's Law***; $F = -kx$ where k is a constant called the* **spring constant**[2]*. Notice $F = -kx > 0$ when $x < 0$ because if the spring is* **compressed** *it pushes rightward. On the other hand, $F = -kx < 0$ when $x > 0$ because if the spring is* **stretched** *it pulls leftward. Lastly, if $x = 0$ the spring is in equilbrium and is neither pushing nor pulling. If the spring undergoes motion which begins with $x = x_1$ and ends with $x = x_2$ we can calculate the work done by the spring*[3]

$$W = \int_{x_1}^{x_2} F\, dx = -\int_{x_1}^{x_2} kx\, dx = \frac{1}{2}k(x_1^2 - x_2^2)$$

Remark 5.2.11. *In one dimensional-motion, the calculation of work is accomplished by plain old integration. The calculation in the footnote holds so long as the force is not a function of time. For any static one-dimensional force F we can calculate the work done by F over the motion $x \to x + dx$ by $dW = F(x)dx$. Then, if F is applied for $x_1 \leq x \leq x_2$ we simply integrate to find*

$$W = \int_{x_1}^{x_2} F(x) dx.$$

[2]surprising name, yes ?

[3]yes this is what follows from our definition, setting $\vec{F} = \langle -kx, 0 \rangle$ the parametrization $\vec{r}(t) = \langle g(t), 0 \rangle$ has $\frac{d\vec{r}}{dt} = \langle dg/dt, 0 \rangle$ for $t_1 \leq t \leq t_2$ where $g(t_1) = x_1$ and $g(t_2) = x_2$ thus $\int_C \vec{F} \bullet d\vec{r} = \int_{t_1}^{t_2} -kg(t)\frac{dg}{dt} dt = \int_{x_1}^{x_2} -kx dx$ making a $g = x$-substitution for the last step.

Next, let us return to the issue of constant force.

Example 5.2.12. *Let $\vec{F}_0$ be constant. Let C be the line-segment path from P_1 to P_2. We can parametrize this path by $\vec{r}(t) = P_1 + t(P_2 - P_1)$ for $0 \leq t \leq 1$. Then $\frac{d\vec{r}}{dt} = P_2 - P_1$ and so the work done by $\vec{F}$ along C is easy to calculate:*

$$W = \int_C \vec{F}_0 \bullet d\vec{r} = \int_0^1 \vec{F}_0 \bullet (P_2 - P_1)dt = \vec{F}_0 \bullet (P_2 - P_1) \int_0^1 dt = \vec{F}_0 \bullet (P_2 - P_1).$$

In other words, the work done by a constant force $\vec{F}_0$ over the displacement $\triangle\vec{r}$ is simply $\vec{F}_0 \bullet \triangle\vec{r}$ since we recognize $\triangle\vec{r} = P_2 - P_1$ for initial point P_1 and terminal point P_2.

We have now covered all the special cases we commonly encounter in work problems. Let me now turn to an example which demonstrates the calculation which is generally necessary to find the work done by a static force in the plane.

Example 5.2.13. *Let $\vec{F}(x,y) = \langle x^2 - y, x + y\rangle$. We wish to calculate the work done by $\vec{F}$ along two curves which are coterminal, but travel different paths. Take C_1 to be the line-segment L_1 from $(0,0)$ to $(1,0)$ followed by the line-segment L_2 from $(1,0)$ to $(1,1)$. For the line-segment L_1 from $(0,0)$ to $(1,0)$ we have*

$$\vec{r}_1 = \langle x, 0\rangle \qquad \& \qquad d\vec{r}_1 = \langle dx, 0\rangle \qquad \& \qquad 0 \leq x \leq 1$$

for L_2 from $(1,0)$ to $(1,1)$ we have

$$\vec{r}_2 = \langle 1, y\rangle \qquad \& \qquad d\vec{r}_1 = \langle 0, dy\rangle \qquad \& \qquad 0 \leq y \leq 1$$

Therefore, as $C_1 = L_1 \cup L_2$ we add the integrals below

$$\begin{aligned} W_1 = \int_{C_1} \vec{F} \bullet d\vec{r} &= \int_0^1 \vec{F}(x,0) \bullet \langle dx, 0\rangle + \int_0^1 \vec{F}(1,y) \bullet \langle 0, dy\rangle \\ &= \int_0^1 \langle x^2 - 0, x + 0\rangle \bullet \langle dx, 0\rangle + \int_0^1 \langle 1^2 - y, 1 + y\rangle \bullet \langle 0, dy\rangle \\ &= \int_0^1 x^2 dx + \int_0^1 (1+y)dy \\ &= \frac{1}{3} + 1 + \frac{1}{2} = \boxed{\frac{11}{6}}. \end{aligned}$$

Next, take C_2 to be the line segment L_3 from $(0,0)$ to $(0,1)$ followed by the line-segment L_4 from $(0,1)$ to $(1,1)$. For the line-segment L_3 from $(0,0)$ to $(0,1)$ we have

$$\vec{r}_3 = \langle 0, y\rangle \qquad \& \qquad d\vec{r}_3 = \langle 0, dy\rangle \qquad \& \qquad 0 \leq y \leq 1$$

for L_4 from $(0,1)$ to $(1,1)$ we have

$$\vec{r}_4 = \langle x, 1\rangle \qquad \& \qquad d\vec{r}_1 = \langle dx, 0\rangle \qquad \& \qquad 0 \leq x \leq 1$$

Therefore, as $C_2 = L_3 \cup L_4$ we add the integrals below, remember $\vec{F}(x,y) = \langle x^2 - y, x+y \rangle$,

$$\begin{aligned} W_2 = \int_{C_2} \vec{F} \bullet d\vec{r} &= \int_0^1 \vec{F}(0,y) \bullet \langle 0, dy \rangle + \int_0^1 \vec{F}(x,1) \bullet \langle dx, 0 \rangle \\ &= \int_0^1 \langle 0^2 - y, 0 + y \rangle \bullet \langle 0, dy \rangle + \int_0^1 \langle x^2 - 1, x+1 \rangle \bullet \langle dx, 0 \rangle \\ &= \int_0^1 y dy + \int_0^1 (x^2 - 1) dx \\ &= \frac{1}{2} + \frac{1}{3} - 1 = \boxed{\frac{-1}{6}}. \end{aligned}$$

Let me recap the calculation here, we find the work done by $\vec{F} = \langle x^2 - y, x+y \rangle$ along C_1 is $W_1 = 11/6$ whereas the work done by $\vec{F}$ along C_2 is $W_2 = -1/6$. This shows the given force is not **path-independent**.

5.2.3 theory of conservative vector fields

Definition 5.2.14. *Let $\vec{F}$ be a vector field. We say $\vec{F}$ is* **path-independent** *on $S \subseteq \mathbb{R}^n$ if for any pair of coterminal paths C_1 and C_2 we have $\int_{C_1} \vec{F} \bullet \vec{r} = \int_{C_2} \vec{F} \bullet \vec{r}$.*

Notice the vector field considered by Example 5.2.13 is not path-independent since the curves C_1 and C_2 both begin at $(0,0)$ and terminate at $(1,1)$ and yet $\int_{C_1} \vec{F} \bullet \vec{r} \neq \int_{C_2} \vec{F} \bullet \vec{r}$. It turns out[4] that if a vector field $\vec{F}$ is path-independent then we can build a *potential function*[5] f with $\nabla f = \vec{F}$ via the formula

$$f(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \vec{F} \bullet d\vec{r}$$

where the notation $\int_{\vec{r}_0}^{\vec{r}}$ indicates calculation of the integral along some curve which begins at $\vec{r}_0$ and ends at $\vec{r}$. In this set-up notice $f(\vec{r}_0) = 0$ hence the formula sets the zero for the potential at the base point of the integration. This leads us to make the following definition (whose terminology is more fully explained in the next page or two of these notes)

Definition 5.2.15. *Let $\vec{F}$ be a vector field. We say $\vec{F}$ is* **conservative** *on $S \subseteq \mathbb{R}^n$ if there is a* **potential function** *$f : S \to \mathbb{R}$ for which $\vec{F} = \nabla f$. Equivalently, $\vec{F}$ is* **conservative** *on $S \subseteq \mathbb{R}^n$ if there is a* **potential energy function** *$U : S \to \mathbb{R}$ for which $\vec{F} = -\nabla U$.*

Notice that if $\vec{F}$ is path-independent on $S \subseteq \mathbb{R}^n$ and $\vec{r}_0$ is a point in S then we can construct the potential energy function via the integral

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \bullet d\vec{r}.$$

It is a theorem of multivariate calculus that if the vector field $\vec{F}$ is path-independent and we define $U(\vec{r})$ as above then $\nabla U = -\vec{F}$ which is to say $\vec{F} = -\nabla U$. Let us pause to appreciate an important theorem of vector calculus (FTC for line integrals):

[4] see my multivariate calculus notes, or any good Calculus III text for proof

[5] I mention this concept in passing since some of you may also be taking Calculus III and I see opportunity for confusion here. We will mostly focus on the construction of the potential energy function due to its physical significance.

Theorem 5.2.16. *Fundamental Theorem of Calculus for Line Integrals.*

Suppose f is differentiable near an oriented curve C from P to Q then

$$\int_C \nabla f \bullet d\vec{r} = f(Q) - f(P).$$

Proof: let $\vec{r} : [t_1, t_2] \to C \subset \mathbb{R}^n$ parametrize C and calculate:

$$\int_C \nabla f \bullet d\vec{r} = \int_{t_1}^{t_2} \nabla f(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}\Big[f(\vec{r}(t))\Big] dt = f(\vec{r}(t_2)) - f(\vec{r}(t_1)) = f(Q) - f(P).$$

The two critical steps above are the application of the multivariate chain-rule and then in the next to last step we apply the FTC from single-variable calculus. □

Notice this means we can easily show that any conservative vector field on S is necessarily path-independent on S. If C_1 and C_2 are any two paths in S from $\vec{r}_1$ to $\vec{r}_2$ then

$$\int_{C_1} \vec{F} \bullet d\vec{r} = -\int_{C_1} \nabla U \bullet d\vec{r} = U(\vec{r}_1) - U(\vec{r}_2) = -\int_{C_1} \nabla U \bullet d\vec{r} = \int_{C_1} \vec{F} \bullet d\vec{r}.$$

Thus $\vec{F} = -\nabla U$ implies $\vec{F}$ is path-independent. Conversely, we also argued that path-independence implies the existence of a potential energy function according to the formula

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \bullet d\vec{r}.$$

Let me exhibit the calculation of a potential energy function from the formula above.

Example 5.2.17. *Let $\vec{F} = \langle x, y \rangle$. Let's set $\vec{r}_0 = (x_0, y_0)$ and write the parametric equation for a line to (x, y) as follows:*

$$\vec{r}(t) = (x_0, y_0) + t\langle x - x_0, y - y_0 \rangle = \langle x_0 + t(x - x_0), y_0 + t(y - y_0) \rangle$$

for $0 \leq t \leq 1$. Notice $\vec{r}(0) = (x_0, y_0)$ whereas $\vec{r}(1) = (x, y)$. Moreover, $\frac{d\vec{r}}{dt} = \langle x - x_0, y - y_0 \rangle$. Hence,

$$U(x, y) = -\int \vec{F} \bullet d\vec{r} = -\int_0^1 \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} dt = -\int_0^1 \vec{r}(t) \bullet \frac{d\vec{r}}{dt} dt.$$

Continuing, getting into the details,

$$\begin{aligned}
U(x, y) &= -\int_0^1 \langle x_0 + t(x - x_0), y_0 + t(y - y_0) \rangle \bullet \langle x - x_0, y - y_0 \rangle \, dt \\
&= -\int_0^1 \left[(x_0 + t(x - x_0))(x - x_0) + (y_0 + t(y - y_0))(y - y_0)\right] dt \\
&= -x_0(x - x_0) - \frac{1}{2}(x - x_0)^2 - y_0(y - y_0) - \frac{1}{2}(y - y_0)^2 \\
&= -\frac{1}{2}(x^2 - 2xx_0 + x_0^2 + 2xx_0 - 2x_0^2) - \frac{1}{2}(y^2 - 2yy_0 + y_0^2 + 2yy_0 - 2y_0^2) \\
&= -\frac{1}{2}(x^2 - x_0^2) - \frac{1}{2}(y^2 - y_0^2).
\end{aligned}$$

You can easily verify $\nabla U = \langle -x, -y \rangle$ hence $\vec{F} = -\nabla U = \langle x, y \rangle$ as desired. Moreover, $U(x_0, y_0) = 0$.

I don't usually teach the method above in lecture since it is far from an efficient mode of calculation. That said, this calculation is of theoretical importance.

Example 5.2.18. *Let $\vec{F}_0$ be constant then*

$$U(\vec{r}) = -\int_0^{\vec{r}} \vec{F}_0 \bullet d\vec{r} = -\vec{F}_0 \bullet \int_0^{\vec{r}} d\vec{r} = -\vec{F}_0 \bullet \vec{r}.$$

In other words, if $\vec{F}_0 = \langle a, b, c \rangle$ and $\vec{r} = \langle x, y, z \rangle$ then

$$U(x, y, z) = -\langle a, b, c \rangle \bullet \langle x, y, z \rangle = -ax - by - cz.$$

Let me hazard a sketchy calculation if we assume the potential is zero at ∞:

Example 5.2.19. *Let $\vec{F}(\vec{r}) = \frac{-1}{r^3}\vec{r}$ then this is the* **attractive inverse square law** *force. Note,*

$$U(\vec{r}) = -\int_\infty^{\vec{r}} \frac{-1}{r^2}\hat{r} \bullet d\vec{r} = \int_\infty^{\vec{r}} \frac{1}{r^2}\hat{r} \bullet \hat{r} dr = \int_\infty^{\vec{r}} \frac{dr}{r^2} = \frac{-1}{r} + \frac{1}{\infty} = \frac{-1}{r}.$$

Of course, you might say the calculation above is pure nonsense, and you might be correct, but perhaps it justifies my assertion that if $\vec{F}(x, y, z) = \frac{-1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z \rangle$ then

$$U(x, y, z) = \frac{-1}{\sqrt{x^2 + y^2 + z^2}}$$

You can check, (following Example 5.1.12) if $r = \sqrt{x^2 + y^2 + z^2}$ then $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$. Thus, using the chain rule for gradients,

$$\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\nabla r = \frac{1}{r^2}\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\langle x, y, z \rangle$$

Returning to Example 5.2.6 we find gravity is a conservative force. In particular, if $\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$ then $U = \frac{Gm_1m_2}{r}$ where $r = \sqrt{x^2 + y^2 + z^2}$ is the position of m_2 and we suppose m_1 is at the origin. In this context, $\vec{F}$ is the force of gravity due to m_1 acting on m_2.

Let us conclude our discussion of the structure of conservative vector fields with a theorem which draws together our thoughts thus far and adds another which should be covered in multivariate calculus.

Theorem 5.2.20.

Suppose S is an open connected subset of $\mathbb{R}^n$ then the following are equivalent

(1.) $\vec{F}$ is conservative; $\vec{F} = -\nabla U$ on all of S

(2.) $\vec{F}$ is path-independent on S

(3.) $\oint_C \vec{F} \bullet d\vec{r} = 0$ for all closed curves C in S

(4.) (add precondition $n = 3$ and S be simply connected) $\nabla \times \vec{F} = 0$ on U.

Let me explain what $\nabla \times \vec{F}$ is for the sake of completeness here. Suppose $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field. We define:

$$\nabla \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Or, if we use notation $\vec{F} = \langle P, Q, R \rangle$ as is common in multivariate calculus texts,

$$\nabla \times \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

If $\vec{F} = \langle P, Q, 0 \rangle$ and P, Q depend only on x, y then $\nabla \times \vec{F} = \left\langle 0, \ 0, \ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$. Thus, in the context of a **planar vector field** $\vec{F} = \langle P, Q \rangle$ we can reduce condition (4.) to the condition that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. This is sometimes called the **closed condition** for $\langle P, Q \rangle$. The logic here is a bit fussy.

(i.) If $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$ for some point(s) in S then $\vec{F} = \langle P, Q \rangle$ is not conservative.

(ii.) If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for all points in S then $\vec{F} = \langle P, Q \rangle$ may or may not be conservative.

Let me explain, if $\vec{F} = \langle P, Q \rangle = \left\langle -\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y} \right\rangle$ then since partial derivatives commute[6]

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{\partial U}{\partial y}\right] = -\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\right] = -\frac{\partial}{\partial y}\left[\frac{\partial U}{\partial x}\right] = \frac{\partial}{\partial y}\left[-\frac{\partial U}{\partial x}\right] = \frac{\partial P}{\partial y}.$$

The calculation above explains logically why (i.) is must hold. If we have conservative vector field then the closed condition is inevitable. To see why (ii.) is subtle we need an example.

Example 5.2.21. *If you think about angle in the right-half plane then* $\theta = \tan^{-1}(y/x)$. *Then after a bit of differentiation we find* $\nabla\theta = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. *We can also define* $\theta = \cot^{-1}(x/y)$ *for the upper-half plane and once more* $\nabla\theta = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. *In fact, we can always use either tangent or cotangent appropriately shifted to calculate the standard angle. This motivates us to study the vector field* $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$. *Let us calculate the work done by* $\vec{F}$ *around a circle* C_R *of radius* R *centered at the origin with a CCW orientation. Use* $\vec{r}(t) = \langle R\cos t, R\sin t \rangle$ *hence* $\frac{d\vec{r}}{dt} = \langle -R\sin t, R\cos t \rangle$ *and note* $x^2 + y^2 = R^2$ *for the circle* C_R *hence as* $\vec{F}(\vec{r}(t)) = \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle$ *and we calculate*

$$\vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} = \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle \bullet \langle -R\sin t, R\cos t \rangle = \sin^2 t + \cos^2 t = 1$$

Therefore,

$$\int_{C_R} \vec{F} \bullet d\vec{r} = \int_0^{2\pi} dt = 2\pi.$$

This makes sense given the origin story of $\vec{F}$. *Moreover,* $\vec{F} = \nabla\theta$ *over some subset of the plane, just not the entire punctured plane. The problem is this, somewhere there has to be a discontinuity in the angle. If we had a differentiable angle function defined on the entire punctured plane then*

[6] this is true for twice continuously differentiable functions, as with most things, if you have a sufficiently weird function this fails to be true, but such functions don't seem to appear in nature outside the minds of trouble-making mathematicians

that implies the angle function is everywhere continuous. But, this is impossible, we cannot help but jump angle somewhere. Likewise, for $\vec{F}$ to be conservative on its domain of $\mathbb{R}^2 - \{(0,0)\}$ we would require a potential energy function which is defined on the whole punctured plane. But, this is impossible since $\int_{C_R} \vec{F} \bullet d\vec{r} = \int_0^{2\pi} dt = 2\pi$ and if there existed a potential energy function U then viewing C_R as a path which begins at $(R,0)$ and ends at $(R,0)$ we would have

$$\int_{C_R} \vec{F} \bullet d\vec{r} = -\int_{C_R} \nabla U \bullet d\vec{r} = U(R,0) - U(R,0) = 0.$$

But, we just calculated the integral above to be 2π. So, apparently it is impossible to find such a potential energy function.

I should mention, $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ satisfies the closed condition on its domain of $\mathbb{R}^2 - \{(0,0)\}$. This is the quintessential example of a space which is **not** simply connected. If a loop encircles the origin, there is no way to smoothly deform the loop to a point without passing the loop through the origin. A *simply connected space* is one in which any loop in the space can be continuously deformed to a point without taking the loop out of space. Ultimately these problems led Poincare and other late 19th century mathematicians and physicists to discover topology. The problem of finding potential energy functions in physics inevitably begs the question of what to do when there are holes or defects in the domain. Such questions lead us to study the problem of classifying holes in space. In some sense, that is the problem of algebraic topology. It is no accident that major advances in topology have happened due to using physical theory to probe possible constructions of space itself[7].

5.2.4 conservation of energy and the work energy theorem

Why "conservative"? Let me address that. The key is a little identity, if m is a constant,

$$\frac{d}{dt}\left[\frac{1}{2}mv^2\right] = \frac{d}{dt}\left[\frac{1}{2}m\vec{v} \bullet \vec{v}\right] = \frac{1}{2}m\left[\frac{d\vec{v}}{dt} \bullet \vec{v} + \vec{v} \bullet \frac{d\vec{v}}{dt}\right] = m\vec{a} \bullet \vec{v}.$$

If $\vec{F}$ is the net-force on a mass m then Newton's Second Law states $\vec{F} = m\vec{a}$ therefore, if C is a curve from $\vec{r}_1$ to $\vec{r}_2$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{t_1}^{t_2} \vec{F} \bullet \frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} \left(\, m\vec{a} \bullet \vec{v} \, \right) dt = \int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{1}{2}mv^2\right] dt = K(t_2) - K(t_1)$$

where $K = \frac{1}{2}mv^2$ is the kinetic energy. This result is known as the **work-energy** theorem. It does not require that $\vec{F}$ be conservative. If $\vec{F}$ is conservative then we can choose a potential energy function U such that $\vec{F} = -\nabla U$. In this case we can use the FTC for line-integrals to once more calculate the work done by the net-force,

$$\int_C \vec{F} \bullet d\vec{r} = -\int_C \nabla U \bullet d\vec{r} = -U(\vec{r}_2) + U(\vec{r}_1)$$

It follows that we have, for a conservative force, $K_2 - K_1 = -U_2 + U_1$ hence $K_1 + U_1 = K_2 + U_2$. The quantity $E = U + K$ is the total mechanical energy and it is a constant of the motion when only conservative forces comprise the net-force[8]

[7] see Seiberg-Witten Theory which was somehow discovered as an outgrowth of so-called instanton physics in the early 1980's. I am surely not giving a complete history here, I merely wish for you to know that the link between math and physics is still active and producing results to the present day.

[8] If $\vec{F}_1 = -\nabla U_1$ and $\vec{F}_2 = -\nabla U_2$ then $\vec{F} = \vec{F}_1 + \vec{F}_2$ has potential energy $U = U_1 + U_2$ since $\nabla(U_1 + U_2) = \nabla U_1 + \nabla U_2$.

5.2.5 on the calculation of potential energy functions

In the previous subsections I illustrated how if we're given $\vec{F}$ then we may calculate U for which $\vec{F} = -\nabla U$ via an integration; $U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \bullet d\vec{r}$. This is almost always the wrong way to calculate U. It is far easier to simply integrate the necessary equations which follow from direct examination of $\vec{F} = -\nabla U$. Generally, if $\vec{F} = \langle P, Q, R \rangle$ where P, Q, R are given functions then we need to solve

$$\frac{\partial U}{\partial x} = -P, \qquad \frac{\partial U}{\partial y} = -Q, \qquad \frac{\partial U}{\partial z} = -R.$$

This is easier than it looks. Let's try guessing.

Example 5.2.22. *If $\vec{F} = \langle x, y^2, z^3 \rangle$ then we need to solve*

$$\frac{\partial U}{\partial x} = -x, \quad \frac{\partial U}{\partial y} = -y^2, \quad \frac{\partial U}{\partial z} = -z^3.$$

Simply integrate each equation whilst holding the remaining variable fixed,

$$U = -\frac{x^2}{2} + C_1(y,z), \quad U = -\frac{y^3}{3} + C_2(x,z), \quad U = -\frac{z^4}{4} + C_3(x,y).$$

In each case the unknown function reflects the variables which were held fixed while we integrated the remaining variable. Comparing all three formulas we deduce,

$$\boxed{U = -\frac{x^2}{2} - \frac{y^3}{3} - \frac{z^4}{4}}.$$

Basically, by way of comparison we realized that

$$C_1(y,z) = -\frac{y^3}{3} - \frac{z^4}{4}, \quad C_2(x,z) = -\frac{x^2}{2} - \frac{z^4}{4}, \quad C_3(x,y) = -\frac{x^2}{2} - \frac{y^3}{3}.$$

This comparison method does require care to not double count functions which are common to more than one integration.

Example 5.2.23. *If $\vec{F} = \langle \sin x - 2xy^2z^2, \cosh y - 2x^2yz^2, -2x^2y^2z \rangle$ then we need to solve*

$$\frac{\partial U}{\partial x} = 2xy^2z^2 - \sin x, \quad \frac{\partial U}{\partial y} = 2x^2yz^2 - \cosh y, \quad \frac{\partial U}{\partial z} = 2x^2y^2z.$$

Simply integrate each equation whilst holding the remaining variable fixed,

$$U = x^2y^2z^2 + \cos x + C_1(y,z), \quad U = x^2y^2z^2 - \sinh y + C_2(x,z), \quad U = x^2y^2z^2 + C_3(x,y).$$

In each case the unknown function reflects the variables which were held fixed while we integrated the remaining variable. Comparing all three formulas we deduce,

$$\boxed{U = x^2y^2z^2 + \cos x - \sinh y}.$$

We can check U is correct by calculating ∇U to see that $-\nabla U = \langle \sin x - 2xy^2z^2, \cosh y - 2x^2yz^2, -2x^2y^2z \rangle$.

The is a potential danger in the method of calculation shown thus far. It is possible to deceive ourselves into thinking $\vec{F}$ is conservative when it is not if we don't check our answer. Let me give an example to illustrate careless thinking.

Example 5.2.24. *If $\vec{F} = \langle 2x+2y, 4y^3 - \cos x\rangle$ then we need to solve*

$$\frac{\partial U}{\partial x} = -2x - 2y, \quad \frac{\partial U}{\partial y} = -4y^3 + \cos x.$$

Simply integrate each equation whilst holding the remaining variable fixed,

$$U = -x^2 - 2xy + C_1, \quad U = -y^4 + y\cos x + C_2$$

Comparing all three formulas we deduce, $\boxed{U = -x^2 - 2xy - y^4 + y\cos x}$. *But, we calculate*

$$\nabla U = \langle -2x - 2y - y\sin x, -x - 4y^3 + \cos x\rangle \quad \Rightarrow \quad -\nabla U \neq \vec{F}.$$

Let me be less careless with the previous example and illustrate a method which should discover if $\vec{F} \neq -\nabla U$ for any choice of U (that is to say, the given force field is not conservative).

Example 5.2.25. *If $\vec{F} = \langle 2x+2y, 4y^3 - \cos x\rangle$ then we need to solve*

$$(1.)\ \frac{\partial U}{\partial x} = -2x - 2y, \quad (2.)\ \frac{\partial U}{\partial y} = -4y^3 + \cos x.$$

Integrate equation (1.) while holding y fixed to obtain

$$U = -x^2 - 2xy + C_1(y)$$

Next substitute the above into (2.),

$$\frac{\partial}{\partial y}\left[-x^2 - 2xy + C_1(y)\right] = -4y^3 + \cos x \quad \Rightarrow \quad -2x + \frac{\partial C_1}{\partial y} = -4y^3 + \cos x.$$

We find $\frac{\partial C_1}{\partial y} + 4y^3 = 2x + \cos x$. This is impossible since the LHS is a function of y-alone whereas the RHS is a non-constant function of x. This is a contradiction which goes to show our assumption that (1.) and (2.) could be solved simultaneously was incorrect. In short, $\vec{F}$ is not conservative so we cannot find a potential energy function for $\vec{F}$.

Let's record a special case of Theorem 5.2.20 for our convenience.

Theorem 5.2.26. *Closed criterion for conservative vector fields in the plane*

If $\vec{F} = \langle P, Q\rangle$ and $\partial_x Q \neq \partial_y P$ then $\vec{F}$ is non conservative.

I explained why this is true in the previous subsection. Notice how it applies to the previous example. Notice $\vec{F} = \langle 2x+2y, 4y^3 - \cos x\rangle$ has $P = 2x+2y$ and $Q = 4y^3 - \cos x$ thus $\partial_y P = 2$ whereas $\partial_x Q = \sin x$. Thus is it unsurprising that our efforts to find a potential energy function failed. If in doubt about whether or not a given two-dimensional vector field is conservative then this is an easy criteria to check.

5.3 energy based physics examples

In the previous section I shared the mathematical backdrop which supports the current section. The basic idea which we use in most examples here is that energy is conserved when the net-force is conservative. Otherwise, the total energy changes according to the work done by non-conservative forces. Calculus is typically required for variable forces.

Example 5.3.1. Problem: *Suppose the net force $F(x) = -2\alpha x + 3\beta x^2$ acts on a mass m. If the mass has an initial velocity v_0 when $x = x_0$ then find the velocity v_f when $x = x_f$.*

Solution: *notice $U = \alpha x^2 - \beta x^3$ gives $-\frac{dU}{dx} = F$ thus F is conservative and so the total energy $E = \frac{1}{2}mv^2 + \alpha x^2 - \beta x^3$ is conserved. Since $E_0 = E_f$ we find*

$$\frac{1}{2}mv_0^2 + \alpha x_0^2 - \beta x_0^3 = \frac{1}{2}mv_f^2 + \alpha x_f^2 - \beta x_f^3 \quad \Rightarrow \quad \boxed{v_f = \sqrt{v_0^2 + \frac{2}{m}\left[\alpha(x_0^2 - x_f^2) - \beta(x_0^3 - x_f^3\right]}}.$$

Example 5.3.2. Problem: *a ball with mass m is thrown directly upward with speed v_0. How high does it go ?*

Solution: *assuming the motion is on Earth near the surface we have potential energy due to gravity of $U = mgh$ and if we ignore air friction then energy is conserved. Here $E = mgh + \frac{1}{2}mv^2$ thus*

$$mgh_0 + \frac{1}{2}mv^2 = mgh_f$$

since $v_f = 0$ at the top of the flight. Thus $\frac{1}{2}mv_0^2 = mgh_f - mgh_0$ thus, dividing by m we find

$$\boxed{h_f - h_0 = \frac{v_0^2}{2g}}.$$

In other words, the ball flies height $\frac{v_0^2}{2g}$ above where it was thrown.

Example 5.3.3. Problem: *suppose a box is given a push up an inclined plane at angle θ. If the coefficient of friction is μ_k the how far up the plane does the box slide given it has initial speed v_0 just after the push ?*

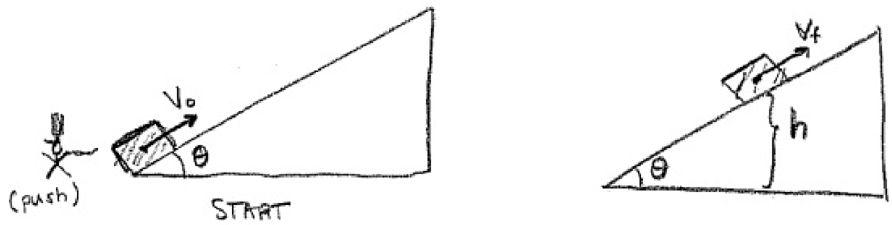
Solution: *the normal force $F_N = mg\cos\theta$ thus $F_f = \mu_k mg\cos\theta$. If the box stops when it reaches height h from the base of the incline then $\sin\theta = \frac{h}{L}$ where L is the distance the box slides up the plane. At the end of the motion, $PE = mgh$ and $KE = 0$. At the beginning of the motion, $PE = 0$ and $KE = \frac{1}{2}mv_0^2$. Notice the work done by friction is $-F_f L$ since F_f is directed opposite the motion. Energy is lost to friction; $E_0 - F_f L = E_f$. In particular,*

$$\frac{1}{2}mv_0^2 - L\mu_k mg\cos\theta = mgh$$

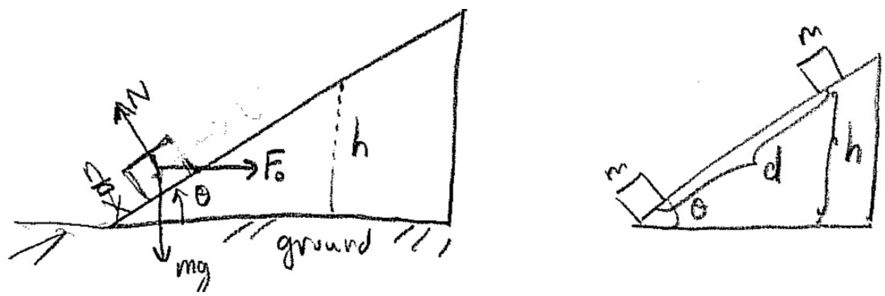
But, $h = L\sin\theta$ thus, dividing by m,

$$\frac{1}{2}v_0^2 - L\mu_k g\cos\theta = gL\sin\theta \quad \Rightarrow \quad \frac{1}{2}v_0^2 = gL\left(\mu_k\cos\theta + \sin\theta\right) \quad \Rightarrow \quad \boxed{L = \frac{v_0^2}{2g(\mu_k\cos\theta + \sin\theta)}}$$

Notice two special cases. First, if $\theta = 90^o$ then $L = v_0^2/2g$ which brings us back to the previous example if you think about it. On the other hand, if $\theta = 0^o$ then $L = \frac{v_0^2}{2g\mu_k}$, this is how far a box slides when given a push. Notice, we can also solve for $h = L\sin\theta$ and obtain $h = \frac{v_0^2}{2g(\mu_k\cot\theta+1)}$. See http://www.supermath.info/physics231lecture17.pdf page 3 for a handwritten solution to this example.



Example 5.3.4. Problem: *Suppose Mr. Tophat pushes with constant force F_0 horizontal to the ground (not parallel to the plane, see picture)S. If he pushes the box to height h then how fast is the box moving. Assume the box is initially at rest.*



Solution: *notice the angle between the direction of motion and F_0 is θ thus*

$$W_{Tophat} = \vec{F_0} \bullet \triangle \vec{r} = F_0 d \cos\theta$$

where d is the distance along the plane the box slides; $d = \frac{h}{\sin\theta}$ thus $W_{Tophat} = F_0 h \cot\theta$. In this problem energy is not conserved since Mr. Tophat does positive work to increase the total mechanical energy during the motion:

$$E_f = E_0 + W_{Tophat}$$

Thus as the initial mechanical energy is $E_0 = \frac{1}{2}mv_0^2 + mgh_0 = 0$ since $h_0 = 0$ and $v_0 = 0$ we find

$$\frac{1}{2}mv_f^2 + mgh = F_0 h \cot\theta$$

Then after some algebra we derive

$$\boxed{v_f = \sqrt{\frac{2(F_0\cot\theta - mg)h}{m}}}.$$

It might be interesting to rewrite the result in terms of the distance d the box has slid up the plane. Since $F_0 \cot\theta h = F_0 d\cos\theta$ and $mgh = mgd\sin\theta$ we find

$$\boxed{v_f = \sqrt{\frac{2[F_0\cos\theta - mg\sin\theta]d}{m}}}.$$

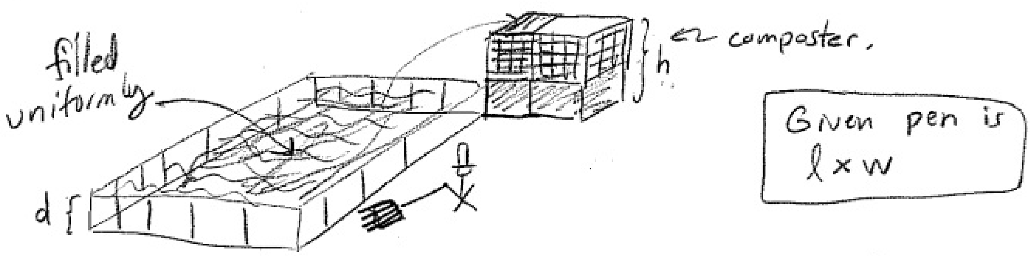
If we think about special cases the formula above is interesting. If $\theta = 0$ then $v_f = \sqrt{\frac{2F_0 d}{m}}$. If $\theta = 90^o$ then the formula gives imaginary v_f, which goes to show the problem is not well-defined in that case.

Remark 5.3.5. *The problem above could also be solved by finding the constant acceleration for the motion and working through the kinematic equations. If you look back at the timeless equation, it is nothing more than energy conservation in disguise. For instance,* $v_f^2 = v_0^2 - 2g(y_f - y_0)$ *when multiplied by* $m/2$ *and rearranged gives* $\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_0^2 + mgy_0$*. Notice if the problem you face asks for acceleration then energy methods are probably not the best approach. You should think about your strategy for solving problems.*

Please forgive my lazy formatting on what follows. I just want to put everything in one place for now. These examples were taken from http://www.supermath.info/physics231lecture16.pdf.

Example 5.3.6. Problem: *Suppose Mr. Tophat shovels a mess of wet leaves out of a rectangular* $\ell \times w$ *chicken pen into a bin which is height* h *above the flat ground on which the leaves lay. Assume the leaves have thickness* d *and constant density* $\rho = \frac{dm}{dV}$*. Find the work done by Mr. Tophat.*

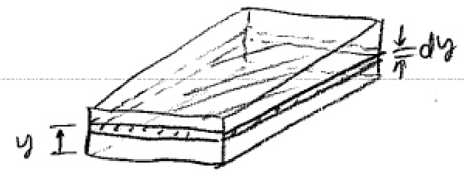
Solution: *integration is required since it takes more work to lift the leaves at one depth in the pile than others. We imagine slicing the leaves into horizontal slabs of volume* dV *so then we can calculate the work* dW *needed to lift such a slab. Finally, integrate* dW *to find the total work:*



Idea: we cannot just use h because leafs at $y = d$ only have to be lifted $\Delta y = h - d$ whereas leaves at base of pen need $\Delta y = h$. ALL the leaves at a particular y need same Δy to make it up to $y = h$.

Let ρ = density of leafs

$$\rho = \frac{dm}{dV}$$

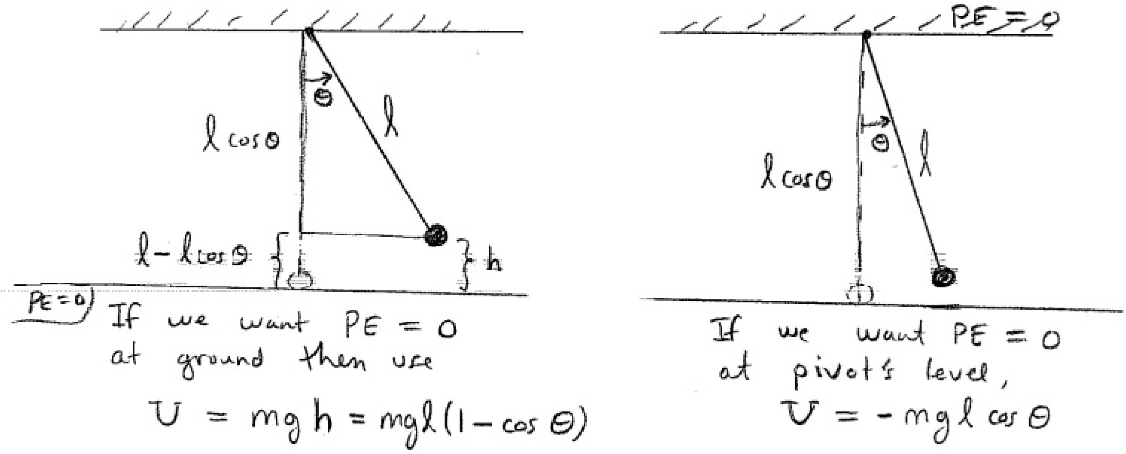


$$dV = (\text{AREA})\,dy = \ell w\,dy$$

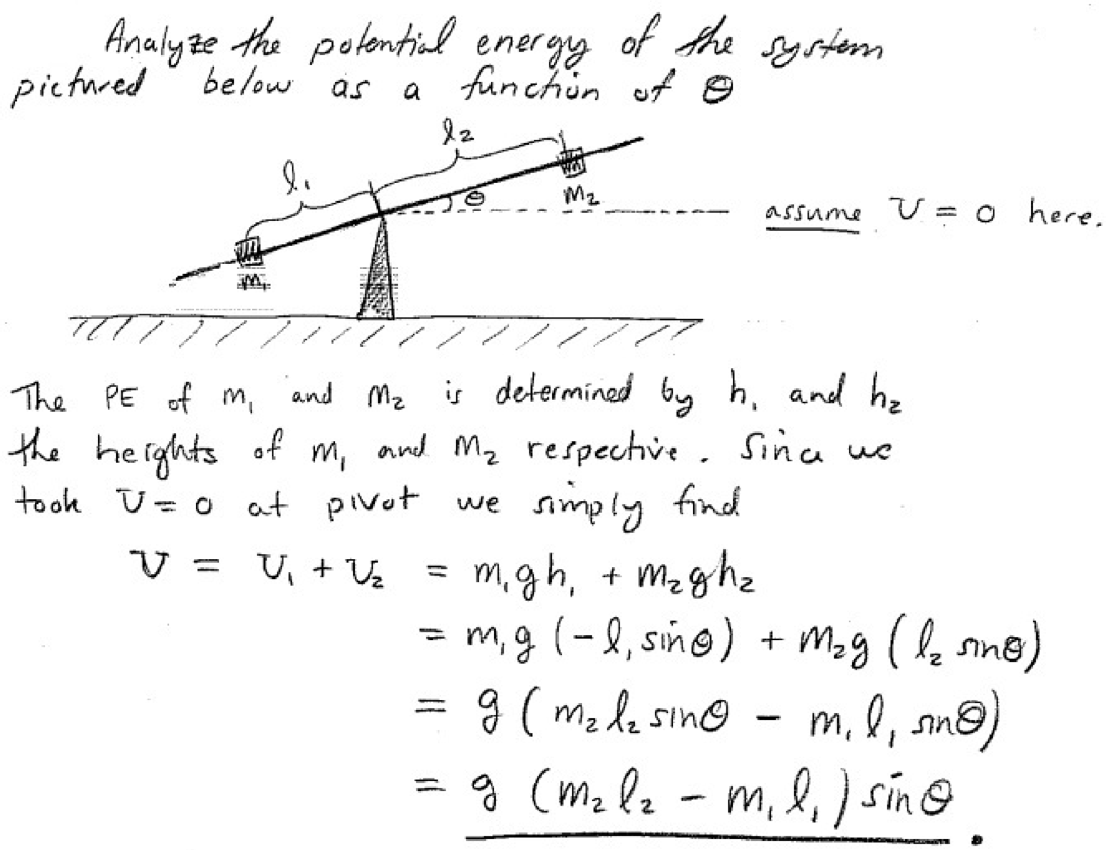
The mass $dm = \rho\,dV = \rho \ell w\,dy$ must be lifted $\Delta y = h - y$ against gravity $\Rightarrow dW = (\rho \ell w\,dy)(y - h)$

$$W = \int_0^d \rho \ell w (y - h)\,dy = \rho \ell w \left(\frac{y^2}{2} - hy\right)\Big|_0^d$$
$$= \rho \ell w \left(\frac{d^2}{2} - hd\right)$$
$$= \rho \ell w d \left(\frac{d}{2} - h\right)$$

Example 5.3.7. *A pendulum is an interesting physical example which energy analysis makes simple to understand. Given mass m as pictured we have* $E = \frac{1}{2}mv^2 + mg\ell(1-\cos\theta)$ *is constant through the motion. Alternatively, we can use* $E = \frac{1}{2}mv^2 - mg\ell\cos\theta$

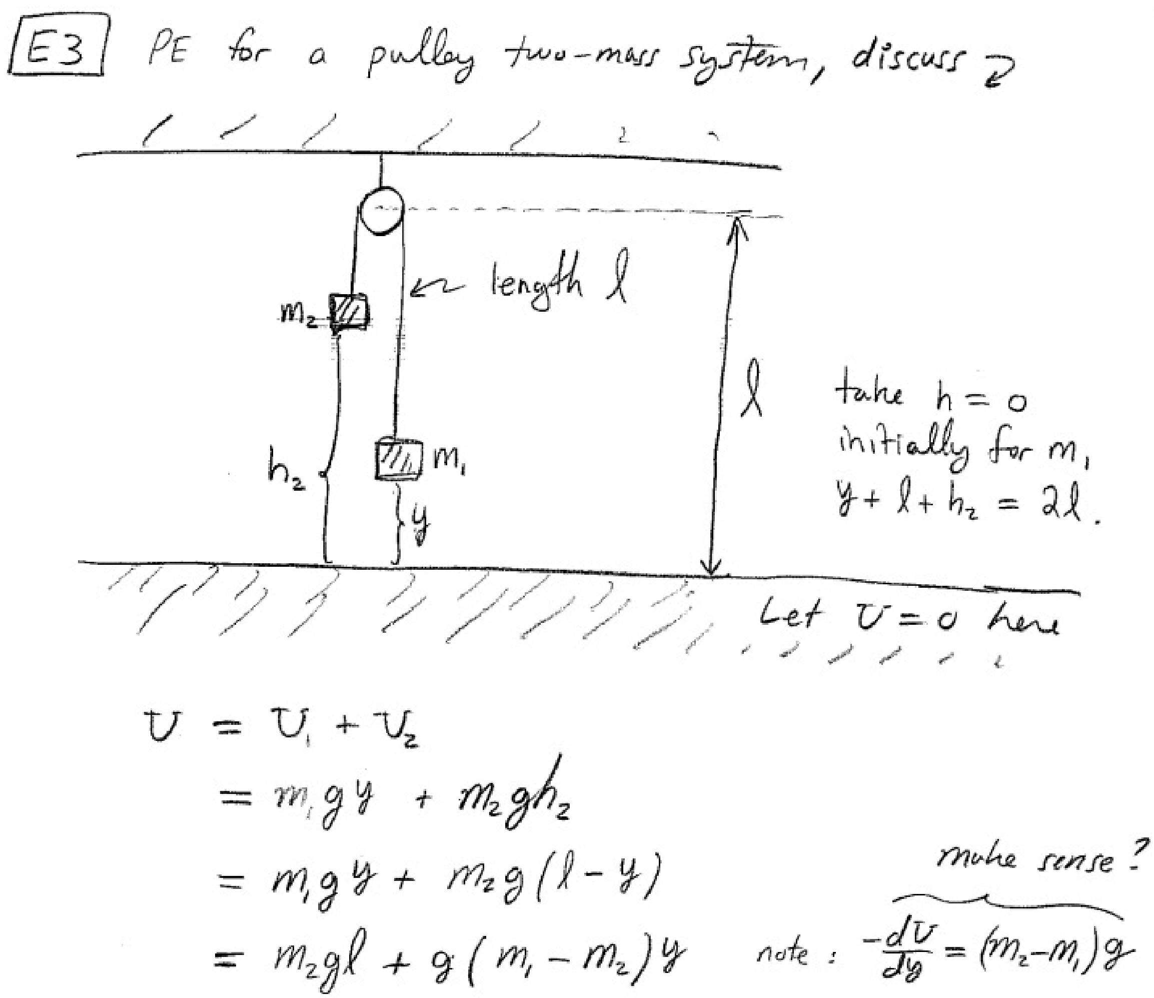


Example 5.3.8. *A teeter-totter is an interesting physical example which energy analysis makes simple to understand.*



Notice, $\theta = 0$ *implies no potential energy is stored. If* $\theta = 90^o$ *implies* $PE = m_2\ell_2 g - m_1\ell_1 g$.

Example 5.3.9. *A pair of masses hung over a pulley with a string is known as the Atwood Machine. Energy analysis allows a simple analysis of the resulting motion:*



If $m_2 > m_1$ then $U(y)$ is made smaller as y increases. A system tends to the state of lowest potential energy so it follows that m_2 falls as m_1 rises in this case.

5.3.1 power

If we consider a path C from some fixed point $\vec{r}_0$ to $\vec{r}(t)$ then

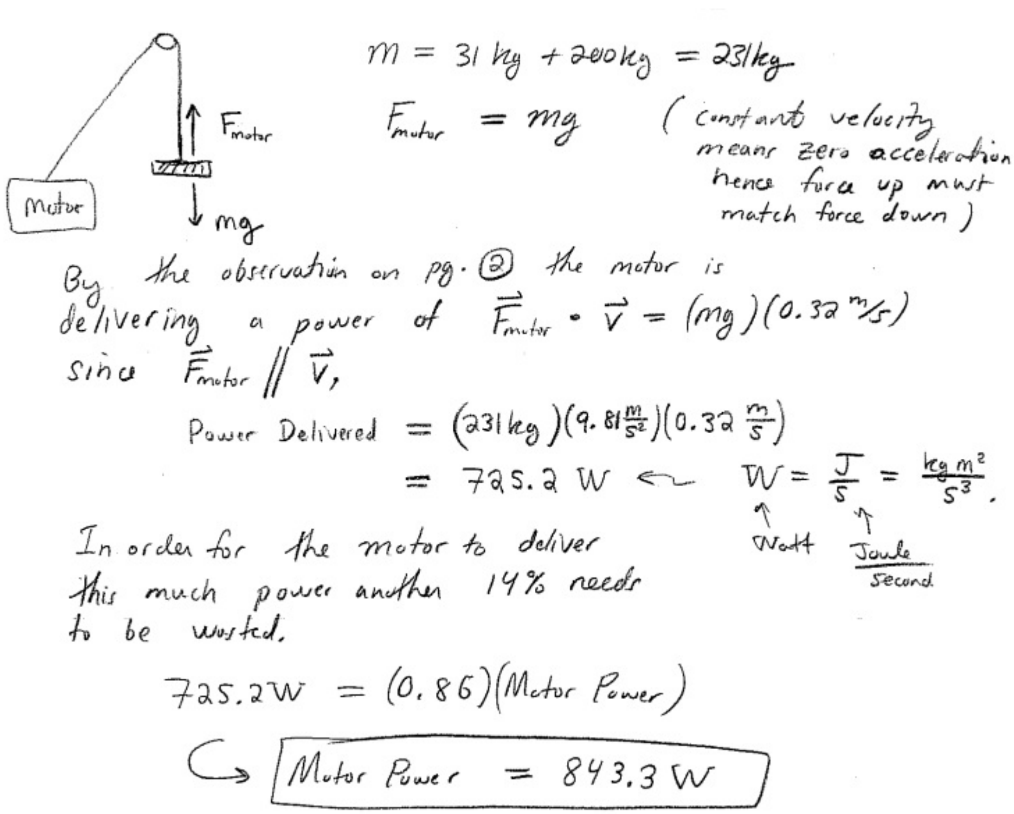
$$W_{net}(t) = \int_C \vec{F}_{net} \bullet d\vec{r} = K(t) - K(t_0)$$

by the work energy theorem. Differentiating gives

$$\frac{dW_{net}}{dt} = \frac{dK}{dt} = \frac{d}{dt}\left[\frac{m}{2}\vec{v} \bullet \vec{v}\right] = \frac{m}{2}\left[\frac{d\vec{v}}{dt} \bullet \vec{v} + \vec{v} \bullet \frac{d\vec{v}}{dt}\right] = m\vec{a} \bullet \vec{v} = \vec{F}_{net} \bullet \vec{v}$$

We define **power** developed by a force $\vec{F}$ as the instantaneous rate of change of the work done by $\vec{F}$. In fact, the power developed by $\vec{F}$ is given by $\boxed{\vec{F} \bullet \vec{v}}$ where $\vec{v}$ is the velocity of the mass to which the force $\vec{F}$ is applied. I suppose I should confess, the concepts of energy and power extend well past the mechanical context, but our current interests are mechanical[9]

Example 5.3.10. Problem: *Suppose you have a sub-basement filled with robot monkeys. An elevator with mass* $31kg$ *connects the monkeys to the upstairs levels. When in motion the elevator moves* $0.32m/s$ *upward, almost entirely without acceleration (ignore the brief start-up motion). The efficiency of the elevator motor is* 86%*. If your maximum load for the elevator is* $200kg$ *then what is the minimum power rating you want for the motor ?*
Solution:



5.4 springs and simple harmonic motion

Let us pause to settle some math we need to study springs. The differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

has general solution[10] $y = A\sin(\omega t + \phi)$ where A, ϕ are constants which depend on the initial conditions for y. It is simple to verify that twice differentiating $y = A\sin(\omega t + \phi)$ gives $\frac{d^2y}{dt^2} = -\omega^2 y$ which means the given solution solves the differential equation.

[9] in Physics 232 we consider electrical power, and in chemistry you could consider power due to a chemical reaction, there are many forms of energy and to each a corresponding concept of power

[10] in the differential equations course we often present this solution as $y = c_1 \cos\omega t + c_2 \sin\omega t$, but this formula is not ideal for physical analysis.

Example 5.4.1. *Given* $y(0) = 10$ *and* $y'(0) = 0$ *we solve*

$$\frac{d^2y}{dt^2} + 9y = 0$$

by identifying $\omega = 3$ *and hence* $y(t) = A\sin(3t + \phi)$. *Now we use the given initial conditions to find the values of the amplitude* A *and phase* ϕ, $y(0) = A\sin\phi = 10$ *and* $\frac{dy}{dt} = 3A\cos(3t + \phi)$ *so* $y'(0) = 3A\cos\phi = 0$. *We deduce* $\phi = \pi/2$ *and* $A = 10$ *thus* $\boxed{y = 10\sin(3t + \pi/2)}$.

Example 5.4.2. *Given* $y(0) = 0$ *and* $y'(0) = 20\pi$ *we solve*

$$\frac{d^2y}{dt^2} + 4\pi^2 y = 0$$

by identifying $\omega = 2\pi$ *and hence* $y(t) = A\sin(2\pi t + \phi)$. *Now we use the given initial conditions to find the values of the amplitude* A *and phase* ϕ, $y(0) = A\sin\phi = 0$ *and* $\frac{dy}{dt} = 2\pi A\cos(2\pi t + \phi)$ *so* $y'(0) = 2\pi A\cos\phi = 20\pi$. *We deduce* $\phi = 0$ *and* $A = 10$ *thus* $\boxed{y = 10\sin(2\pi t)}$.

The force of a spring with spring constant k on a mass m in one-dimensional motion in the x-direction is given by Hooke's Law $F = -kx$. Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = -kx \quad \Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Identify that $\omega = \sqrt{\frac{k}{m}}$ for the above differential equation hence

$$\boxed{x(t) = A\sin(\omega t + \phi)}.$$

gives the motion of the spring at time t. We call ω the **angular frequency** of the solution. Notice $\omega = \frac{2\pi}{T}$ where T is the **period** of the motion:

$$x(t + T) = x(t)$$

for all t. In other words, the motion returns to where it was in time T.

Example 5.4.3. *Suppose a spring has mass* $m = 0.5\,kg$ *and* $k = 50\,N/m$ *then* $k/m = 100\,\frac{1}{s^2}$ *and it follows that*[11] $\omega = \sqrt{k/m} = 10\ rad/s$. *Notice* $T = \frac{2\pi}{\omega} \cong 0.6283\,s$.

We should study the conservation of energy for the spring-mass system without friction. Since $F = -kx = -\frac{d}{dx}\left[\frac{1}{2}kx^2\right]$ we identify $U(x) = \frac{1}{2}kx^2$ is the potential energy and hence the total energy is given by

$$\boxed{E(x, v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2}.$$

On the other hand, we know for $\omega = \sqrt{\frac{k}{m}}$ the equation of motion for the spring is

$$x(t) = A\sin(\omega t + \phi) \quad \Rightarrow \quad \boxed{U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)}.$$

[11]since we are using sine and cosine with radian measure it follows that we ought to think of angular frequency as carrying units of radians per second. In contrast, cycles per second is denoted Hz for Hertz.

Differentiate to find the velocity at time t and hence the KE,

$$v(t) = \frac{dx}{dt} = \omega A \cos(\omega t + \phi) \quad \Rightarrow \quad \boxed{K = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi)}.$$

Notice that $\omega^2 = \frac{k}{m}$ thus $m\omega^2 = k$ hence

$$E(x, v) = \frac{1}{2} kA^2 \sin^2(\omega t + \phi) + \frac{1}{2} kA^2 \cos^2(\omega t + \phi) = \frac{1}{2} kA^2$$

Of course this had to happen, we know that the total mechanical energy is conserved when the net-force is conservative. We should observe the following nice formulas:

$$\boxed{E(x, v) = \frac{1}{2} kA^2 = \frac{1}{2} m\omega^2 A^2}.$$

Notice further the maximum distance the spring is compressed or stretched is given by A. Furthermore, from the velocity equation $v(t) = \omega A \cos(\omega t + \phi)$ we can deduce the maximum speed is ωA. In other words,

$$\boxed{E(x, v) = \frac{1}{2} kx_{max}^2 = \frac{1}{2} mv_{max}^2}$$

where $x_{max} = A$ and $v_{max} = A\omega$. The formula above reflects the fact that the PE is largest when $KE = 0$ and vice-versa. The KE is largest when $PE = 0$. The motion of a spring can be naturally broken into four parts:

(i.) The spring is maximally stretched, $x = A$ and $v = 0$. At this instant the energy is purely potential and the spring is momentarily at rest before it begins its contraction.

(ii.) The spring is in equilbrium position with $x = 0$ and $v_{max} = -A\omega$. At this instant the energy is purely kinetic and the motion is leftward.

(iii.) The spring is fully compresses, $x = -A$ and $v = 0$. At this instant the energy is purely potential and the spring is momentarily at rest before it begins its expansion.

(iv.) The spring is in equilbrium position with $x = 0$ and $v_{max} = A\omega$. At this instant the energy is purely kinetic and the motion is rightward.

Let us consider a simple problem to further appreciate the interconnected concepts we've discussed in this section thus far.

Example 5.4.4. Problem: *A spring attached to a* 350 *gram mass has* $100\,J$ *stored when it is maximally compressed at* $23.0\,cm$. *Find the angular frequency, period, spring constant and maximum speed for the given spring-mass system.*

Solution: *given* $m = 0.35\,kg$ *and* $E = 100\,J = \frac{1}{2} m\omega^2$ *hence*

$$\omega = \sqrt{\frac{2(100\,J)}{0.35\,kg}} \cong 23.90\,\frac{rad}{s}.$$

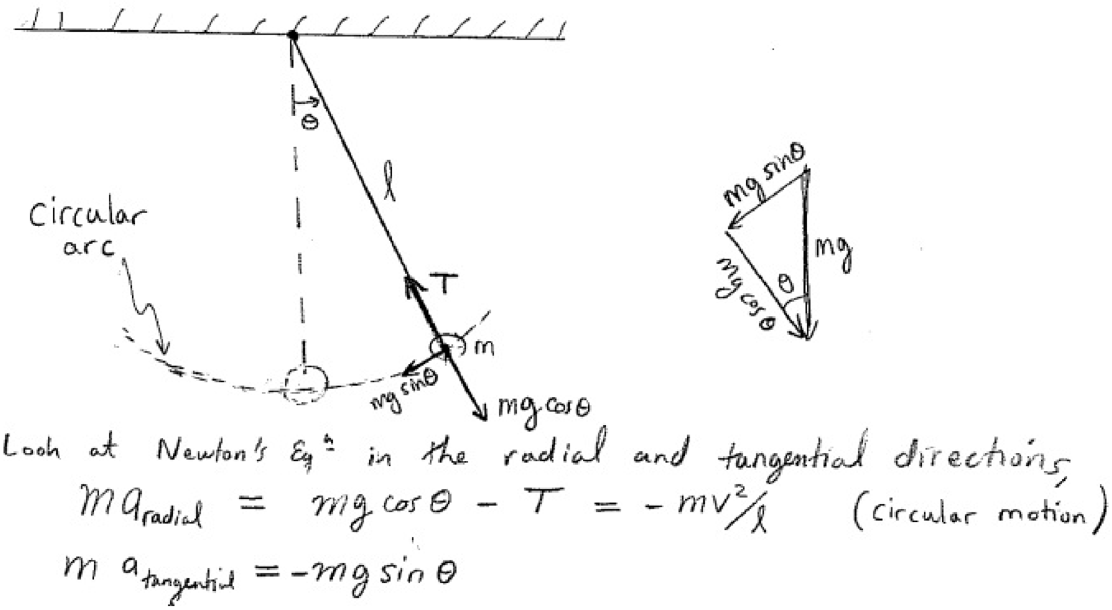
Then $T = \frac{2\pi}{\omega} \cong 0.2628\,s$. *Next, since* $E = \frac{1}{2} kA^2$ *thus*

$$k = \frac{2E}{A^2} = \frac{200\,J}{(0.23\,m)^2} = 3781\,N/m$$

Finally, $v_{max} = A\omega = (0.23)(23.90)\,m/s \cong 5.497\,m/s$.

Many physical systems have motion which is similar to the spring-mass problem. If the motion is oscillatory with constant period and amplitude given by some sinuisoidal function then we say the motion is **simple harmonic motion** (SHM). Often a more complicated system in a particular limit has motion which is close to SHM. Motion of a pendulum is a good example of this phenomenon.

Example 5.4.5. *If we attach a mass m to an essentially massless string of length ℓ and we imagine the string swings from a frictionless pivot then we can analyze the motion as pictured below:*



Here $a_T = \frac{dv}{dt} = \frac{d}{dt}\left(\ell\frac{d\theta}{dt}\right) = \ell\frac{d^2\theta}{dt^2}$. Hence the tangential component of Newton's Second Law yields

$$m\ell\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

If the θ is small[12] *then $\sin\theta \cong \theta$ and the differential equation above gives*

$$\ell\frac{d^2\theta}{dt^2} + g\theta = 0 \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$$

Observe $\omega^2 = \frac{g}{\ell}$ thus $\omega = \sqrt{\frac{g}{\ell}} = \frac{2\pi}{T}$ thus $\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}}$. The simple pendulum gives us a nice way to measure the value of g. We can measure ℓ and use many periods to gain good accuracy on the measurement of T.

5.5 energy analysis

In this section we consider a one-dimensional problem where the net-force is conservative with potential energy U. Notice $F = -\frac{dU}{dx}$ and so $m\frac{dv}{dt} = -\frac{dU}{dx}$ then

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}mv^2 + U\right] = mv\frac{dv}{dt} + \frac{dU}{dt} = mv\frac{dv}{dt} + \frac{dx}{dt}\frac{dU}{dx} = v\left[ma + \frac{dU}{dx}\right] = 0.$$

[12]for angles within $\pm 20^o$ the equation $\sin\theta = \theta$ is true to within about 1% accuracy. In Calculus II we learn $\sin\theta = \theta - \frac{1}{6}\theta^3 + \cdots$ and roughly speaking the term $|\theta^3/6|$ quantifies an upper bound on $|\sin\theta - \theta|$.

Of course this is not surprising, we know energy is conserved for a conservative system. This means that we can plot $E(x,v) = \frac{1}{2}mv^2 + U(x)$ in the xv-plane and the resulting curve shows possible motions for the given energy-level. To plot such curves we note they satisfy a couple defining features:

$$v = \frac{dx}{dt}$$

means the curve goes rightward for $v > 0$ and leftward for $v < 0$. Furthermore, a critical point of the energy-level curve we need both $\frac{dx}{dt} = 0$ and $\frac{dv}{dt} = 0$. Points with $\frac{dx}{dt} = 0$ are on the x-axis in the xv-plane. Points with $\frac{dv}{dt} = 0$ correspond to points where x is a critical number of the potential energy. Why ? Consider:

$$0 = \frac{dv}{dt} = \frac{F}{m} = \frac{-1}{m}\frac{dU}{dx}$$

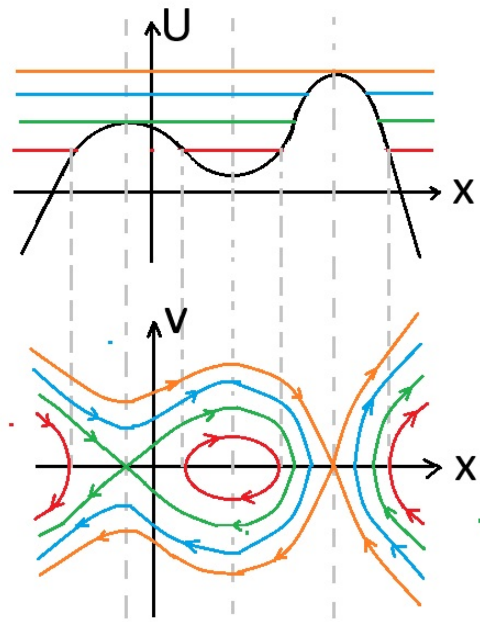
thus points with zero $0 = \frac{dv}{dt}$ are the same points with $\frac{dU}{dx} = 0$. This motivates the method of analysis given in the examples below. The procedure is as follows:

(i.) Given $U(x)$ find its critical numbers and plot U.

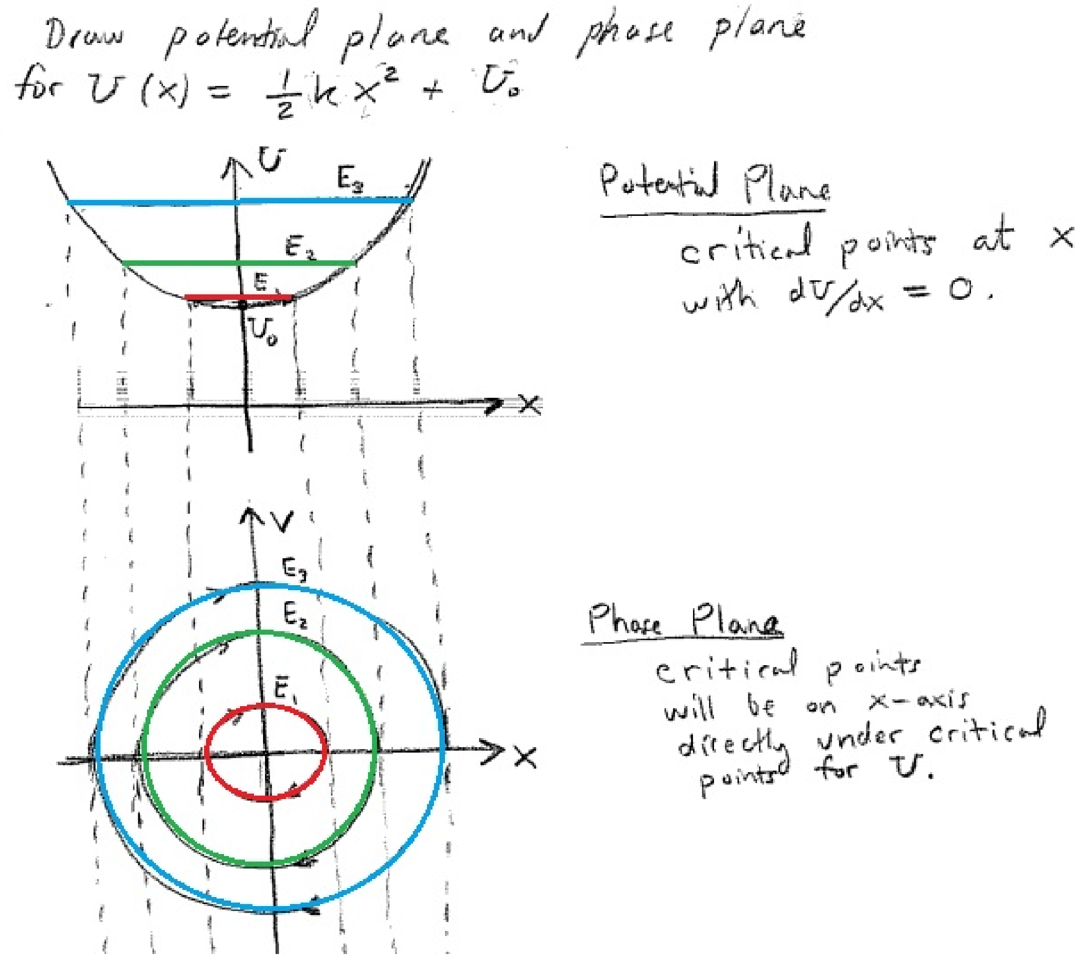
(ii.) Make a xv-graph below the plot in the previous step. Try to line-up the x-axis with the same scale so you can match-up critical points and other data,

(iii.) Graph several representative energy levels in the U-graph, key idea, for a given energy level the line must not go below the PE plot since KE cannot be negative

(iv.) From the energy level plots, create corresponding trajectories in the Poincare Plane (xv-plot



Example 5.5.1. *The Poincare plot for the spring without friction is as follows:*



Chapter 6

momentum

Momentum gives us another way to understand Newton's Laws. The concept is especially helpful in the analysis of collisions. Basically there are two types of collisions; elastic and inelastic. By definition, an elastic collision is one in which the net kinetic energy is conserved. In contrast, in an inelastic collision the net kinetic energy is not conserved. Heuristically, an elastic collision is a perfectly bouncy collision whereas an inelastic collision has some stickyness involved. A purely inelastic collision is one in which the masses colliding stick together after the collision.

To begin we need to develop some ideas which allow us to treat a system of masses as a single mass at the center of mass. This is the *particle model* and we've been using it all along. You've probably been bothered we ignored where the force was applied to various masses we've considered. Pretty much every time we ignore the physical extent of a body we're probably using the particle model implicitly. To take into acount the size of an object and the difference between pushing on the middle, top or base of a box would require us to provide additional mathematics which describes the position of all the atoms comprising the box. There are techniques which accomplish such an analysis for rigid bodies. We have a course in statics which is in part about what it takes to knock over objects. There is a Chapter in most introductory books about the problem of *mechanical equilbrium*. Sadly, we do not have time to cover all that in this course, but, rest assured the material we do cover is fundamental in understanding more complicated problems which don't use the simplistic particle model.

We begin by studying center of mass. This concept is given first for a finite collection of point masses. Then we generalize to a mass distributed along a curve, plane, surface or volume. When we consider a continuous distribution of mass we use a **mass density** function to describe the mass per unit length, area or volume. Integration over a curve, plane, surface or volume is necessary to find the center of mass as well as the total mass of a continuous distribution.

Once the center of mass is defined we then go on to define momentum for a system of particles and we likewise define the velocity and momentum of the system as a whole. In short, a system of particles can be thought of as a single particle with total mass M at the center of mass $\vec{R}$. The velocity of the center of mass is $\vec{V} = \frac{d\vec{R}}{dt}$ and $\vec{P} = M\vec{V}$ is the total momentum of the system. We show that the if $\vec{f}^{ext}$ is the sum of the external forces on a system then $\frac{d\vec{P}}{dt} = \vec{f}^{ext}$. In particular, if the net-external force is zero for a system then the total momentum is constant. On the other hand, for $\triangle t$ small we also have the approximation $\triangle\vec{P} = \vec{f}^{ext}\triangle t$. Thus the momentum is conserved if the duration $\triangle t$ is very small. On the other hand, if the duration is not small then we must integrate; $\triangle\vec{P} = \int_{t_1}^{t_2} \vec{f}^{ext} dt$ to calculate the change in momentum for the system from t_1 to t_2. Incidentally, the $\triangle\vec{P}$ delivered by a particular force $\vec{F}$ is known as the **impulse** of the force.

Collisions are usually a problem which requires careful use of vector math. Net momentum is a vector so both the x, y and z components of the total momentum are conserved. Note we assume $\triangle t \to 0$ whenever we think about a collision, we're thinking about the *before* and the *after* as if they are nearly the same time. I should mention, and we will derive, in the case of one-dimensional motion, we can conserve momentum using our usual $\pm$ conventions to describe direction along a line. There are special formula known for elastic one or two-dimensional or collisions. Beware, such formulas only apply in the very special case of an elastic collision. Usually the collision is inelastic and we only have conservation of momentum as a guide.

6.1 momentum for a point particle

Let m be the mass of a particle found at position $\vec{r}$ then recall $\vec{v} = \frac{d\vec{r}}{dt}$ is the **velocity** of m. Let us introduce a new physical quantity:

Definition 6.1.1. *If mass m has velocity $\vec{v}$ the define the* **momentum** *of m to be $\vec{p} = m\vec{v}$.*

Notice momentum is a vector quantity with units of $kg\, m/s$. Furthermore, recall $\vec{a} = \frac{d\vec{v}}{dt}$ is the **acceleration** of m and if m is constant in time we may reformulate Newton's Second Law as follows:

$$\vec{F}_{net} = m\frac{d\vec{v}}{dt} \quad \Rightarrow \quad \vec{F}_{net} = \frac{d}{dt}(m\vec{v}) \quad \Rightarrow \quad \boxed{\vec{F}_{net} = \frac{d\vec{p}}{dt}}.$$

Furthermore, by the Fundamental Theorem of Calculus we find:

$$\int_{t_1}^{t_2} \vec{F}_{net} dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt} dt = \vec{p}(t_2) - \vec{p}(t_1) \quad \Rightarrow \quad \boxed{\triangle\vec{P} = \int_{t_1}^{t_2} \vec{F}_{net} dt}.$$

Often we use notation like $\vec{p}(t_2) = \vec{p}_2$ and $\vec{p}(t_1) = \vec{p}_1$ where $\triangle\vec{p} = \vec{p}_2 - \vec{p}_1$. The change in momentum has a special name, it is known as the **impulse** which is sometimes denoted $\vec{J}$. For a one-dimensional problem we simply write $J = p_2 - p_1 = \int_{t_1}^{t_2} F_{net} dt$. **If the force is constant** then notice

$$\triangle\vec{P} = \int_{t_1}^{t_2} \vec{F}_{net} dt = \vec{F}_{net} \int_{t_1}^{t_2} dt = \vec{F}_{net} \triangle t \quad \Rightarrow \quad \boxed{\triangle\vec{P} = \vec{F}_{net} \triangle t}.$$

On the other hand, even if the force is not constant, we can calculate the **average force** by dividing the **impulse** $\triangle\vec{p} = \vec{p}_2 - \vec{p}_1$ by the duration $\triangle t = t_2 - t_1$

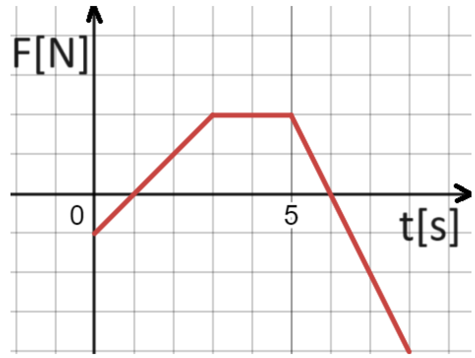
$$\boxed{\vec{F}_{avg} = \frac{\triangle\vec{p}}{\triangle t}}.$$

Example 6.1.2. *Imagine a noble father throws a $5kg$ cat horizontally against a wall with a speed of $10m/s$. Then the evil cat rebounds after hitting a wall with a speed of $20m/s$ at an angle of 30^o above the horizontal. If it took the cat $\triangle t = 0.1\, s$ to rebound then we can calculate the average force of the wall on the cat as follows: initially $\vec{p}_o = \langle (10m/s)(5kg), 0\rangle = \langle 50kgm/s, 0\rangle$ and after the rebound $\vec{p}_f = \langle -\cos 30^o(20m/s)(5kg), \sin 30^o(20m/s)(5kg)\rangle = \langle -86.60, 50\rangle kgm/s$. Thus,*

$$\vec{F}_{avg} = \frac{\triangle\vec{p}}{\triangle t} = \frac{1}{0.1s}(\langle -86.60, 50\rangle - \langle 50, 0\rangle)kgm/s = \langle -1366, 500\rangle N.$$

The magnitude of the average force is $1455N$ at standard angle $\theta = 159.9^o$.

Example 6.1.3. *Given that each square represents a* $(1N)(1s) = (kgm/s^2)s = kgm/s$ *we find the area under the given time vs. force graph indicates:*



$\triangle p = -0.5\, kgm/s$ *signed-area for* $0 \leq t \leq 1s$,

$\triangle p = 2\, kgm/s$ *signed-area for* $1s \leq t \leq 3s$,

$\triangle p = 4\, kgm/s$ *signed-area for* $3s \leq t \leq 5s$,

$\triangle p = 1\, kgm/s$ *signed-area for* $5s \leq t \leq 6s$,

$\triangle p = -4\, kgm/s$ *signed-area for* $6s \leq t \leq 8s$.

In total, we find the net-impulse delivered by the force graphed above is $2.5\, kgm/s$.

Sometimes it is nice to formulate everything in terms of momentum. We know $KE = \frac{1}{2}mv^2$. Since $p = mv$ gives $v = p/m$ note

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(p/m)^2 = \frac{p^2}{2m} \quad \Rightarrow \quad \boxed{KE = \frac{p^2}{2m}}.$$

Example 6.1.4. *Suppose a particle quadruples its kinetic energy. What can we say about its momentum ? Well, if* $KE_f = 4KE_o$ *then*

$$\frac{p_f^2}{2m} = 4\frac{p_o^2}{2m} \quad \Rightarrow \quad p_f^2 = 4p_o^2 \quad \Rightarrow \quad p_f = 2p_o.$$

The example below illustrates how Newton's Second Law in momentum form allows us to treat problems where the mass is not constant.

Example 6.1.5. *Suppose the mass of a street cleaning truck is giving by* $m = m_o + \alpha t$. *If the motor creates a constant force* F_o *then we can derive the resulting acceleration by solving Newton's Second Law in momentum form:*

$$\frac{dP}{dt} = F_o \quad \Rightarrow \quad \frac{d}{dt}(mv) = F_o \quad \Rightarrow \quad \frac{dm}{dt}v + m\frac{dv}{dt} = F_o \quad \Rightarrow \quad a = \frac{dv}{dt} = \frac{F_o - \alpha v}{m}$$

as $\frac{dm}{dt} = \alpha$. *Largest acceleration occurs at time zero;* $a = \frac{F_o}{m_o}$. *Separate and integrate*

$$\int_{v_o}^{v_f} \frac{dv}{F_o - \alpha v} = \int_0^{t_f} \frac{dt}{m_o + \alpha t} \quad \Rightarrow \quad \frac{-1}{\alpha}\ln\left|\frac{F_o - \alpha v_f}{F_o - \alpha v_o}\right| = \frac{1}{\alpha}\ln\left|\frac{m_o + \alpha t_f}{m_o}\right|.$$

Then by properties of the logarithm and algebra we find

$$\left|\frac{F_o - \alpha v_f}{F_o - \alpha v_o}\right| = \left|\frac{m_o}{m_o + \alpha t_f}\right| \quad \Rightarrow \quad v = \frac{F_o}{\alpha} + \frac{m_o}{m_o + \alpha t}\left(v_o - \frac{F_o}{\alpha}\right)$$

where we've set $t_f = t$ *and* $v_f = v$.

6.2 center of mass

Definition 6.2.1. *Suppose masses $m_1, m_2, \dots, m_n$ are at positions $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ then $M = \sum_{i=1}^n m_i$ is the* **total mass** *of the system and $\vec{R}$ is the* **center of mass** *as defined below:*

$$\vec{R} = \frac{1}{M}\sum_{i=1}^n m_i\vec{r}_i = \frac{1}{M}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n\right).$$

Similarly, we define the **velocity of the center of mass** *by $\vec{V} = \frac{d\vec{R}}{dt}$.*

Most real world objects concern some continuous three dimensional distribution of mass. However, we also find conceptual use for one or two dimensional distributions. Every one of the integrations given below amount to a continuous extension of the center of mass of finitely many particles. Formally, we replace $\sum$ with $\int$ to go from the finite to the continuous and we replace m_i with dm as follows:

$$M = \int dm \qquad \& \qquad x_{cm} = \frac{1}{M}\int x\,dm, \qquad y_{cm} = \frac{1}{M}\int y\,dm, \qquad z_{cm} = \frac{1}{M}\int z\,dm.$$

So $\vec{R} = \langle x_{cm}, y_{cm}, z_{cm}\rangle$. Now, the details of how the integral over dm should be calculated varies with context. There are five common contexts. Let us make the calculational methods explicit:

(1.) Mass along a line with coordinate x; we set $\lambda = \frac{dm}{dx}$ to denote the mass per unit length. If the object is found from x_1 to x_2 then since $dm = \lambda dx$ we calculate the total mass M and the center of mass x_{cm} via:

$$M = \int_{x_1}^{x_2} \lambda dx \qquad \& \qquad x_{cm} = \frac{1}{M}\int_{x_1}^{x_2} x\lambda dx$$

Notice the formulas above are simply calculational methods to form the integrals $\int dm$ and $\int xdm$ along the line-segment $[x_1, x_2]$.

(2.) Mass along a curve C parametrized by $t \mapsto \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $t_1 \leq t \leq t_2$ with **linear mass density** $\lambda = \frac{dm}{ds}$ is calculated by integrals with respect to arclength,

$$M = \int_C \lambda\,ds \qquad \& \qquad x_{cm} = \frac{1}{M}\int_C x\lambda\,ds, \qquad y_{cm} = \frac{1}{M}\int_C y\lambda\,ds, \qquad z_{cm} = \frac{1}{M}\int_C z\lambda\,ds.$$

To be explicit, the calculation of $\int_C f ds$ is found as follows:

$$\int_C f ds = \int_{t_1}^{t_2} f(\vec{r}(t))\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\,dt.$$

We've done this integral before, note $\int_C ds$ give the distance travelled along C, or the arclength of C. Integration to find M, x_{cm}, y_{cm} and z_{cm} are nearly the same calculation as the arclength.

(3.) If a mass is spread over a planar region P with **area mass density** $\sigma = \frac{dm}{dA}$ then we use area integrals[1] to calculate the total mass M and center of mass (x_{cm}, y_{cm}). We assume $z = 0$ and ignore z in this application. Notice $dm = \sigma dA$ hence:

$$M = \int_P \sigma dA \qquad \& \qquad x_{cm} = \frac{1}{M}\int_P x\sigma dA, \qquad y_{cm} = \frac{1}{M}\int_P y\sigma dA.$$

[1]these do not necessarily require the full calculational arsenal of third semester calculus. Indeed, the statics course in Mechanical Engineering also has similar integration and no prerequisite of Calculus III.

(4.) If a mass is spread over a surface S with parametrization $(u,v) \mapsto \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for $(u,v) \in D \subseteq \mathbb{R}^2$ and **surface mass density** $\sigma = \frac{dm}{dS}$ where $dS = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\, dv$. Then,

$$M = \int_S \sigma\, dS \qquad \& \qquad x_{cm} = \frac{1}{M}\int_S x\sigma\, dS, \qquad y_{cm} = \frac{1}{M}\int_S y\sigma\, dS, \qquad z_{cm} = \frac{1}{M}\int_S z\sigma\, dS.$$

The integral with respect to surface area is explicitly found as follows:

$$\int_S f\, dS = \int_D f(\vec{r}(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\, dv.$$

(5.) If a mass is spread over a three dimensional region B with **volume mass density** $\rho = \frac{dm}{dV}$ then $dm = \rho dV$ and we may calculate total mass and center of mass via volume integrals[2]:

$$M = \int_B \rho\, dV \qquad \& \qquad x_{cm} = \frac{1}{M}\int_B x\rho\, dV, \qquad y_{cm} = \frac{1}{M}\int_B y\rho\, dV, \qquad z_{cm} = \frac{1}{M}\int_B z\rho\, dV.$$

Example 6.2.2. *Suppose mass M is uniformly distributed over $[x_1, x_2]$ then $\lambda = \frac{M}{x_2 - x_1}$ and*

$$x_{cm} = \frac{1}{M}\int_{x_1}^{x_2} \left(\frac{M}{x_2 - x_1}\right) x\, dx = \frac{1}{x_2 - x_1}\left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right) \quad \Rightarrow \quad \boxed{x_{cm} = \frac{x_1 + x_2}{2}}.$$

Example 6.2.3. *Suppose mass M is uniformly distributed over $R = [x_1, x_2] \times [y_1, y_2]$ then area$(R) = (x_2 - x_1)(y_2 - y_1)$ and $\sigma = \frac{M}{(x_2 - x_1)(y_2 - y_1)}$. Imagine slicing the rectangle R into vertical strips where $y_1 \leq y \leq y_2$ and x ranges from x to $x + dx$. By the previous example, the center of mass of such a vertical strip is found at $(x, \frac{1}{2}(y_1 + y_2))$. Notice $dA = (y_2 - y_1)dx$ for the strip. Thus the total mass of the strip is simply*

$$dm = \sigma dA = \frac{M}{(x_2 - x_1)(y_2 - y_1)}(y_2 - y_1)dx = \frac{M dx}{x_2 - x_1}$$

Now we find the center of mass for the rectangle by integrating over dx,

$$x_{cm} = \frac{1}{M}\int_{x_1}^{x_2} x \frac{M dx}{x_2 - x_1} = \frac{1}{x_2 - x_1}\left(\frac{x_2^2}{2} - \frac{x_2^2}{2}\right) = \frac{x_1 + x_2}{2}.$$

$$y_{cm} = \frac{1}{M}\int_{x_1}^{x_2} \left[\frac{1}{2}(y_1 + y_2)\right] \frac{M dx}{x_2 - x_1} = \left[\frac{1}{2}(y_1 + y_2)\right] \frac{x_2 - x_1}{x_2 - x_1} = \frac{y_1 + y_2}{2}.$$

Let me repeat, we are using the previous example to inform us that the center of mass of vertical strip is found at $(x, \frac{1}{2}(y_1 + y_2))$. This is why we integrated $\frac{1}{2}(y_1 + y_2)$ for y of each vertical strip. In summary, we found the very unsurprising result that the center of mass for a rectangle with uniform mass distribution is at the center of rectangle which is at $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$.

Example 6.2.4. *Suppose the helix H with parametric equations $x = R\cos t, y = R\sin t$ and $z = bt$ has mass density $\lambda = \gamma z$. Given $0 \leq t \leq 2\pi$ let us calculate the mass and center of mass in terms of the given constants R, b, γ. Notice $dx = -R\sin t dt$ and $dy = R\cos t dt$ and $dz = bdt$ hence*

[2] again, this does not necessarily indicate the whole arsenal of third semester calculus is needed, we have methods of calculating volume from even first semester calculus which are suitably modified here.

$ds^2 = dx^2 + dy^2 + dz^2 = R^2dt^2 + b^2dt^2$ and we may use $ds = \sqrt{R^2+b^2}dt$ for this constant speed helix. Hence calculate:

$$M = \int_H dm = \int_H \gamma z ds = \int_0^{2\pi} \gamma bt\sqrt{R^2+b^2}dt = 2\pi^2 b\gamma\sqrt{R^2+b^2}$$

Let us work out the integrals needed first: using integration by parts with $u = t$ and $dv = \cos t dt$ hence $v = \sin t$,

$$\int_0^{2\pi} t\cos t dt = t\sin t\Big|_0^{2\pi} - \int_0^{2\pi} \sin t dt = 0$$

Likewise, using integration by parts with $u = t$ and $dv = \sin t dt$ hence $v = -\cos t$,

$$\int_0^{2\pi} t\sin t dt = -t\cos t\Big|_0^{2\pi} + \int_0^{2\pi} \cos t dt = -2\pi.$$

Consequently,

$$x_{cm} = \frac{1}{M}\int_H x\, dm = \frac{\gamma bR\sqrt{R^2+b^2}}{M}\int_0^{2\pi} t\cos t dt = 0$$

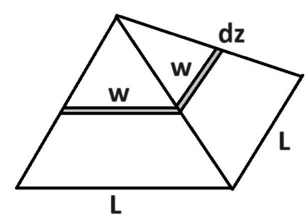
$$y_{cm} = \frac{1}{M}\int_H y\, dm = \frac{\gamma bR\sqrt{R^2+b^2}}{M}\int_0^{2\pi} t\sin t dt = \frac{-2\pi\gamma bR\sqrt{R^2+b^2}}{2\pi^2 b\gamma\sqrt{R^2+b^2}} = \frac{-R}{\pi}.$$

Last, we calculate

$$z_{cm} = \frac{1}{M}\int_H z dm = \frac{1}{M}\int_H \gamma z^2 ds = \frac{\sqrt{R^2+b^2}}{M}\int_0^{2\pi} \gamma(bt)^2 dt = \frac{\gamma b^2\sqrt{R^2+b^2}}{2\pi^2 b\gamma\sqrt{R^2+b^2}}\frac{(2\pi)^3}{3} = \frac{4b\pi}{3}.$$

In summary, the mass of the helix is $M = 2\pi^2 b\gamma\sqrt{R^2+b^2}$ and its center of mass is at $(0, -R/\pi, 4b\pi/3)$.

Example 6.2.5. *Consider a square pyramid P with total mass M and base side length of L and height H. Suppose the mass density is given by $\rho = \alpha z^2$ for $0 \leq z \leq H$. Imagine a square slice of side-length w and thickness dz. Since $\rho = \frac{dm}{dV}$ and $dV = w^2dz$ we find $dm = \alpha z^2w^2dz$ for the pictured slice. To calculate much else we need to notice that w depends linearly on z such that $w(0) = L$ whereas $w(H) = 0$. Notice this forces us to write $w = L - \frac{L}{H}z = \frac{L(H-z)}{H}$. Calculate,*



$$dm = \alpha z^2w^2dz = \frac{\alpha L^2(H-z)^2z^2}{H^2}dz = \frac{\alpha L^2}{H^2}\left(H^2z^2 - 2Hz^3 + z^4\right)dz$$

To find the total mass we add up all the little dm's by integrating over z,

$$M = \int_P dm = \int_0^H \frac{\alpha L^2}{H^2}\left(H^2z^2 - 2Hz^3 + z^4\right)dz = \frac{\alpha L^2H^3}{30}$$

Notice the dimensional analysis checks out, if $\rho = \alpha z^2$ then this requires[3] that α carry units of kg/m^5 hence the formula above is quite reasonable. By symmetry we find $x_{cm} = y_{cm} = 0$ supposing the peak of the pyramid is on the z-axis. To calculate z_{cm} we need to integrate:

$$z_{cm} = \frac{1}{M}\int_P z dm = \frac{1}{M}\int_0^H \frac{\alpha L^2}{H^2}\left(H^2z^3 - 2Hz^4 + z^5\right)dz = \frac{\alpha L^2H^4}{60M} = \frac{\alpha L^2H^4}{60}\frac{30}{\alpha L^2H^3} = \frac{H}{2}.$$

[3] $[\rho] = kg/m^3 = [\alpha z^2] = [\alpha]m^2$ thus $[\alpha] = kg/m^5$.

Example Problem 6.2.6. *Suppose we have a $m_a = 10kg$ rod placed such that it begins at $(1, 1, 1)m$ and ends at $(3, 3, 2)m$. Then $m_b = 20kg$ is evenly distributed across a square in the $z = 3m$ plane where $0 \leq x, y \leq 4m$. Finally a mass $m_c = 30kg$ is uniformly distributed over a cube described by $-10m \leq x, y, z \leq -2m$. Find the center of mass of this collection of masses.*

Solution: *we find the equivalent point mass to each given distribution. For m_a we can replace the rod with a single mass of $m_a = 10kg$ placed at the midpoint of the rod:*

$$\vec{r}_a = \frac{1}{2}((1, 1, 1)m + (3, 3, 2)m) = \langle 2, 2, 1.5\rangle m$$

Likewise, the square can be replaced by a single mass $m_b = 20kg$ at its center $\vec{r}_b = \langle 2, 2, 3\rangle m$. The cube is also equivalent to a mass $m_c = 30kg$ placed at the center of the cube $\vec{r}_c = \langle -6, -6, -6\rangle m$. In total we have mass $M = m_a + m_b + m_c = 60kg$ and

$$\begin{aligned}
\vec{R} &= \frac{1}{M}(m_a\vec{r}_a + m_b\vec{r}_b + m_c\vec{r}_c) \\
&= \frac{1}{60kg}\left(10kg\langle 2, 2, 1.5\rangle m + 20kg\langle 2, 2, 3\rangle m + 30kg\langle -6, -6, -6\rangle m\right) \\
&= \frac{1}{60}\left(\langle 20, 20, 15\rangle + \langle 40, 40, 60\rangle + \langle -180, -180, -180\rangle\right) m \\
&= \boxed{\langle -2m, -2m, -1.75m\rangle}.
\end{aligned}$$

6.3 conservation of momentum

Now let us return to the theoretical importance of the center of mass. I'll give a derivation for the case of finitely many masses, but this could easily be adapted to continuous distributions provided the shape of the solid was not rotating or deforming under the motion. If a distribution of mass is rotating or deforming as it moves then other mathematics must be used to capture the energy bound up in the rotation and deformation of the mass. For deformation we would need to study the **stress energy tensor** and its formulation of forces within a solid. Fortunately, if the body is rigid then we can understand a fair amount about its rotational motion using the mathematics which is available to us in this course. We study rotational motion in a later chapter.

Definition 6.3.1. *Let masses $m_1, m_2, \dots, m_n$ be at positions $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ with velocities $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and momenta $\vec{p}_1 = m_1\vec{v}_1, \vec{p}_2 = m_2\vec{v}_2, \dots, \vec{p}_n = m_n\vec{v}_n$ respective. Then the* **total momentum** *of the system is denoted $\vec{P}$ and is defined by $\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n$.*

The total momentum of a system is related to the center of mass and its velocity as follows:

$$M\vec{V} = M\frac{d\vec{R}}{dt} = M\frac{d}{dt}\left[\frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i\right] = \sum_{i=1}^{n} m_i\frac{d\vec{r}_i}{dt} = \sum_{i=1}^{n} m_i\vec{v}_i = \sum_{i=1}^{n}\vec{p}_i = \vec{P}.$$

In other words, it is as if the whole system of n-particles was just a single mass M at position $\vec{R}$ with velocity $\vec{V} = \frac{d\vec{R}}{dt}$ and with momentum $\vec{P} = M\vec{V}$. Yet, this single mass also carries information about the collection of masses which it represents.

Suppose that the n-masses under consideration act on each other with certain forces. We call such forces **internal forces**. By Newton's Third Law internal forces necessarily come in pairs which act with equal magnitude in opposite directions. If we denote $\vec{F}_{ij}$ to mean the force placed on mass m_j by mass m_i then Newton's Third Law requires $\vec{F}_{ij} = -\vec{F}_{ij}$. In addition, let us denote $\vec{f}_i^{ext}$ for the forces on m_i which are not from the other masses in the distribution, these are the **external forces** on the distribution. Let us also define the **total external force** on the system by $\vec{f}^{ext} = \sum_{i=1}^{n} \vec{f}_i^{ext}$. In summary, if we consider the i-th mass then Newton's Second Law in momentum form yields

$$\frac{d\vec{p}_i}{dt} = \vec{f}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij}$$

Take the vector sum of each such equation as i ranges from 1 to n to derive:

$$\sum_{i=1}^{n} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{n} \left(\vec{f}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij} \right)$$

Consequently, by linearity of d/dt and the definition of the total external force,

$$\frac{d}{dt} \sum_{i=1}^{n} \vec{p}_i = \vec{f}^{ext} + \sum_{i=1}^{n} \sum_{j \neq i} \vec{F}_{ij} \quad \Rightarrow \quad \boxed{\frac{d\vec{P}}{dt} = \vec{f}^{ext}}$$

since the double sum must vanish thanks to the calculation below:

$$\sum_{i=1}^{n} \sum_{j \neq i} \vec{F}_{ij} = \sum_{i<j} \vec{F}_{ij} + \sum_{i>j} \vec{F}_{ij} = \sum_{i<j} \vec{F}_{ij} + \sum_{l>k} \vec{F}_{lk} = \sum_{i<j} \vec{F}_{ij} - \sum_{l>k} \vec{F}_{kl} = \sum_{i<j} \vec{F}_{ij} - \sum_{k<l} \vec{F}_{kl} = 0$$

The cancellation above is due to Newton's Third Law.

Thus we derive the important theorem of classical mechanics; if the net-external force on a system is zero then the total momentum of the system is conserved. Simply put:

$$\boxed{\frac{d\vec{P}}{dt} = 0}$$

This means that in such a case, $\vec{P}_o = \vec{P}_f$; the total initial momentum and the total final momentum must be equal. This is conservation of a vector quantity and we must use vectors to understand its proper application to reality.

Example Problem 6.3.2. *Suppose a system with masses $m_A = M$ and $m_B = 2M$ with initial velocities $\vec{V}_{Ao} = v_o\hat{\mathbf{x}}$ and $\vec{V}_{Bo} = v_o\hat{\mathbf{y}}$. If in the end we observe $\vec{V}_{Bf} = v_o(\hat{\mathbf{x}} + 2\hat{\mathbf{y}})$ then what is the final velocity of m_A given that there are no external forces on this system ?*

Solution: *since the external force is zero we know $\vec{P}_{Ao} + \vec{P}_{Bo} = \vec{P}_{Af} + \vec{P}_{Bf}$ thus*

$$m_A\vec{V}_{Ao} + m_B\vec{V}_{Bo} = m_A\vec{V}_{Af} + m_B\vec{V}_{Bf}$$

Solving for $\vec{V}_{Af}$ we find

$$\vec{V}_{Af} = \frac{1}{m_A}\left(m_A\vec{V}_{Ao} + m_B\vec{V}_{Bo} - m_B\vec{V}_{Bf}\right) = \frac{1}{M}\left(M\vec{V}_{Ao} + 2M\vec{V}_{Bo} - 2M\vec{V}_{Bf}\right)$$

thus

$$\vec{V}_{Af} = \vec{V}_{Ao} + 2\vec{V}_{Bo} - 2\vec{V}_{Bf} = v_o\hat{\mathbf{x}} + 2v_o\hat{\mathbf{y}} - 2v_o(\hat{\mathbf{x}} + 2\hat{\mathbf{y}}) \quad \Rightarrow \quad \boxed{\vec{V}_{Af} = -v_o(\hat{\mathbf{x}} + 2\hat{\mathbf{y}})}.$$

Definition 6.3.3. *Given masses $m_1, m_2, \dots, m_n$ we define the* **kinetic energy of the system** *by*

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots + \frac{1}{2}m_nv_n^2$$

If the masses undergo a collisition then the collision is called **elastic** *if $KE_i = KE_f$. If the collision is not elastic then it is called an* **inelastic** *collision.*

Example Problem 6.3.4. *A car with mass $m_1 = 1000\,kg$ collides with a truck of unknown mass m_2 and the resulting composite mass travels at 50^o with respect to the initial path of the car. Supposing that the paths were perpendicular and that initially the car had $v_{1,i} = 20\,m/s$ whereas $v_{2,i} = 40\,m/s$ for the truck. Determine the mass of the truck from given data. Also find the speed v_f of the car and truck which are stuck together just after the collision.*

Solution: *conservation of momentum:*

$$\vec{p}_{1,i} + \vec{p}_{2,i} = \vec{p}_{1,f} + \vec{p}_{2,f} \quad \Rightarrow \quad m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}.$$

By assumption $\vec{v}_{1,f} = \vec{v}_{2,f} = \vec{v}_f$. Thus,

$$m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i} = (m_1 + m_2)\vec{v}_f.$$

With coordinates which assume the car travels right and the truck travels vertically we find $\vec{v}_f = v_f\langle \cos 50^o, \sin 50^o\rangle$. Thus,

$$(1000\,kg)\langle 20\,m/s, 0\rangle + m_2\langle 0, 40\,m/s\rangle = (1000\,kg + m_2)v_f\langle \cos 50^o, \sin 50^o\rangle$$

This is a two-dimensional vector equation which gives us two scalar equations. This is good news since we have two unknowns m_2, v_f. Let us do the algebra. The x-component yields:

$$(1000\,kg)(20\,m/s) = (1000\,kg + m_2)v_f\cos 50^o.$$

The y-component yields:

$$m_2(40\,m/s) = (1000\,kg + m_2)v_f\sin 50^o$$

Clearly $v_f \neq 0$ so we can divide equations to derive

$$\frac{m_2(40\,m/s)}{(1000\,kg)(20\,m/s)} = \frac{(1000\,kg + m_2)v_f\sin 50^o}{(1000\,kg + m_2)v_f\cos 50^o} = \tan 50^o$$

Solve for m_2,

$$m_2 = \frac{(1000\,kg)(20\,m/s)\tan 50^o}{40\,m/s} \quad \Rightarrow \quad \boxed{m_2 = 595.9\,kg}.$$

Hence,

$$v_f = \frac{(1000\,kg)(20\,m/s)}{(1000\,kg + 595.9\,kg)\cos 50^o} \quad \Rightarrow \quad \boxed{v_f = 19.50\,m/s}.$$

You can check the kinetic energy before the collision and contrast it to the kinetic energy after the collision:

$$KE_i = \frac{1}{2}(1000kg)(20m/s)^2 + \frac{1}{2}(595.9kg)(40m/s)^2 = 676.7kJ$$

whereas

$$KE_f = \frac{1}{2}(1000kg + 595.9kg)(19.5m/s)^2 = 303.4kJ$$

About 373.3kJ of energy were lost in the collision. Now, the theory is that the energy was not truly lost, the energy merely changed form into heat, sound etc. in the collision process. Conservation of total energy includes objects outside of the car-truck system. For the system including the car and the truck energy was lost in the collision.

Theorem 6.3.5. *If m_a and m_b undergo a one-dimensional elastic collision then*

$$\boxed{v_{ai} - v_{bi} = -(v_{af} - v_{bf})}.$$

where v_{bf}, v_{af} are the final velocities and v_{bi}, v_{ai} are the initial velocities of m_b and m_a respective.

Proof: Suppose both kinetic energy and momentum are conserved in a collision: we use the coordinate system where $v_{bi} = 0$ for convenience[4],

$$\frac{1}{2}m_a v^2 = \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2 \ \ \& \ \ m_a v = m_a v_a + m_b v_b$$

I'll use $v = v_{ai}$ and $v_a = v_{af}$ and $v_b = v_{bf}$ for brevity. From the kinetic energy equation we find:

$$v^2 = v_a^2 + \frac{m_b}{m_a} v_b^2$$

Likewise from momentum conservation we find $v_b = \frac{m_a(v - v_a)}{m_b}$ thus and $\frac{m_a}{m_b} = \frac{v_b}{v - v_a}$

$$v^2 = v_a^2 + \frac{m_b}{m_a}\left[\frac{m_a(v - v_a)}{m_b}\right]^2 = v_a^2 + \frac{m_a}{m_b}(v - v_a)^2 = v_a^2 + \frac{v_b}{v - v_a}(v - v_a)^2 = v_a^2 + v_b(v - v_a).$$

Subtracting v_a^2 and factor

$$v^2 - v_a^2 = (v - v_a)(v + v_a) = v_b(v - v_a)$$

Either $v = v_a$ which means the collision did not really happen, or $v - v_a \neq 0$ hence we derive

$$v + v_a = v_b.$$

Thus $v = -(v_a - b_b)$. We have shown $v_{ai} - v_{bi} = -(v_{af} - v_{bf})$ in the frame of reference where $v_{bi} = 0$ and $v_{ai} = v$ and $v_{af} = v_a$ and $v_{bf} = v_b$ hence the result holds in all other inertially related frames. $\square$.

Remark 6.3.6. *The relative velocity theorem for elastic collisions only holds in the context of one-dimensional collisions. If you try to derive the result in higher dimensions then roughly speaking, the derivation fails since we cannot divide by a vector. We'd have*

$$m_a\vec{v} = m_a\vec{v}_a + m_b\vec{v}_b \qquad \& \qquad \frac{1}{2}m_a\vec{v}\bullet\vec{v} = \frac{1}{2}m_a\vec{v}_a\bullet\vec{v}_a + \frac{1}{2}m_b\vec{v}_b\bullet\vec{v}_b$$

Multiply by two and divide by m_a,

$$\vec{v} = \vec{v}_a + \frac{m_b}{m_a}\vec{v}_b \qquad \& \qquad \vec{v}\bullet\vec{v} = \vec{v}_a\bullet\vec{v}_a + \frac{m_b}{m_a}\vec{v}_b\bullet\vec{v}_b$$

[4] notice for another inertially related coordinate system S' we have $v' = v + v_o$ for the velocity v_o of the origin of the S' system, but then $v'_{ai} - v'_{bi} = -(v'_{af} - v'_{bf})$ since all the velocities in the analysis are shifted by the same v_o.

Therefore,

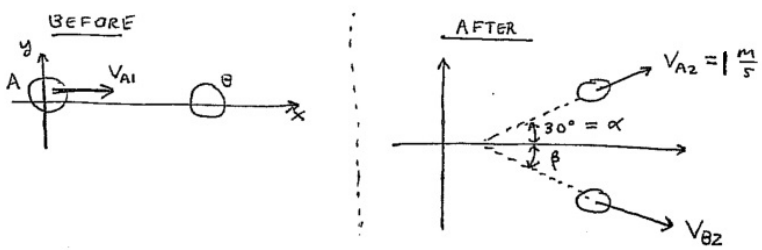
$$\vec{v} \bullet \left(\vec{v}_a + \frac{m_b}{m_a}\vec{v}_b\right) = \vec{v}_a \bullet \vec{v}_a + \frac{m_b}{m_a}\vec{v}_b \bullet \vec{v}_b \quad \Rightarrow \quad \vec{v}_a \bullet (\vec{v} - \vec{v}_a) = \frac{m_b}{m_a}\vec{v}_b \bullet \vec{v}_b - \frac{m_b}{m_a}\vec{v} \bullet \vec{v}_b$$

Hence, using $\frac{m_b}{m_a}\vec{v}_b = \vec{v} - \vec{v}_a$ *to eliminate the masses,*

$$\vec{v}_a \bullet (\vec{v} - \vec{v}_a) = (\vec{v} - \vec{v}_a) \bullet \vec{v}_b - \vec{v} \bullet (\vec{v} - \vec{v}_a) = (\vec{v}_b - \vec{v}) \bullet (\vec{v} - \vec{v}_a).$$

If we could cancel $(\vec{v} - \vec{v}_a)$ *then we'd find* $\vec{v}_a = \vec{v}_b - \vec{v}$ *(which is what we'd like to prove in principle). Unfortunately, no such cancellation is possible since the dot-product equation* $\vec{A} \bullet \vec{C} = \vec{B} \bullet \vec{C}$ *does not imply* $\vec{A} = \vec{B}$.

Example Problem 6.3.7. *Two students are sitting on an ice pond in lawn chairs. One of them has an umbrella and a strong wind sends student* A *on a collision course with student* B. *Suppose* $m_A = 60kg$ *has an intial speed of* $2m/s$ *in the Eastern direction whereas* $m_B = 30kg$ *is initially at rest and after the collision* m_A *has speed* $1m/s$ *at a standard angle* $\alpha = 30^o$. *What is the final velocity of* m_B *? Also find* β *as pictured.*



Solution: *we are given* $\vec{v}_{A1} = \langle 2m/s, 0\rangle$ *and* $\vec{v}_{B1} = 0$ *and* $\vec{v}_{A2} = (1m/s)\langle \cos 30^o, \sin 30^o\rangle = \langle 0.866m/s,\ 0.5m/s\rangle$. *Conservation of momentum gives*

$$(60kg)\langle 2\,m/s, 0\rangle = (60kg)\langle 0.866\,m/s,\ 0.5\,m/s\rangle + (30kg)\vec{v}_{B2}.$$

Thus,

$$\vec{v}_{B2} = 2\langle 2\,m/s, 0\rangle - 2\langle 0.866\,m/s,\ 0.5\,m/s\rangle \quad \Rightarrow \quad \boxed{\vec{v}_{B2} = \langle 2.268\,m/s, -1\,m/s\rangle}.$$

Moreover, we find $\tan\beta = \frac{2.268}{1}$ *hence* $\boxed{\beta = 66.21^o}$.

Example Problem 6.3.8. *Two cars of equal mass slam into each other with equal speed* v_o *at an intersection where perpendicular roads meet. After the collision the cars stick together and slide a distance L until they come to rest under the force of friction. Find the coefficient of kinetic friction as a function of the given speed and sliding distance.*

Solution: *conservation of momentum gives* $m\langle v_o, 0\rangle + m\langle 0, v_o\rangle = 2m\vec{v}_f$ *hence* $\vec{v}_f = \langle v_o/2, v_o/2, \rangle$. *That is, we have* $\vec{v}_f = \frac{v_o}{2}\langle 1, 1\rangle$ *thus* $v_f = v_o\sqrt{2}/2 = v_o/\sqrt{2}$. *After the collision the force of friction* $F_f = \mu(2mg)$ *acts opposite the motion and the work done by friction reduces the initial kinetic energy after the collision to zero at the end of the slide. Thus,*

$$\frac{1}{2}(2m)v_f^2 = \mu(2mg)L \quad \Rightarrow \quad \mu = \frac{v_f^2}{2gL} \quad \Rightarrow \quad \boxed{\mu = \frac{v_o^2}{4gL}}.$$

The relative velocity theorem for collisions doesn't hold for two dimensional collisions. However, in the case of equal masses there is an interesting result we can derive.

Theorem 6.3.9. *If two equal masses undergo a glancing elastic collision in the plane then after the collision the masses go in perpendicular directions.*

Proof: suppose m_a and m_b undergo an elastic collision in the plane. Also, suppose $m_a = m_b = m$. Choose a frame of reference in which m_b is initially at rest and let the velocity of m_a point in the positive x-direction. Conserve momentum,

$$m\langle v_o, 0\rangle = mv_a\langle \cos\alpha, \sin\alpha\rangle + mv_b\langle \cos\beta, -\sin\beta\rangle.$$

Conservation of kinetic energy,

$$\frac{1}{2}mv_o^2 = \frac{1}{2}mv_a^2 + \frac{1}{2}mv_b^2.$$

Therefore, cleaning up momentum conservation and kinetic energy conservation we have:

$$v_o = v_a\cos\alpha + v_b\cos\beta \qquad \& \qquad 0 = v_a\sin\alpha + v_b\sin\beta \qquad \& \qquad v_o^2 = v_a^2 + v_b^2$$

Square and add the equations above,

$$\begin{aligned} v_o^2 = v_o^2 + 0^2 &= (v_a\cos\alpha + v_b\cos\beta)^2 + (v_a\sin\alpha + v_b\sin\beta)^2 \\ &= v_a^2(\cos^2\alpha + \sin^2\alpha) + 2v_av_b\cos\alpha\cos\beta + 2v_av_b\sin\alpha\sin\beta + v_b^2(\cos^2\alpha + \sin^2\beta) \\ &= v_a^2 + 2v_av_b\cos\alpha\cos\beta + 2v_av_b\sin\alpha\sin\beta + v_b^2. \end{aligned}$$

Consequently,

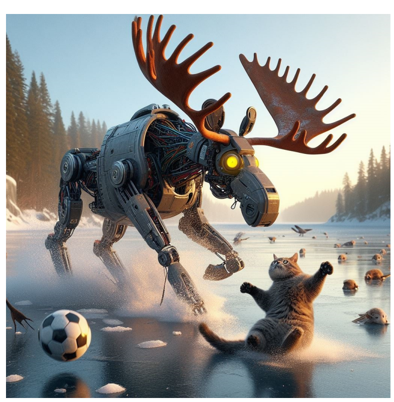
$$v_a^2 + v_b^2 = v_a^2 + 2v_av_b\cos\alpha\cos\beta + 2v_av_b\sin\alpha\sin\beta + v_b^2$$

and we deduce $\cos\alpha\cos\beta + \sin\alpha\sin\beta = 0$ since $v_av_b \neq 0$. Thus, by trigonometry,

$$\cos(\alpha+\beta) = 0 \quad \Rightarrow \quad \boxed{\alpha + \beta = 90^o} \qquad \square.$$

In retrospect, if we think about Problem 6.3.7 then we see $\alpha + \beta = 96.21^o$. Logically, this means that the collision considered in that example was not elastic ? FALSE. It may or may not have been elastic, the masses in that problem were not equal. In general, the theorem only holds for equal masses. Trust me, I've tried to prove it when $m_a \neq m_b$, it leads to much suffering.

Question: how can you discern if the collision was elastic or not ?



Example Problem 6.3.10. *Two cats of equal mass are shot into the air with speed v_o at angle θ. Then, at the top of their flight a bomb explodes and throws both of them forward at the same angle above the horizontal for the top cat and below the horizontal for the bottom cat. If the kinetic energy of the cats increased by a factor of 4 in the explosion then find the angle of the feline motion just after the bomb.*

Solution: *At the top of the flight the motion is horizontal right before the explosion. Let m be the cat mass then conservation of momentum gives*

$$2m\langle v_o \cos\theta, 0\rangle = mv_t\langle \cos\alpha, \sin\alpha\rangle + mv_b\langle \cos\alpha, -\sin\alpha\rangle$$

From the y-component we derive $v_t = v_b$. Let's set $v_f = v_t = v_b$. From the x-component we find

$$2v_o \cos\theta = 2v_f \cos\alpha$$

Thus $v_f = v_o \cos\theta / \cos\alpha$. We're given $KE_f = 4KE_o$ thus

$$\frac{1}{2}mv_f^2 + \frac{1}{2}mv_f^2 = 4\left(\frac{1}{2}(2m)(v_o\cos\theta)^2\right) \quad \Rightarrow \quad v_f^2 = 4v_o^2\cos^2\theta \quad \Rightarrow \quad \frac{v_o^2\cos^2\theta}{\cos^2\alpha} = 4v_o^2\cos^2\theta$$

Thus $\cos^2\alpha = \frac{1}{4}$ and we find $\cos\alpha = \frac{1}{2}$ since the motion is forward. Thus $\boxed{\alpha = 60^o}$.

6.4 energy and momentum in special relativity

If we study motion with speeds approaching the speed of light it turns out that the laws of classical mechanics require modification. For example, high energy particle physics involve experiments with particles that are travelling near c and cosmic rays are also commonly made of particles going near c (these rays account for a measurable number of computer errors for a real world application). Einstein's **Special Relativity** explains how to modify classical mechanics in such a way that the speed of light is the speed limit[5]. One of the lessons of special relativity is that the total energy of a particle is given by $E = \gamma mc^2$ where m is the **rest mass** and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$ where v is the speed of the object. Power series arguments from calculus show:

$$\gamma = (1-\beta^2)^{-1/2} = 1 + \frac{1}{2}\beta^2 + \cdots$$

[5]but, not like on the roads, $c \approxeq 3 \times 10^8 m/s$ is a speed which cannot be suprpassed by massive objects, and light all goes the same speed in a given medium

Thus, setting $\beta^2 = v^2/c^2$ and cancelling c^2 from the second term,

$$E = \gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \cdots$$

We should recognize the second term as the usual formula for kinetic energy. The constant term is something new. Indeed, the famous equation $E = mc^2$ is just the zeroth order term in the power series I give above[6]. From the viewpoint of special relativity the correct formula for kinetic energy is $KE = (\gamma - 1)mc^2$.

Momentum is also modified. The **relativistic momentum** is given by $\vec{p} = \gamma m\vec{v}$ where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ once more. Only for speeds which are small compared to c do we simply have $\vec{p} = m\vec{v}$. Probably the right way to think about relativistic momentum is using four dimensional **space time** where we consider the so-called 4-momentum $P_\mu = (E/c, \vec{p})$ then one can show $P_\mu P^\mu$ is an invariant in different inertially related frame. Most of the formulas we study this semester are modified by appropriate injections of the γ-factor, but, the rest of that story is for a different course. If you're curious feel free to ask me about it in office hours, I have things you can read. I usually discuss relativity, modern physics and quantum mechanics and other modern topics towards the conclusion of this course.

[6]I take the viewpoint that m is rest mass, others put γ into m and discuss a variable mass, but this viewpoint is antiquated in my view

Chapter 7

rotational physics

Motion based on forming closed curves around some axis is generally known as **rotational motion**. We study the basic kinematics of such motion and relate some new rotational variables such as **angle**, **angular velocity** and **angular acceleration** to our earlier discussions of centripetal and tangential acceleration. We discuss how to decompose motion into a radial and tangential part for a given origin and axis. Kinetic energy for a rigid body is studied and used as a motivation for the introduction of the **moment of inertia**. The analog of mass for rotational motion is the moment of inertia. Calculus allows us to continuously extend the moment calculation from the discrete to the continuum. We find the moments of inertia for several common shapes such as rods, cylinders, spheres and spherical shells. As before, the infinitesimal method guides our calculation approach.

Then we turn to the problems which involve a composition of rotational and linear motion. Here the axis of rotation is itself in motion. It turns out the mathematics is relatively easy in the end. We simply decouple the rotational and linear parts of the motion and ascribe energy to each separately. This allows us to solve interesting problems about balls rolling without slipping down hills and such. We are able to account for the shape of the object and the distinction between how a bowling ball would roll verses a hula-hoop or a big wheel of cheese.

Then we turn to the problem of understanding how momentum generalizes to **angular momentum**. We find the angular analog to Newton's Second Law requires us to relate the **torque** to the derivative of angular momentum. This is a genuinely three dimensional problem as the cross-product and right-hand-rule are needed to sort out directions properly. Furthermore, it turns out a system of particles has a net-angular momentum which is conserved whenever the net-torque on the system is zero.

In summary, everything we did in the linear Newtonian physics of previous chapters has a natural analog here. In some sense, this chapter allows us to review the course with new material.

7.1 rotational kinematics

Let us begin with motion in a two-dimensional context. Rotation about the origin is naturally covered by polar coordinates r, θ which are defined implicitly by the usual equations:

$$x = r\cos\theta \qquad \& \qquad y = r\sin\theta.$$

Supposing x, y are functions of time t then likewise r, θ are also functions of time t. The rates of change of x, y and r, θ are related:

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} \qquad \& \qquad \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

A better notation, one which Newton used in his writing on calculus and physics, is

$$\dot{x} = \dot{r}\cos\theta - (r\sin\theta)\dot{\theta} \qquad \& \qquad \dot{y} = \dot{r}\sin\theta + (r\cos\theta)\dot{\theta}$$

Thus,

$$\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= \Big(\dot{r}\cos\theta - (r\sin\theta)\dot{\theta}\Big)^2 + \Big(\dot{r}\sin\theta + (r\cos\theta)\dot{\theta}\Big)^2 \\
&= \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta \\
&= \dot{r}^2(\cos^2\theta + \sin^2\theta) + r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) \\
&= \dot{r}^2 + r^2\dot{\theta}^2
\end{aligned}$$

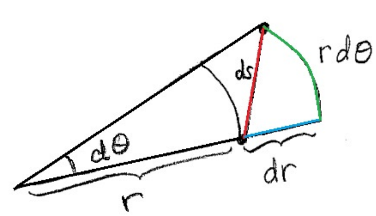
Definition 7.1.1. *Given a particle with position (x, y) and polar coordinates (r, θ) we define* **radial velocity** *as $v_r = \frac{dr}{dt}$ and* **angular velocity** *$\omega = \frac{d\theta}{dt}$. Notice the* **tangential velocity** *is related to the angular velocity by $v_t = r\omega$.*

Observe the speed is the magnitude of the velocity $\vec{v} = \langle \dot{x}, \dot{y}\rangle$ hence $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$. Since $\frac{ds}{dt} = v$ we also may write:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}.$$

Formally multiplying by dt we find

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2 d\theta^2}.$$



We can integrate ds along a given trajectory and calculate the distance travelled. It is also helpful to remember such formulas without the squareroot;

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2.$$

Notice these can be intuitively derived from applying the Pythagorean Theorem infinitesimally. I'll let you draw the picture for $ds^2 = dx^2 + dy^2$. My picture for $ds^2 = dr^2 + r^2 d\theta^2$ has $d\theta$ which is rather large. If you can imagine it, the better picture makes $d\theta$ small enough so it is clear that the green segment with length $rd\theta$ is essentially linear. In any event, we've derived the arclength in polar coordinates via calculus so these intuitive geometric comments are superfluous.

When motion is circular about the origin then in polar coordinates the equation of a circle is simply $r = R$ where R is the **radius** of the circle. In that case we have very nice equations which relate arclength and the corresponding subtended angle $\triangle\theta$,

$$\triangle s = R\,\triangle\theta$$

If we divide by $\triangle t$ and consider $\triangle t \to 0$ we obtain

$$\frac{ds}{dt} = R\frac{d\theta}{dt}$$

In other words, for **circular motion** on a circle of radius R we find $v_t = R\omega$. Here positive v_t corresponds to positive ω. This is a special case of $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$. In particular, as $\dot{r} = 0$ since $r = R$ is constant we find $v = \sqrt{R^2\dot{\theta}^2} = R|\dot{\theta}|$. Since the radial velocity is zero we note $v_t = \pm v = \pm|R\omega|$ and it follows[1] $v_t = R\omega$.

Let's calculate the vector formulas for velocity and acceleration as they relate to polar coordinates. Let us review what we've already derived in terms of $\vec{r} = \langle x, y\rangle$ and $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{a} = \frac{d\vec{v}}{dt}$. Let us introduce unit-vectors which point in the direction of increasing r and θ as $\hat{r} = \langle \cos\theta, \sin\theta\rangle$ and $\hat{\theta} = \langle -\sin\theta, \cos\theta\rangle$. Calculate, $\frac{d\hat{r}}{dt} = \langle -\sin\theta, \cos\theta\rangle\frac{d\theta}{dt} = \dot{\theta}\hat{\theta}$ whereas $\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$.

$$\vec{r} = r\langle \cos\theta, \sin\theta\rangle = r\hat{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}\hat{r} + r\hat{\theta}\frac{d\theta}{dt} \quad \Rightarrow \quad \boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}.$$

Since $\hat{r}, \hat{\theta}$ are perpendicular unit-vectors we once more derive $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$. Differentiate once more to find acceleration,

$$\begin{aligned} \vec{a} = \frac{d\vec{v}}{dt} &= \frac{d}{dt}\left[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right] \\ &= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\ &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r}) \quad \Rightarrow \quad \boxed{\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}}. \end{aligned}$$

Once more, in the case of circular motion we find simplified formulas which we've already seen in different a different notational scheme. If $r = R$ is constant then $\dot{r} = \ddot{r} = 0$ hence

$$\boxed{\vec{a} = (\,-R\dot{\theta}^2\,)\hat{r} + (\,R\ddot{\theta}\,)\hat{\theta}} \quad \Rightarrow \quad \boxed{a = R\sqrt{\omega^4 + \alpha^2}}.$$

where we used $\omega = \dot{\theta}$ and $\alpha = \ddot{\theta}$. In this circular case, $v_t = R\omega$ where $\omega = \dot{\theta}$ thus $\frac{dv_t}{dt} = R\ddot{\theta}$. If we rewrite the formula above in terms of the tangential velocity v_t and the tangential acceleration,

$$\boxed{\vec{a} = \left(-\frac{v_t^2}{R}\right)\hat{r} + \left(\frac{dv_t}{dt}\right)\hat{\theta}} \quad \Rightarrow \quad \boxed{a = \sqrt{\frac{v_t^4}{R^2} + a_t^2}}$$

Since $\hat{r}$ points away from origin and $\hat{\theta}$ points in the tangential direction to the circular motion the formula above we have seen before in Section 3.5 where we derived these formulas via vector-theoretic arguments. Here we took a more calculus-based approach.

[1]this probably is a definition if you wish to be picky

Example Problem 7.1.2. *Suppose we are accelerating around a turn with radius $R = 30\,m$. If our speed is increasing at $1\,m/s^2$ then what is the maximum speed we can reach without losing traction on a surface with coefficient of friction $\mu = 0.8$? How many revolutions per minute are possible if the track is circular ?*

Solution: *if we consider Newton's Second Law then the vertical forces cancel as the normal force and the gravity force balance. It follows the force of friction has magnitude μmg and the force of friction is directed in the direction of $\vec{a}$ which has both a radial and tangential components.*

$$\mu mg = m\sqrt{\frac{v_t^4}{R^2} + a_t^2}$$

Solve for v_t.

$$\mu^2 g^2 - a_t^2 = \frac{v_t^4}{R^2} \quad \Rightarrow \quad v_t = \sqrt{R}\left(\mu^2 g^2 - a_t^2\right)^{1/4} = \boxed{15.27\,m/s}.$$

where I've used $\mu = 0.8$ and $R = 30\,m$ and $g = 9.8 m/s^2$ and $a_t = 1\,m/s^2$. Then,

$$\omega = \frac{v_t}{R} = \frac{15.27\,\frac{m}{s}}{30\,m} = 0.509\,\frac{rad}{s} \cdot \frac{60\,s}{1\,min} \cdot \frac{rev}{2\pi\,rad} = \boxed{4.86\,rpm}.$$

Sometimes we consider rotational motion as an end unto itself without direct connection to an underlying Cartesian coordinate system[2]

Definition 7.1.3. *If θ denotes the angle an object rotates with respect to some given axis then we define* **angular velocity** *by $\omega = \frac{d\theta}{dt}$ and the* **angular acceleration** *by $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$.*

Angular displacement is $\triangle\theta = \theta_f - \theta_o$. We can measure angles in radians, degrees or even revolutions. Sometimes angular velocity is given in revolutions per minute (rpm).

Example 7.1.4. *Suppose $\triangle\theta = 10\pi\,rad$ then since $\pi rad. = 180^o$ we have*

$$\triangle\theta = (10\pi rad)\left(\frac{180^o}{\pi rad}\right) = 1800^o.$$

Likewise, since $360^o = 1\,rev$ (rev is short for revolution),

$$\triangle\theta = (1800^o)\left(\frac{rev}{360^o}\right) = 5\,rev.$$

Suppose $\omega = 100\,rpm$ then to convert to rad/s

$$\omega = 100\,rpm = 100\,\frac{rev}{min} = 100\,\frac{rev}{min} \cdot \frac{min}{60\,s} \cdot \frac{2\pi\,rad}{rev} = 10.47\,\frac{rad}{s}.$$

7.2 constant angular acceleration motion

In general, $\omega = \frac{d\theta}{dt}$ and $\alpha = \frac{d\omega}{dt}$ and if α is not a constant then we have to work out the calculus. That said, in the very special case that α is constant then

$$\frac{d\omega}{dt} = \alpha \quad \Rightarrow \quad \boxed{\omega(t) = \omega_1 + \alpha(t - t_1)}$$

[2] in many examples we could still set-up Cartesian coordinates and think as we were in the last page, but in other contexts, for example the angle rotated by a yo-yo with respect to its axis, this would be quite ackward.

where $\omega_1 = \omega(t_1)$. Integrating $\omega = \frac{d\theta}{dt}$ we derive

$$\frac{d\theta}{dt} = \omega \quad \Rightarrow \quad \boxed{\theta(t) = \theta_1 + \omega_1(t - t_1) + \frac{1}{2}\alpha(t - t_1)^2}.$$

The timeless equation follows from the identity $\alpha = \frac{d\omega}{dt} = \frac{d\theta}{dt}\frac{d\omega}{d\theta} = \omega\frac{d\omega}{d\theta}$ which gives $\alpha d\theta = \omega d\omega$ which integrates to reveal

$$\boxed{\omega_f^2 = \omega_o^2 + 2\alpha(\theta_f - \theta_o)}.$$

We also can relate the average angular velocity to the average of the angular velocities:

$$\boxed{\frac{\triangle\theta}{\triangle t} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\omega_1 + \omega_2}{2}}.$$

Examples we worked in the context of constant acceleration linear motion have analogs in the constant angular acceleration rotational context. We will soon learn our energy methods also have natural rotational generalizations.

Example Problem 7.2.1. *A car accelerates from rest to $40\,m/s$ using wheels with a radius of $20\,cm$. If the linear acceleration was constant and took 5 s then find the angular velocity and angular acceleration of the wheels. Also, how many revolutions to the wheels make during the given motion ? Assume the wheels roll without slipping.*

Solution: *if the wheels roll without slipping then we have $s = R\theta$ and $v = R\omega$ and $a = R\alpha$. Notice $\omega_o = 0$ whereas $\omega_f = v_f/R = \frac{40\,m/s}{0.2\,m} = 200\,\frac{rad}{s}$. Noting $\omega_f = \omega_o + \alpha(5\,s)$ we calculate*

$$\alpha = \frac{200\,rad/s}{5\,s} = 40\,\frac{rad}{s^2}.$$

Thus $\boxed{\alpha = 40\,rad/s^2}$ and $\boxed{\omega = (40 rad/s^2)t}$. Likewise, $\theta = (20\,rad/s^2)t^2$ thus when $t = 5s$ we find that $\theta_f = 500\,rad$. The number of revolutions made is thus given by $500/2\pi$; the number of revolution is $\boxed{79.58 \ revolutions}$.

Example Problem 7.2.2. *Suppose grinding wheel accelerates from rest at $2\,rad/s^2$ for $10\,s$ then it continues to spin without friction. How many revolutions will the wheel go through in the first minute ?*

Solution: *we treat the problem in two stages. For the first $10\,s$ we find:*

$$\triangle\theta_1 = \frac{1}{2}\alpha t^2 = \frac{1}{2}(2\,rad/s^2)(10\,s)^2 = 100\,rad.$$

On the other hand, $\omega(10) = (2\,rad/s^2)(10\,s) = 20\frac{rad}{s}$. Since $\alpha = 0$ for $10 \leq t \leq 60$ seconds we calculate

$$\triangle\theta_2 = \left(20\frac{rad}{s}\right)(50\,s) = 1000\,rad$$

Thus in total we find $\triangle\theta_1 + \triangle\theta_2 = 1100\,rad$. Since each revolution corresponds to 2π radians, we find a total of approximately $\boxed{175.1 \ revolutions}$.

Example Problem 7.2.3. *Suppose a wheel rolls to a stop over a distance of* $300\,m$. *If the initial velocity of the wheel was* $10\,m/s$ *then find the angular acceleration of the wheel supposing the acceleration was constant and the diameter of the wheel is* $1.2\,m$.

Solution: *notice* $R = 1.2\,m/2 = 0.6\,m$ *and* $\omega_o = v_o/R = \frac{10\,m/s}{0.6\,m} = 16.67\,\frac{rad}{s}$. *The angle subtended during the motion is given by*

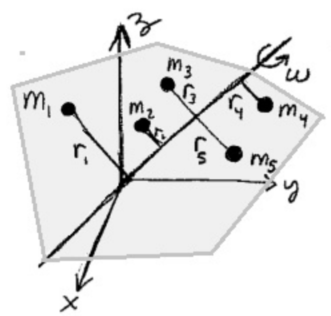
$$\theta = \frac{s}{R} = \frac{300\,m}{0.6\,m} = 500\,rad.$$

Timeless equation is convenient here since $\omega_f = 0$ *we have*

$$0 = \omega_o^2 + 2\theta\alpha \quad \Rightarrow \quad \alpha = -\frac{\omega_o^2}{2\theta} = -\frac{(16.67 rad/s)^2}{2(500\,rad)} = \boxed{-0.2779\,\frac{rad}{s^2}}.$$

7.3 moments of inertia

In this section we study kinetic energy which is related to a given object spinning around some axis of rotation. Suppose a mass M is distributed over some region[3] S then $M = \int_S dm$. Suppose further that the mass is solid and that it spins at rate ω around an axis.



Pictured above is a solid object and some representative bits of mass inside the object. Each little bit of mass dm is located at distance r from the axis pictured. Notice that the speed of dm is thus given by $v = r\omega$ since the motion in question involves no radial motion. The kinetic energy of the dm is given by

$$dK = \frac{1}{2}dm \cdot v^2 = \frac{1}{2}r^2\omega^2 dm.$$

To calculate the total kinetic energy of M we integrate dK over S,

$$K = \int_S \frac{1}{2}r^2\omega^2 dm = \frac{1}{2}\left(\int_S r^2 dm\right)\omega^2.$$

We are free to factor ω^2 out of the integral since the M all rotates at the same angular velocity ω.

Definition 7.3.1. *The* **moment of inertia** *of a mass* M *distributed over* S *is given by* $I = \int_S r^2 dm$ *where* r *is the distance from the axis of rotation and* dm.

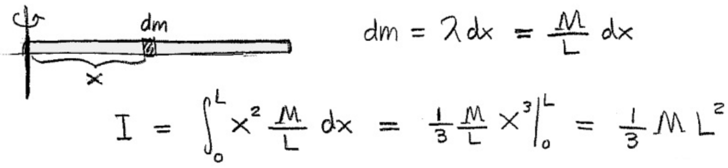
Thus, in view of the calculation in this section, $\boxed{K = \frac{1}{2}I\omega^2}$. Comparing to $K = \frac{1}{2}mv^2$, we see that the moment of inertia plays a role analogus to mass for rotational motion.

[3] I'm being deliberately vague, this could be a distribution of mass along a line, curve, planar region, curved surface or a volume. We learn how to integrate over all such regions in the calculus sequence.

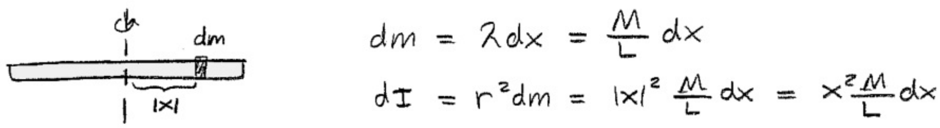
Example 7.3.2. *A point mass m travelling in a circle of radius r with angular velocity ω has kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$ since $v = r\omega$. Observe $I = mr^2$ for a point mass.*

For extended objects we usually need to work out an integral to find the moment of inertia. The integration concept here is much inline with our work in Section 6.2.

Example 7.3.3. *Suppose a mass M is uniformly distributed over a length L and imagine it rotates around one end of its length. In this case we find* $\boxed{I = \frac{1}{3}ML^2}$ *since:*



Example 7.3.4. *Suppose a mass M is uniformly distributed over a length L, let us find the moment of inertia for M with respect to an axis through its middle.*



By symmetry, we calculate

$$I = 2\int_0^{L/2} x^2 \frac{M}{L} dx = \frac{2M}{L}\int_0^{L/2} x^2 dx = \frac{2M}{L}\frac{1}{3}\left(\frac{L}{2}\right)^3 = \boxed{\frac{1}{12}ML^2}.$$

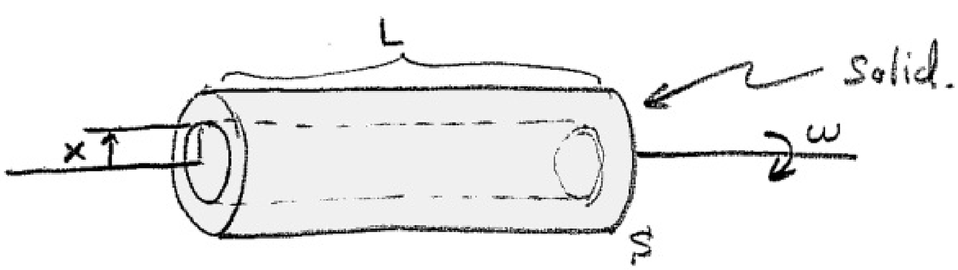
Alternatively you can derive this formula by thinking of the rod as two rods rotated around the center, each would have mass M/2 and length L/2

$$I = 2 \cdot \frac{1}{3}\frac{M}{2}\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2.$$

Example 7.3.5. *If we had a ring with mass M and radius R then the moment of inertia about the axis through the center of the ring is simply* $\boxed{I = MR^2}$ *since every bit of the ring is the distance R from the axis of rotation.*

Example 7.3.6. *A cylindrical shell with mass M and radius R has moment of inertia of* $\boxed{I = MR^2}$ *about the axis through the center axis of the cylinder. Just as in the previous example, all the mass is distance R from the axis of rotation.*

Example 7.3.7. *Suppose a mass M is uniformly distributed over a solid cylinder of radius R and length L. To calculate the moment of inertia about the axis through the center of the cylinder we can imagine slicing the cylinder into cylindrical shells at radius x for $0 \leq x \leq R$.*



To find the mass dm of such a slice we solve the density $\rho = \frac{M}{\pi R^2 L} = \frac{dm}{dV}$ to see $dm = \frac{M}{\pi R^2 L}dV$. But, notice $dV = (2\pi x L)dx$ if we imagine cutting the the cylindrical shell of thickness dx lengthwise and laying it flat with length L and width $2\pi x$. The moment of inertia dI from our shell is given by $dI = x^2 dm$ which gives:

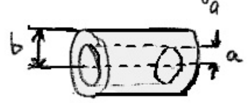
$$dI = x^2 \cdot \frac{M}{\pi R^2 L}(2\pi x L)dx = \frac{2Mx^3dx}{R^2}$$

Then integrate to find the total inertia,

$$I = \int_0^R \frac{2Mx^3dx}{R^2} = \frac{2M}{R^2}\frac{R^4}{4} = \boxed{\frac{1}{2}MR^2}.$$

Example 7.3.8. *If we uniformly distribute a mass M over a thick cylindrical shell of inner-radius a and outer radius b then the moment of inertia about the axis through the center of the cylinder is given by $\boxed{I = \frac{1}{2}M(a^2+b^2)}$. Here is a sketchy solution:*

$$I = \int_a^b x^2 \underbrace{\left(\frac{M}{\pi(b^2-a^2)L}\right)}_{\text{Mass/Volume}} 2\pi x L\, dx = \left(\frac{2M}{b^2-a^2}\right)\left(\frac{b^4}{4} - \frac{a^4}{4}\right)$$
$$= \frac{1}{2}M\frac{1}{b^2-a^2}(b^2-a^2)(b^2+a^2) = \boxed{\frac{1}{2}M(a^2+b^2)}$$



Remark 7.3.9. *Another way to derive the moment of inertia in the previous example is to subtract the moment of inertia of a solid cylinder of radius a from that of a solid cylinder of radius b. However, it is tricky because to be fair you have to give the mass more than M in order that the mass of the thick shell be M.*

$$\rho = \frac{M}{\pi L(b^2 - a^2)}$$

then we should consider

$$M_a = \frac{M(\pi L a^2)}{\pi L(b^2-a^2)} = \frac{Ma^2}{b^2-a^2} \qquad \& \qquad M_b = \frac{M(\pi L b^2)}{\pi L(b^2-a^2)} = \frac{Mb^2}{b^2-a^2}$$

Then, to find moment of inertia for the thick cylinder we calculate:

$$I = I_b - I_a = \frac{1}{2}\left(\frac{Mb^2}{b^2-a^2}\right)b^2 - \frac{1}{2}\left(\frac{Ma^2}{b^2-a^2}\right)a^2 = \frac{M}{2}\left[\frac{b^4-a^4}{b^2-a^2}\right] = \frac{1}{2}M(a^2+b^2).$$

Example 7.3.10. *If a mass M is uniformly distributed over a solid sphere of radius R then the moment of inertia about an axis through a diameter of the sphere is $\boxed{I = \frac{2}{5}MR^2}$. There are a variety of ways to calculate this result.*

Derivation by Disks: *slice the sphere into disks of mass dm. Suppose the diameter lies along the z-axis where the sphere ranges from $(0,0,-R)$ and $(0,0,R)$. The slice at z with thickness dz has radius r with $r^2 = x^2 + y^2$ and since $x^2 + y^2 + z^2 = R^2$ describes the boundary of the sphere we find $r^2 = R^2 - z^2$. The volume of the disk at z with thickness dz is*

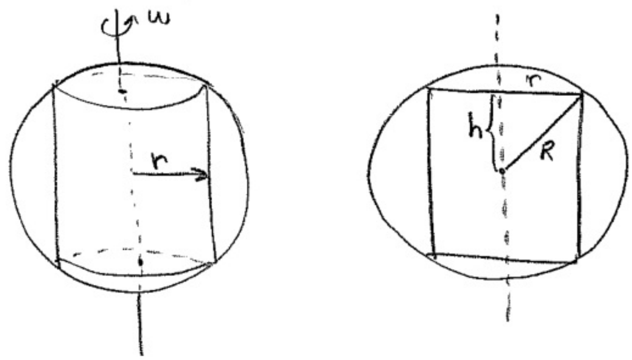
$$dV = \pi r^2 dz$$

Since $\rho = \frac{dm}{dV} = \frac{M}{\frac{4}{3}\pi R^3}$ *we derive* $dm = \frac{M}{\frac{4}{3}\pi R^3}\pi r^2 dz = \frac{3M}{4R^3} r^2 dz$*. Hence,* $dI = \frac{1}{2} r^2 dm = \frac{3M}{8R^3} r^4 dz$*. However,* $r^2 = R^2 - z^2$ *so we must make this dependence explicit to integrate over* z *correctly. By symmetry we calculate by doubling the inertia for the upper hemisphere,*

$$I = 2\int_0^R \frac{3M}{8R^3}(R^2 - z^2)^2 dz = \frac{3M}{4R^3}\int_0^R \left(R^4 - 2R^2 z^2 + z^4\right) dz = \frac{3M}{4R^3}\left[R^5 - 2R^2\frac{R^3}{3} + \frac{R^5}{5}\right]$$

Therefore, $I = \frac{3M}{4R^3}\frac{8}{15}R^5 = \frac{2}{5}MR^2$*.*

Derivation by Shells: *I'll offer a sketchy solution:*



$$h = \sqrt{R^2 - r^2}$$

$$dV = \underbrace{(2\pi r)(2h)}_{\text{area of shell}}\underbrace{dr}_{\text{thickness}} \rightsquigarrow dm = 4\rho\pi r\sqrt{R^2 - r^2}\, dr$$

all at radius r from rotation axis, $dI = r^2 dm$.

$$I = \int_0^R 4\rho\pi r^3 \sqrt{R^2 - r^2}\, dr$$

: Let $u = R^2 - r^2$ then $du = -2r\,dr \to r\,dr = -\frac{du}{2}$ and $u(0) = R^2$, $u(R) = 0$ note $r^2 = R^2 - u$

$$= \int_{R^2}^0 4\rho\pi r^2 \sqrt{u}\left(\frac{-du}{2}\right)$$

$$= \int_0^{R^2} 2\rho\pi (R^2 - u)\sqrt{u}\, du$$

$$= 2\left[\frac{M}{\frac{4}{3}\pi R^3}\right]\pi\left(\frac{2}{3}R^2 u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^{R^2}$$

$$= \frac{3M}{2R^3}\left(\frac{2}{3}R^5 - \frac{2}{5}R^5\right) \qquad \frac{2}{3} - \frac{2}{5} = \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$$

$$= \boxed{\frac{2}{5}MR^2 = I_{sphere}} \qquad \frac{4}{15}\cdot\frac{3}{2} = \frac{2}{5}$$

Example 7.3.11. *Consider a conical top with uniformly distributed mass M. We consider a cone with radius R and height h. We can envision this shape as a stack of washers at z for $0 \leq z \leq h$. The radius of a washer would linearly decrease from $r = R$ at $z = 0$ to $r = 0$ at $z = h$. The formula for this is $r = R(1 - \frac{z}{h})$. Notice $dV = \pi r^2 dz$ hence the volume of the cone is given by:*

$$V = \int_0^h \pi R^2 \left(1 - \frac{z}{h}\right)^2 dz = \int_0^h \pi R^2 \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) dz = \pi R^2 \left(h - \frac{2h^2}{2h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}\pi R^2 h$$

I derived this in case the reader is unfamilar with the formula for the volume of a cone. This is a fairly standard problem from introductory calculus and most students will have seen this in elementary school mathematics. Notice,

$$\rho = \frac{dm}{dV} = \frac{M}{\frac{1}{3}\pi R^2 h} \quad \Rightarrow \quad dm = \frac{M}{\frac{1}{3}\pi R^2 h} dV = \frac{M}{\frac{1}{3}\pi R^2 h} \pi r^2 dz = \frac{3Mr^2 dz}{R^2 h}.$$

Consequently, (using the inertia of a disk) $dI = \frac{1}{2} r^2 dm$ integrated yields

$$I = \int_0^h \frac{3Mr^4 dz}{2R^2 h} = \frac{3MR^4}{2R^2 h} \int_0^h \left(1 - \frac{z}{h}\right)^4 dz \quad \Rightarrow \quad I = \frac{3MR^2}{2h} \int_1^0 u^4(-h du) = \boxed{\frac{3}{10} MR^2}.$$

we set $u = 1 - z/h$ then $du = -dz/h$ and $u(0) = 1$ whereas $u(h) = 0$.

Example 7.3.12. *A spherical shell of mass M with radius R has moment of inertia of $I = \frac{2}{3} MR^2$ about an axis through a diameter. The calculation of this result rests either on the problem of surface area calculation, or on the use of spherical coordinates or both. As such, I don't typically test on it in this course.*

Imagine the sphere is cut into bands of radius r at each z for $-R \leq z \leq R$. Each band has mass dm. If the mass M is uniformly distributed over the surface area $4\pi R^2$ then the area mass density is $\sigma = \frac{dm}{dS} = \frac{M}{4\pi R^2}$ hence $dm = \frac{MdS}{4\pi R^2}$. To calculate dS it is convenient to use spherical coordinates[4]*:*

$$x = \rho \cos\theta \sin\phi, \quad y = \rho \cos\theta \sin\phi, \quad z = \rho \cos\phi$$

The spherical shell is given by $\rho = R$ where θ, ϕ serve as coordinates on the sphere. By convention, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. Notice that if we fix ϕ on the sphere then sweep through $d\theta$ then the arclength subtended is $ds_1 = R \sin\phi d\theta$. On the other hand, if we fix θ on the sphere and sweep through $d\phi$ we find the arclength subtended is $ds_2 = R d\phi$. Thus $dS = ds_1 ds_2 = R^2 \sin\phi d\theta d\phi$. Integrating this quantity over $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$ yields the well-known formula of $4\pi R^2$ for the surface area of the sphere. If we wish to calculate the moment of inertia with respect to the z-axis then $r = \sqrt{x^2 + y^2} = R \sin\phi$ and so $dI = r^2 dm$ yields

$$dI = \frac{Mr^2 dS}{4\pi R^2} = \frac{M(R\sin\phi)^2 R^2 \sin\phi \, d\theta d\phi}{4\pi R^2} = \frac{MR^2}{4\pi} \sin^3\phi \, d\theta d\phi$$

Thus,

$$I = \frac{MR^2}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin^3\phi \, d\theta d\phi = \frac{MR^2}{4\pi} \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\pi} \sin^3\phi \, d\phi\right) = \frac{MR^2}{4\pi} \cdot 2\pi \cdot \frac{4}{3} = \boxed{\frac{2}{3} MR^2}.$$

[4] yes, I'm using spherical conventions typical to math, there are at least three common conventions, so, sorry, also $\rho = \sqrt{x^2 + y^2 + z^2}$ is not density here.

Example Problem 7.3.13. *Find the moment of inertia for a wagon wheel about its center if it has 6 spokes each with mass m and a rim with mass M at radius R. If the spokes have mass $m = 2\,kg$ each and the rim has mass $M = 40\,kg$ and the wheel has radius $R = 1m$ then how much kinetic energy is stored in the wheel as it rotates with $\omega = 360$ rpm.*

Solution: *moment of inertia is additive; we can find the total moment of inertia as a sum of the inertias of each part. Using Example 7.3.3 we find each spoke gives $\frac{1}{3}mR^2$. Thus, in total:*

$$I = 6\left(\frac{1}{3}mR^2\right) + MR^2 = \boxed{(2m + M)R^2}.$$

Setting $m = 2\,kg$ and $M = 40\,kg$ and $R = 1m$

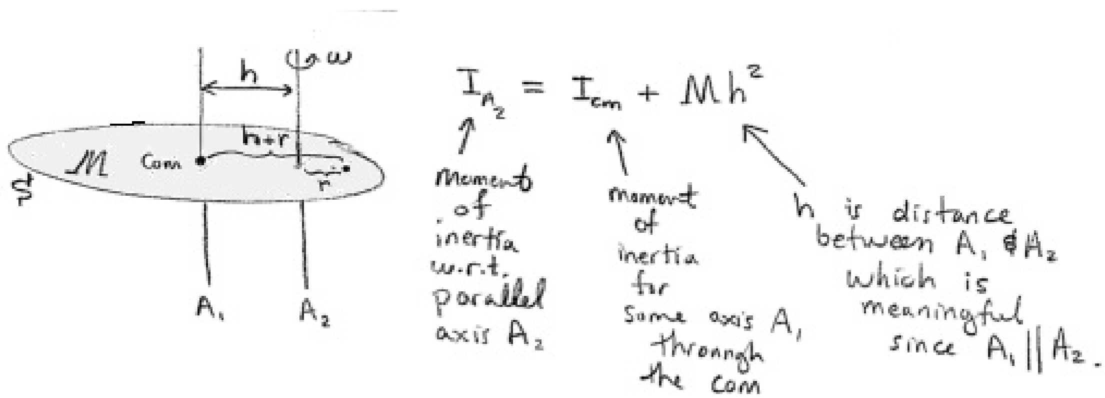
$$I = (2(2kg) + 40kg)(1m)^2 = 44\,kgm^2.$$

Since $KE = \frac{1}{2}I\omega^2$ and $\omega = 360\,\frac{rev}{min}\cdot\frac{min}{60\,s}\cdot\frac{2\pi\,rad}{rev} = 37.70\,\frac{rad}{s}$ we calculate[5]

$$KE = \frac{1}{2}(44\,kgm^2)\left(37.70\,\frac{rad}{s}\right)^2 = \boxed{31.27\,kJ}.$$

7.3.1 parallel axis theorem

The parallel axis theorem gives a simple formula to calculate moment of inertia of a mass M with respect to axis A_2 if you already know the moment of inertia relative to an axis through the center of mass (I_{cm}) of a rigid body; $I_{A_2} = I_{cm} + Mh^2$ where h is the distance between the center of mass axis and the parallel axis.



See http://www.supermath.info/physics231lecture26.pdf page 9 if you desire a proof.

Example 7.3.14. *The moment of inertia of a barbell with a pair of radius R spherical masses M separated distance L about an axis through the center of the barbell is calculated using the parallel axis theorem with $h = L/2$ and $I_{cm} = \frac{2}{5}MR^2$,*

$$I = 2\left(\frac{2}{5}MR^2 + Mh^2\right) = \frac{4}{5}MR^2 + 2M(L/2)^2 = \boxed{M\left(\frac{4}{5}R^2 + L^2/2\right)}$$

Here we have assumed the inertia of the bar holding the masses in place is neglible. We could account for that by adding $\frac{1}{12}mL^2$ if the mass of the bar was m. (Example 7.3.4).

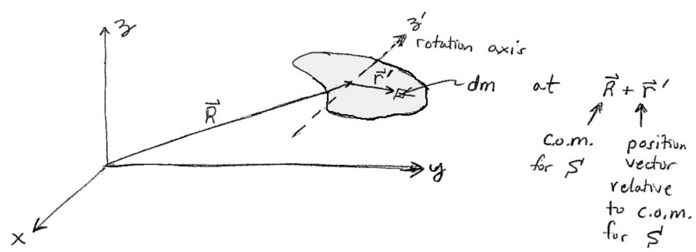
[5] the k in the answer is shorthand for **kilo** which is a multiplier of 10^3. In other words, the answer is approximately $31,270\,J$.

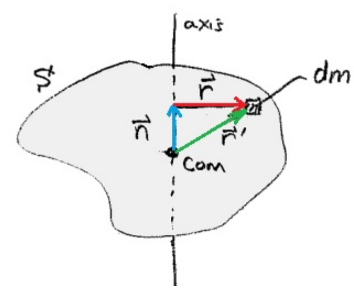
7.4 composite motion

If we have a rigid body of total mass M at center of mass $\vec{R}$ which is rotating about an axis through $\vec{R}$ then it turns out the total energy for the body decouples into a **translational** part and a **rotational** part

$$\boxed{KE = \underbrace{\frac{1}{2}MV^2}_{\text{translational KE}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{rotational KE}}}$$

where V is the speed of the center of mass and ω is the angular velocity of the rotation of the mass. To derive this result we consider a typical mass dm and find how its kinetic energy dK is related to the center of mass motion as well as the motion relative to the center of mass. Let me share some sketchy arguments which hopefully are convincing:





The position of dm is given by $\vec{R} + \vec{r}\,'$ as pictured. Notice that the position relative to the center of mass (c.o.m.) can be further decomposed into two parts. First, the $\vec{n}$ (in blue) points parallel the axis through the c.o.m.. Second, the $\vec{r}$ (in red) points perpendicular to the axis. Together, $\vec{r}\,' = \vec{n} + \vec{r}$. Since we consider a **rigid body** we have $\vec{n}$ is a constant vector and the magnitude $r = \|\vec{r}\|$ is likewise constant. Thus,

$$\frac{d\vec{r}\,'}{dt} = \frac{d}{dt}\left[\vec{n} + \vec{r}\right] = \frac{d\vec{r}}{dt}.$$

Let $\vec{\gamma} = \vec{R} + \vec{r}\,'$ denote the position of dm relative to the origin of the fixed frame of reference which I have illustrated with with x, y, z-axes. Then,

$$\frac{d\vec{\gamma}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt} = \vec{V} + \vec{u}$$

where we've defined $\vec{V} = \frac{d\vec{R}}{dt}$ as is our usual custom and $\vec{u} = \frac{d\vec{r}}{dt}$ is the velocity of dm relative to the c.o.m.. We assume that $\|\vec{u}\| = \omega r$ since $\vec{u}$ serves as the tangential velocity with respect to the rotation with angular velocity ω. Calculate,

$$\left\|\frac{d\vec{\gamma}}{dt}\right\|^2 = (\vec{V} + \vec{u}) \bullet (\vec{V} + \vec{u}) = \vec{V} \bullet \vec{V} + 2\vec{V} \bullet \vec{u} + \vec{u} \bullet \vec{u} = V^2 + 2\vec{V} \bullet \vec{u} + \omega^2 r^2$$

Since the kinetic energy of mass dm at position $\vec{\gamma}$ is $dK = \frac{1}{2}(dm)\left\|\frac{d\vec{\gamma}}{dt}\right\|^2$ we find the total kinetic energy of the rotating and translating mass distributed over S by the integral below:

$$\begin{aligned} KE &= \frac{1}{2}\int_S \left(V^2 + 2\vec{V}\bullet\vec{u} + \omega^2 r^2\right) dm \\ &= \frac{1}{2}\left(\int_S dm\right)V^2 + \int_S \vec{V}\bullet\vec{u}\, dm + \frac{1}{2}\left(\int_S r^2 dm\right)\omega^2 \\ &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 \end{aligned}$$

since $\int_S \vec{V}\bullet\vec{u}\, dm = 0$. Let's see why this term is zero. Notice that $\vec{R}$ is defined by the integral $\vec{R} = \frac{1}{M}\int_S \vec{\gamma}\, dm$ in our current context. Notice,

$$\int_S \left(\vec{\gamma} - \vec{R}\right) dm = M\left(\frac{1}{M}\int_S \vec{\gamma}\, dm\right) - \left(\int_S dm\right)\vec{R} = M\vec{R} - M\vec{R} = 0.$$

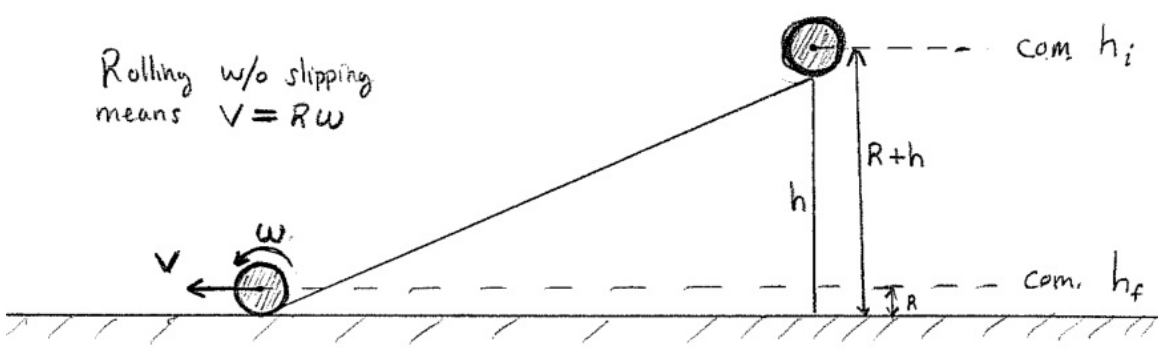
Notice $\vec{u} = \frac{d\vec{\gamma}}{dt} - \vec{V} = \frac{d\vec{\gamma}}{dt} - \frac{d\vec{R}}{dt}$ thus

$$\int_S \vec{V}\bullet\vec{u}\, dm = \int_S \vec{V}\bullet\left(\frac{d\vec{\gamma}}{dt} - \vec{V}\right) dm = \vec{V}\bullet\frac{d}{dt}\int_S\left(\vec{\gamma} - \vec{R}\right) dm = 0.$$

This concludes the derivation of the boxed formula at the beginning of this section.

Remark 7.4.1. *The general study of mechanical systems involves choosing a system of coordinates for a given system which captures the natural dynamics of the problem. In the abstract this can be very challenging and the set-up of Newtonian Mechanics is not well-suited to deal with many natural choices of coordinates. For example, if we considered a pendulum hanging off another pendulum which was attached to a turn-table on a train. Or, a robot arm with n-points of control etc... It turns out that Lagrangian Mechanics or Hamiltonian Mechanics gives a more robust formalism to naturally deal with non-Cartesian coordinate choices as well as geometrically constrained problems. The rotating rigid body problem covers many applications, but a student of mechanics should not be content with the formalism we cover in this course. There are deeper things to learn.*

Example Problem 7.4.2. *Suppose a cylindrical mass M and radius R is at rest above an inclined plane of height h. If the mass rolls without slipping down the plane then what is the speed of the mass when it reaches the base of the incline ?*



Solution: *total mechanical energy is conserved in the process of rolling without slipping if there is no rolling friction. Since we were not given any information about rolling friction we assume it is* *neglible. The total mechanical energy has three parts; translational KE, rotational KE, gravitational PE. We have*

$$E = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + Mgy.$$

Since the mass rolls without slipping we find $V = R\omega$ and since the cylinder is rotating around its center of mass $I = \frac{1}{2}MR^2$. Therefore,

$$E = \frac{1}{2}MV^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\left(\frac{V}{R}\right)^2 + Mgy = \frac{3}{4}MV^2 + Mgy.$$

Notice $E_i = Mgh$ and $E_f = \frac{3}{4}MV_f^2$ thus $E_i = E_f$ yields $Mgh = \frac{3}{4}MV_f^2$ thus $\boxed{V_f = \sqrt{\frac{4gh}{3}}}$.

7.5 angular momentum and torque

Given a mass m at position $\vec{r}$ we define **velocity** by $\vec{v} = \frac{d\vec{r}}{dt}$ and **momentum** by $\vec{p} = m\vec{v}$. To describe motion which is twisting away from the direction of $\vec{v}$ it is natural to consider the cross-product with position.

Definition 7.5.1. *We define* **angular momentum** *of m by $\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v}$.*

Differentiate the definition of angular momentum to derive

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left[\vec{r}\times\vec{p}\right] = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} = m\vec{v}\times\vec{v} + \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt}$$

where we have used the fact that $\vec{v}\times\vec{v} = 0$. Recall that Newton's Second Law in momentum form states $\vec{F} = \frac{d\vec{p}}{dt}$ where $\vec{F}$ is the net-force on m. Thus,

$$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}$$

We see that angular momentum is constant if $\vec{r}\times\vec{F} = 0$.

Definition 7.5.2. *We define* **torque** *due to force $\vec{F}$ applied at $\vec{r}$ to be $\vec{\tau} = \vec{r}\times\vec{F}$. We define the* **net-torque** *on m to be the torque of the net-force on m applied at the position of m; $\vec{\tau}_{net} = \vec{r}\times\vec{F}_{net}$.*

Observe,

$$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}_{net} = \vec{\tau}_{net} \quad\Rightarrow\quad \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{net}}. \tag{7.1}$$

Angular momentum is natural for describing rotational motion. However, it can be calculated for trajectories which are not usually thought of as rotational:

Example 7.5.3. *Consider a projectile m with initial velocity $\vec{v}_o$ and position $\vec{r}_o$ then if the net-force $\vec{F} = -mg\hat{z}$ then solving $m\vec{a} = -mg\hat{z}$ yields*

$$\vec{v} = \vec{v}_o - gt\hat{z} \qquad \& \qquad \vec{r} = \vec{r}_o + t\vec{v}_o - \frac{1}{2}gt^2\hat{z}.$$

We can calculate the angular momentum for the projectile with respect to $\vec{r}_o$,

$$\vec{L} = m(\vec{r} - \vec{r}_o)\times\vec{v} = m\left(t\vec{v}_o - \frac{1}{2}gt^2\hat{z}\right)\times(\vec{v}_o - gt\hat{z}) = -mgt^2\vec{v}_o\times\hat{z} - \frac{1}{2}mgt^2\,\hat{z}\times\vec{v}_o$$

Simplifying we find: $\vec{L} = \frac{1}{2}mgt^2\,\hat{z} \times \vec{v}_o$. *Differentiate,*

$$\frac{d\vec{L}}{dt} = mgt\hat{z} \times \vec{v}_o$$

The net-torque on m *with respect to the origin* $\vec{r}_o$ *is given by*

$$\vec{\tau}_{net} = (\vec{r} - \vec{r}_o) \times \vec{F} = \left(t\vec{v}_o - \frac{1}{2}gt^2\hat{z}\right) \times (-mg\hat{z}) = mgt\,\hat{z} \times \vec{v}_o.$$

We've verified $\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$.

Example 7.5.4. *Suppose mass* $M = 10\,kg$ *at position* $\vec{r} = \langle 1, 1, 0\rangle m$ *has velocity* $\vec{v} = \langle 1, -1, 3\rangle m/s$ *then the angular momentum of* M *with respect to the origin is:*

$$\vec{L} = m\vec{r} \times \vec{v} = (10kgm^2/s)\langle 1, 1, 0\rangle \times \langle 1, -1, 3\rangle = (10kgm^2/s)\langle 3, -3, -2\rangle$$

If we were to apply a net-force $\vec{F} = F_o\langle 1, 1, 0\rangle$ *then*

$$\vec{\tau}_{net} = mF_o\langle 1, 1, 0\rangle \times \langle 1, 1, 0\rangle = 0$$

and the angular momentum would stay constant for small times. If we were to apply a net-force $\vec{F} = F_o\langle 1, -1, 0\rangle$ *then*

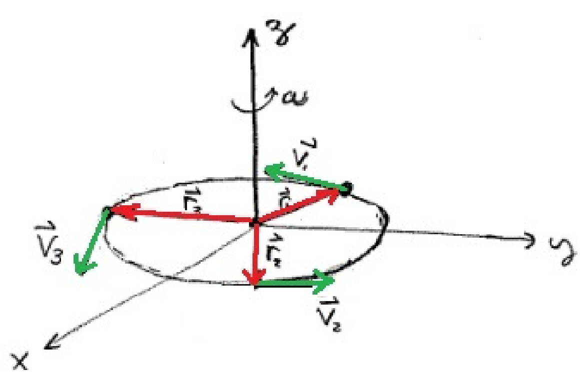
$$\vec{\tau}_{net} = F_o m\langle 1, 1, 0\rangle \times \langle 1, -1, 0\rangle = F_o m\langle 0, 0, -2\rangle$$

and the change in $\vec{L}$ *would be directed in* $-\hat{z}$*-direction.*

Example 7.5.5. *Imagine a ball of mass* M *on a rope which slides without friction on an icy plane. Suppose it travels in a circle of radius* R *with speed* v*. In this case the net-torque is zero since gravity, the normal force and the tension force are all perpendicular to the position vector provided we use the center of the circle as the origin. Thus* $\vec{L}$ *is constant,*

$$\vec{L} = \vec{r} \times (M\vec{v}) = MRv\hat{z}$$

where I have assumed that the circular motion is Counter-Clock-Wise (CCW) as viewed from above the plane where the positive z-axis points. Below is a sketchy picture of the ball at three different positions.



At every time the angular momentum vector points in the positive z-direction. In this context, we have $\vec{\omega} = \vec{r} \times \vec{v}$ where we can think of angular velocity as a vector in the direction of the axis corresponding to a rotation in the positive sense as given by the right-hand-rule. Here, $\vec{\omega} = \omega\hat{z}$. In fact,

$$\vec{L} = I\vec{\omega} = MR^2\omega\hat{z}$$

Definition 7.5.6. *We define* **angular velocity** *of $\vec{\omega}$ of m by $\vec{v} = \vec{\omega} \times \vec{r}$.*

Notice the angular velocity depends on out choice of origin directly. In contrast, if we merely translate coordinate systems then the linear velocity is not altered. This three dimensional concept of angular velocity naturally reduces to our earlier work in two dimensions with the convention that we simply take positive and negative values to indicate the direction of the rotation since the axis of the rotation is understood from context. In contrast, for the three-dimensional concept, there is no fixed axes choice in general hence we require a vector quantity to describe the angular velocity.

We need something called the **inertia tensor** to properly understand the motion of rigid bodies where the torque doesn't happen to happily align with a principle axis of symmetry for the object. Consider,

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad \Rightarrow \quad \triangle\vec{L} \approx \vec{\tau}\,\triangle t$$

If the torque is **not** pointed in the same direction as $\vec{L}$ then we see the direction of the angular momentum will change as time evolves. This certainly is outside the context of our fixed origin two-dimensional rotation discussion from the start of this Chapter.

Notice we may integrate $\vec{\tau} = \frac{d\vec{L}}{dt}$ to derive

$$\vec{L}_2 = \vec{L}_1 + \int_{t_1}^{t_2} \vec{\tau}(t)\,dt$$

If $\vec{\tau}$ is in the same direction as the initial angular momentum $\vec{L}_1$ then the equation above clearly shows $\vec{L}_2$ is in the same direction as $\vec{L}_1$.

The study of the rotational dynamics flows from analyzing the equations:

$$\vec{L} = I\vec{\omega} \qquad \text{and} \qquad \vec{\tau} = \frac{d\vec{L}}{dt}$$

where for a given rigid body with mass density $\rho = \frac{dm}{dV}$ the inertia tensor I_{ij} is defined by

$$I_{ij} = \int (\delta_{ij}\|r\|^2 - x_i x_j)\rho(r)dV.$$

We can also think of the inertia tensor as a 3×3 matrix which generalizes the moment of inertia for general shapes. We probably don't have the mathematics to solve interesting problems in this course, but perhaps in a good Junior level mechanics course we could study the motion of spinning tops and precession. I'll include an optional section at the end of this Chapter which explains how the mysterious integral above is motivated.

7.5.1 conservation of angular momentum

Let us begin with the definition.

Definition 7.5.7. *If masses $m_1, m_2, \dots, m_n$ are at positions $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ with velocities $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ then the* **total angular momentum** *of the system is*

$$\vec{L} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2 + \cdots + m\vec{r}_n \times \vec{v}_n$$

Equivalently, denoting $\vec{L}_j = m_j\vec{r}_j \times \vec{v}_j = \vec{r}_j \times \vec{p}_j$ we define the total momentum of the system as

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_n.$$

We need to repeat much of the argument seen in Section 6.3 where we proved $\frac{d\vec{P}}{dt} = \vec{f}^{ext}$.

Suppose that the n-masses under consideration act on each other with certain forces. We call such forces **internal forces**. By Newton's Third Law internal forces necessarily come in pairs which act with equal magnitude in opposite directions. If we denote $\vec{F}_{ij}$ to mean the force placed on mass m_j by mass m_i then Newton's Third Law requires $\vec{F}_{ij} = -\vec{F}_{ij}$. In addition, let us denote $\vec{f}_i^{ext}$ for the forces on m_i which are not from the other masses in the distribution, these are the **external forces** on the distribution. Let us also define the **total external force** on the system by $\vec{f}^{ext} = \sum_{i=1}^n \vec{f}_i^{ext}$. In summary, the net-force on m_i is given by

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i = \vec{f}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij}$$

Much as in the arguments supporting Equation 7.1, we differentiate $\vec{L}_i = \vec{r}_i \times \vec{p}_i$

$$\frac{d\vec{L}_i}{dt} = \frac{d}{dt}\left[\vec{r}_i \times \vec{p}_i\right] = \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \vec{v}_i \times (m\vec{v}_i) + \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \vec{r}_i \times \vec{F}_i.$$

Therefore, expanding the net-force on m_i, we have

$$\frac{d\vec{L}_i}{dt} = \vec{r}_i \times \left(\vec{f}_i^{ext} + \sum_{j \neq i} \vec{F}_{ij}\right) = \vec{\tau}_i^{\ ext} + \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}$$

where I've introduced the **net-external torque** on m_i by $\vec{\tau}_i^{\ ext} = \vec{r}_i \times \vec{f}_i^{ext}$. Take the vector sum of each such equation as i ranges from 1 to n to derive:

$$\sum_{i=1}^n \frac{d\vec{L}_i}{dt} = \sum_{i=1}^n \left(\vec{\tau}_i^{\ ext} + \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}\right) = \sum_{i=1}^n \vec{\tau}_i^{\ ext} + \sum_{i=1}^n \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij}$$

Let $\vec{\tau}^{\ ext} = \sum_{i=1}^n \vec{\tau}_i^{\ ext}$. Then, by linearity of d/dt we derive

$$\frac{d}{dt}\sum_{i=1}^n \vec{L}_i = \vec{\tau}^{\ ext} + \sum_{i=1}^n \sum_{j \neq i} \vec{r}_i \times \vec{F}_{ij} \quad \Rightarrow \quad \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}^{ext}}$$

since the double sum must vanish thanks to Newton's 3rd Law,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j\neq i}\vec{r}_i\times\vec{F}_{ij} &= \sum_{i<j}\vec{r}_i\times\vec{F}_{ij}+\sum_{i>j}\vec{r}_i\times\vec{F}_{ij} \\
&= \sum_{i<j}\vec{r}_i\times\vec{F}_{ij}+\sum_{l>k}\vec{r}_l\times\vec{F}_{lk} \\
&= \sum_{i<j}\vec{r}_i\times\vec{F}_{ij}-\sum_{l>k}\vec{r}_l\times\vec{F}_{kl} \\
&= \sum_{i<j}\vec{r}_i\times\vec{F}_{ij}-\sum_{k<l}\vec{r}_l\times\vec{F}_{kl} \\
&= \sum_{i<j}\vec{r}_i\times\vec{F}_{ij}-\sum_{i<j}\vec{r}_j\times\vec{F}_{ij} \\
&= \sum_{i<j}(\vec{r}_i-\vec{r}_j)\times\vec{F}_{ij} \\
&= 0.
\end{aligned}$$

Notice $\vec{r}_i-\vec{r}_j$ is the displacement vector to travel from mass m_j to mass m_i. Newton's 3rd Law required F_{ij} to be colinear to the line connecting m_i and m_j, thus $(\vec{r}_i-\vec{r}_j)\times\vec{F}_{ij}=0$ for all $i<j$. This concludes our proof of the conservation of angular momentum for a system of n-point masses.

Remark 7.5.8. *The derivation for a finite collection of point masses can be extended to the context of a continuous distribution of masses by appropriately replacing sums with integrals. I will not make such arguments explicit here, however, we will work examples which assume such results are known. I should also mention, the proper discussion here requires the introduction of the* **inertia tensor** *as well as the analysis of eigenvectors and eigenvalues of the inertia tensor which are identified as the natural axes and moments associated with the motion of such an object. I include an optional section at the end of the Chapter with a brief introduction to the inertia tensor.*

7.6 rotational dynamics in two-dimensions

I've discussed some somewhat complicated three dimensional issues in previous sections. Now we turn to focus on the much simpler case where the torque and angular momentum are aligned and we can surpress the vector notation and adopt a one-dimensional approach where the coordinate is θ in radians. Let me be absurdly explicit:

(i.) positive change in θ indicates a Counter-Clock-Wise (CCW) rotation

(ii.) negative change in θ indicates a Clock-Wise (CW) rotation

Similarly, angular velocity follows:

(i.) $\omega=\frac{d\theta}{dt}>0$ indicates motion in a Counter-Clock-Wise (CCW) direction

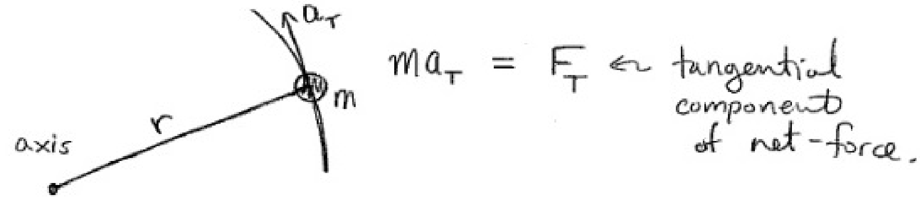
(ii.) $\omega=\frac{d\theta}{dt}<0$ indicates motion in a Clock-Wise (CW) direction

We have $L=I\omega$ where $I>0$ so angular momentum follows the same directional comments as ω. Also, angular acceleration:

(i.) $\alpha=\frac{d\omega}{dt}>0$ indicates ω is increasing in the Counter-Clock-Wise (CCW) direction

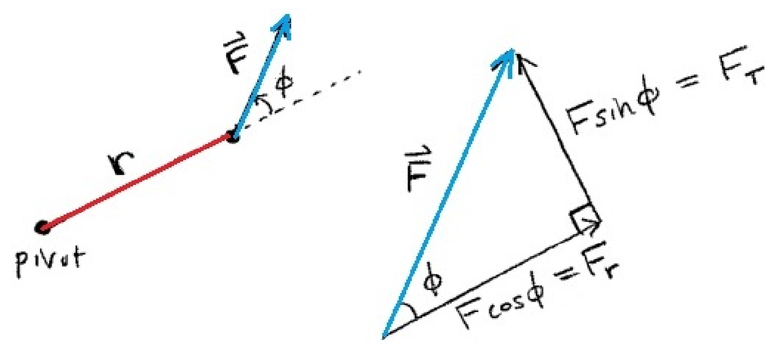
(ii.) $\alpha = \frac{d\omega}{dt} < 0$ indicates ω is decreasing in the Counter-Clock-Wise (CCW) direction

Here $\tau = I\alpha$ so motion with a CCW-angular acceleration is caused by a CCW-torque. Likewise, CW-torque gives CW-angular acceleration. Consider a mass m travelling an arc distance r from a pivot point



In this case $\tau = rF_T$ and $a_T = r\alpha$. I've illustrated for a point mass, but the same equation holds for a rigid body provided we consider the pivot point about the center of mass.

Example 7.6.1. *Suppose we apply a force $\vec{F}$ at angle ϕ off the radial line at distance r from the pivot point then $F_T = F\sin\phi$ and $\tau = rF\sin\phi$ and the radial component F_r produces no torque.*



In our previous work we could not treat massive pulleys. Now we can[6] .

Example Problem 7.6.2. *Suppose masses m_1 and m_2 hang on opposite sides of a massive pulley with mass M and radius R. The masses are connected to a rope with very small mass which pulls on the pulley without slipping. Find the acceleration of the system.*

Solution: *let T_1 be the tension on the rope connecting m_1 to the pulley and T_2 be the tension on the rope connecting m_2 to the pulley. . Newton's Second law for m_1, m_2 and the pulley yield:*

$$m_1a = m_1g - T_1, \qquad \& \qquad m_2a = T_2 - m_2g, \qquad \& \qquad I\alpha = RT_1 - RT_2$$

where I is the moment of inertia of the pulley and I've assumed the tension forces are tangential to their point of application on the pulley. Assume the pulley is a solid disk so $I = \frac{1}{2}MR^2$ and use $a = R\alpha$ to obtain

$$\frac{1}{2}MR^2 \cdot \frac{a}{R} = RT_1 - RT_2 \quad \Rightarrow \quad \frac{1}{2}Ma = T_1 - T_2$$

Now add all three equations for a,

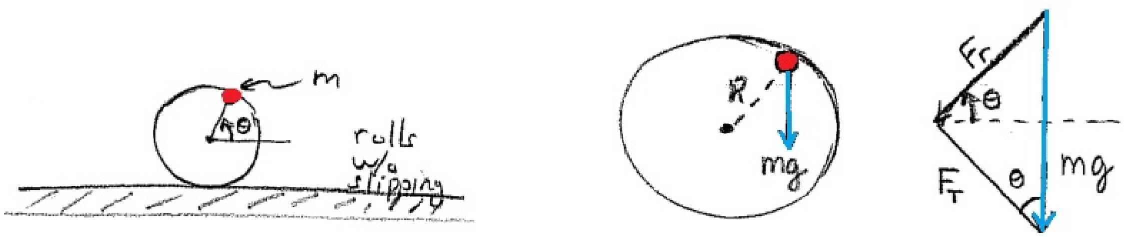
$$m_1a + m_2a + \frac{1}{2}Ma = m_1g - T_1 + T_2 - m_2g + T_1 - T_2$$

[6] why is E3 in Lecture 27 incorrect ?

and solve for a,

$$\boxed{a = \frac{(m_1 - m_2)g}{m_1 + m_2 + M/2}}.$$

Example 7.6.3. *Consider cylinder of mass M and radius R if a small mass m is placed on the rim at an angle θ as pictured.*



In this case, $\tau = RF_T = mgR\cos\theta$ and the moment of inertia for the system is given by $I = mR^2 + \frac{1}{2}MR^2$ provided the cylinder is a solid cylinder with uniformly distributed mass M. Notice gravity does not give a net torque on the cylinder because the gravitational force on the right and left halves of the cylinder give equal and opposite torques. Thus, noting $\alpha = \ddot{\theta}$,

$$(m + M/2)R^2\frac{d^2\theta}{dt^2} = mgR\cos\theta \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = \frac{mg}{(m + M/2)R}\cos\theta$$

Notice the torque $mgR\cos\theta$ is zero for $\theta = \pi/2$ as we should expect. Also, the torque is mgR when $\theta = 0$. Naturally, the torque is $-mgR$ when $\theta = \pi$. These cases double-check our set-up. Suppose $\beta = \theta - \pi/2$ then $\ddot{\beta} = \ddot{\theta}$ and $\cos\theta = \cos(\beta + \pi/2) = -\sin\beta$. If we define $\omega = \sqrt{\frac{mg}{(m+M/2)R}}$ then for small β we have $\sin\beta \approx \beta$ and the given differential equation[7] reads

$$\frac{d^2\beta}{dt^2} = -\omega^2\beta \quad \Rightarrow \quad \beta(t) = \beta_o\sin(\omega t + \delta)$$

In other words, the motion is oscillatory with period $T = \frac{2\pi}{\omega}$. In particular, if the little mass m is near the ground and we give it a little push then the cylinder and mass should rock back and forth with a period of

$$T = \frac{2\pi}{\sqrt{\frac{mg}{(m+M/2)R}}} = \sqrt{\frac{4\pi^2(m + M/2)R}{mg}}$$

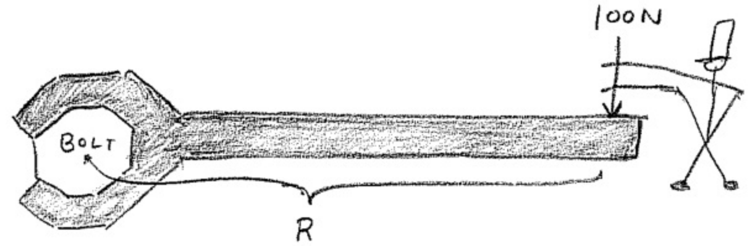
Small angle approximations for sine and cosine are known from calculus. More generally we learn in second semsester calculus that

$$\sin\theta = \theta - \frac{1}{6}\theta^3 + \cdots \qquad \& \qquad \cos\theta = 1 - \frac{1}{2}\theta^2 + \cdots$$

Often in physics just the first term is needed due to a simplifying assumption.

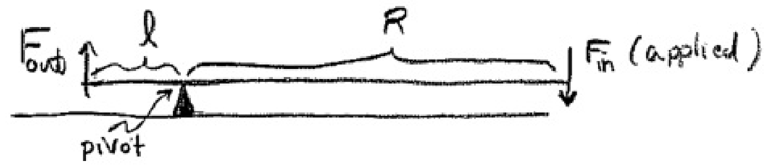
Example 7.6.4. *If you have a wrench that is infinitely long then in principle you can create an infinite torque. If Mr. Tophat applies $100\,N$ as pictured then the torque $\tau = R(100N) \to \infty$ as $R \to \infty$.*

[7] Here we're using the same mathematics we saw in our previous study of simple harmonic motion of a spring. Anytime you encounter a differential equation of the form $\ddot{y} = -\omega^2 y$ you can anticipate oscillatory solutions with an angular frequency ω. The frequency f and period T are also related by $f = \frac{1}{T}$ and $\omega = 2\pi f = \frac{2\pi}{T}$.



Example Problem 7.6.5. *How long a lever do you need to increase your applied force by a factor of 100 times ? Assume there is length ℓ on the left side of the pivot point and neglect mass of lever for ease of argument here.*

Solution: *we desire $F_{out} = 100F_{in}$. Consider the picture below:*

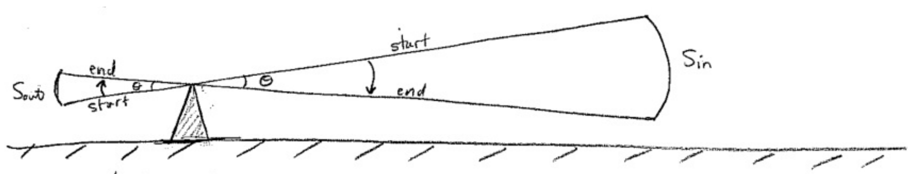


Assume the force is applied at $\alpha = 0$ then we need net-torque zero.

$$F_{out}\ell - F_{in}R = 0 \quad \Rightarrow \quad R = \frac{F_{out}\ell}{F_{in}} = 100\ell$$

Therefore, we need a lever of length 101ℓ to accomplish this feat.

At first glance the example above seems to violate conservation of energy. Isn't the work put into the system smaller than the work put out ? Well, no. The issue is that the larger force is applied over a proproportionally smaller arc and the smaller force is applied over a proportionally larger arc. Here are the sketchy details:



Note that $W_{out} = -S_{out}F_{out} = -S_{out}\,100F_{in}$ whereas $W_{in} = S_{in}F_{in}$ to relate W_{out} & W_{in} we need to link the distances S_{out} and S_{in}. This is accomplished by $S_{out} = \ell\theta$ and $S_{in} = 100\ell\theta$ as the angle is the same in both wedges. Thus $S_{in} = 100\,S_{out}$ and it follows $W_{out} = -W_{in}$ hence $W_{net} = 0$ which is logical since we insisted $\tau_{net} = 0 \Rightarrow \alpha = 0 \Rightarrow a_T = 0 \Rightarrow \Delta KE = 0 \Rightarrow W_{net} = 0$.

Remark 7.6.6. *Lectures 28 and 30 have great examples. I should probably work those out in lecture.*

7.7 inertia tensor

Warning: this section stolen from my Linear Algebra notes. I use terminology here you will not find tested nor in the rest of the course. In the finished version of these notes this will probably be relegated to an Appendix. You can skip it if you're not interested.

We can use quadratic forms to elegantly state a number of interesting quantities in classical mechanics. For example, the translational kinetic energy of a mass m with velocity v is

$$T_{trans}(v) = \frac{m}{2}v^T v = [v_1, v_2, v_3]\begin{bmatrix} m/2 & 0 & 0 \\ 0 & m/2 & 0 \\ 0 & 0 & m/2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

On the other hand, the rotational kinetic energy of an object with moment of inertia I and angular velocity ω with respect to a particular axis of rotation is

$$T_{rot}(v) = \frac{I}{2}\omega^T \omega.$$

In addition you might recall that the force F applied at radial arm r gave rise to a torque of $\tau = r \times F$ which made the angular momentum $L = I\omega$ have the time-rate of change $\tau = \frac{dL}{dt}$. In the first semester of physics this is primarily all we discuss. We are usually careful to limit the discussion to rotations which happen to occur with respect to a particular axis. But, what about other rotations? What about rotations with respect to less natural axes of rotation? How should we describe the rotational physics of a rigid body which spins around some axis which doesn't happen to line up with one of the nice examples you find in an introductory physics text?

The answer is found in extending the idea of the moment of inertia to what is called the inertia tensor I_{ij} (in this section I is not the identity). To begin I'll provide a calculation which motivates the definition for the inertia tensor.

Consider a rigid mass with density $\rho = dm/dV$ which is a function of position $r = (x_1, x_2, x_3)$. Suppose the body rotates with angular velocity ω about some axis through the origin, however it is otherwise not in motion. This means all of the energy is rotational. Suppose that dm is at r then we define $v = (\dot{x_1}, \dot{x_2}, \dot{x_3}) = dr/dt$. In this context, the velocity v of dm is also given by the cross-product with the angular velocity; $v = \omega \times r$. Using the einstein repeated summation notation the k-th component of the cross-product is nicely expressed via the Levi-Civita symbol; $(\omega \times r)_k = \epsilon_{klm}\omega_l x_m$. Therefore, $v_k = \epsilon_{klm}\omega_l x_m$. The infinitesimal kinetic energy due to this little bit of rotating mass dm is hence

$$\begin{aligned}
dT &= \frac{dm}{2} v_k v_k \\
&= \frac{dm}{2}(\epsilon_{klm}\omega_l x_m)(\epsilon_{kij}\omega_i x_j) \\
&= \frac{dm}{2}\epsilon_{klm}\epsilon_{kij}\omega_l\omega_i x_m x_j \\
&= \frac{dm}{2}(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\omega_l\omega_i x_m x_j \\
&= \frac{dm}{2}(\delta_{li}\delta_{mj}\omega_l\omega_i x_m x_j - \delta_{lj}\delta_{mi}\omega_l\omega_i x_m x_j) \\
&= \omega_l \frac{dm}{2}(\delta_{li}\delta_{mj} x_m x_j - \delta_{lj}\delta_{mi} x_m x_j)\omega_i \\
&= \omega_l \left[\frac{dm}{2}(\delta_{li}||r||^2 - x_l x_i)\right]\omega_i.
\end{aligned}$$

Integrating over the mass, if we add up all the little bits of kinetic energy we obtain the total kinetic energy for this rotating body: we replace dm with $\rho(r)dV$ and the integration is over the volume of the body,

$$T = \int \omega_l \left[\frac{1}{2}(\delta_{li}||r||^2 - x_l x_i)\right]\omega_i \rho(r) dV$$

However, the body is rigid so the angular velocity is the same for each dm and we can pull the components of the angular velocity out of the integration[8] to give:

$$T = \frac{1}{2}\omega_j \underbrace{\left[\int (\delta_{jk}||r||^2 - x_j x_k)\rho(r) dV\right]}_{I_{jk}} \omega_k$$

This integral defines the inertia tensor I_{jk} for the rotating body. Given the inertia tensor I_{lk} the kinetic energy is simply the value of the quadratic form below:

$$T(\omega) = \frac{1}{2}\omega^T I \omega = [\omega_1, \omega_2, \omega_3] \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

The matrix above is not generally diagonal, however you can prove it is symmetric (easy). Therefore, we can find an orthonormal eigenbasis $\beta = \{u_1, u_2, u_3\}$ and if $P = [\beta]$ then it follows by orthonormality of the basis that $[I]_{\beta,\beta} = P^T[I]P$ is diagonal. The eigenvalues of the inertia tensor (the matrix $[I_{jk}]$) are called the **principle moments of inertia** and the eigenbasis $\beta = \{u_1, u_2, u_3\}$ define the **principle axes** of the body.

The study of the rotational dynamics flows from analyzing the equations:

$$L_i = I_{ij}\omega_j \qquad \text{and} \qquad \tau_i = \frac{dL_i}{dt}$$

If the initial angular velocity is in the direction of a principle axis u_1 then the motion is basically described in the same way as in the introductory physics course provided that the torque is also

[8] I also relabled the indices to have nicer final formula, nothing profound here

in the direction of u_1. The moment of inertia is simply the first principle moment of inertia and $L = \lambda_1 \omega$. However, if the torque is not in the direction of a princple axis or the initial angular velocity is not along a principle axis then the motion is more complicated since the rotational motion is connected to more than one axis of rotation. Think about a spinning top which is spinning in place. There is wobbling and other more complicated motions that are covered by the mathematics described here.

Example 7.7.1. *The inertia tensor for a cube with one corner at the origin is found to be*

$$I = \frac{2}{3}Ms^2 \begin{bmatrix} 1 & -3/8 & -3/8 \\ -3/8 & 1 & -3/8 \\ -3/8 & -3/8 & 1 \end{bmatrix}$$

Introduce $m = M/8$ to remove the fractions,

$$I = \frac{2}{3}Ms^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

You can calculate that the e-values are $\lambda_1 = 2$ and $\lambda_2 = 11 = \lambda_3$ with principle axis in the directions

$$u_1 = \frac{1}{\sqrt{3}}(1,1,1), \ \ u_2 = \frac{1}{\sqrt{2}}(-1,1,0), \ \ u_3 = \frac{1}{\sqrt{2}}(-1,0,1).$$

The choice of u_2, u_3 is not unique. We could just as well choose any other orthonormal basis for span$\{u_2, u_3\} = W_{11}$.

Finally, a word of warning, for a particular body there may be so much symmetry that no particular eigenbasis is specified. There may be many choices of an orthonormal eigenbasis for the system. Consider a sphere. Any orthonormal basis will give a set of principle axes. Or, for a right circular cylinder the axis of the cylinder is clearly a principle axis however the other two directions are arbitrarily chosen from the plane which is the orthogonal complement of the axis. I think it's fair to say that if a body has a unique (up to ordering) set of principle axes then the shape has to be somewhat ugly. Symmetry is beauty but it implies ambiguity for the choice of certain princple axes.

Chapter 8

Gravitation

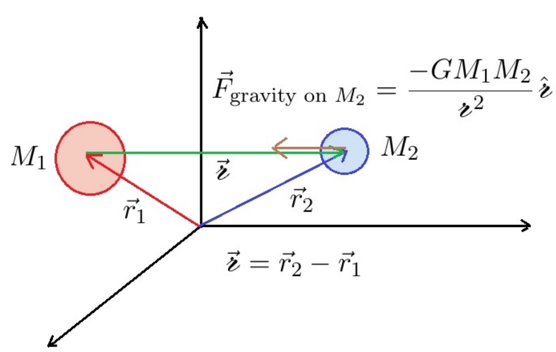
In Example 5.2.6 we looked briefly at the formula for Newton's Universal Law of Gravitation. We take a deeper look at this force law as well as its elementary applications to orbital motion of planets, moons, etc. Isaac Newton

8.1 Newton's Universal Law of Gravitation

If mass M_1 is at $\vec{r}_1$ and mass M_2 is at $\vec{r}_2$ the force of gravity from M_1 on M_2 at $\vec{r}_2$ is given by

$$\vec{F}_{\text{gravity on } M_2} = \frac{-GM_1M_2}{\mathscr{r}^2}\,\hat{\mathscr{r}}.$$

where $\vec{\mathscr{r}} = \vec{r}_2 - \vec{r}_1$ is the displacement vector from $\vec{r}_1$ to $\vec{r}_2$. We assume the masses M_1, M_2 are pointlike. Gravitation for extended objects can be obtained through appropriate integration. It turns out we can think of extended objects as point masses at their centers of mass provided the shape of the objects is sufficiently symmetric or the distance between the objects is sufficiently large that inhomogeneity in shapes can be ignored relative to the desired precision. I will forego proof of this assertion here. Example 8.1.5 illustrates a specific example of how the shape of an object forbids naive thinking for close-by objects, but allows naive thinking for the distant case.



Notice the positions of M_1 and M_2 are taken to be their respective centers of mass $\vec{r}_1$ and $\vec{r}_2$. In terms of mathematical convenience, it is useful to rewrite the formula without use of the unit-vector $\hat{\mathscr{r}}$ as follows:

$$\vec{F}_{\text{gravity on } M_2} = \frac{-GM_1M_2}{\mathscr{r}^3}\,\vec{\mathscr{r}}.$$

Or, to continue in this litany of redundant formulae,

$$\vec{F}_{\text{gravity on } M_2} = \frac{-GM_1M_2}{\|\vec{r}_2 - \vec{r}_1\|^3}(\vec{r}_2 - \vec{r}_1).$$

On the other hand, the gravitation of M_2 on M_1 is given by:

$$\vec{F}_{\text{gravity on } M_1} = \frac{-GM_2M_1}{\|\vec{r}_1 - \vec{r}_2\|^3}(\vec{r}_1 - \vec{r}_2).$$

Notice these gravitational forces comprise a third law pair:

$$\vec{F}_{\text{gravity on } M_2} = -\vec{F}_{\text{gravity on } M_1}.$$

These are equal and opposite reactions within the system comprised of masses M_1 and M_2. Finally, notice that if we place M_1 at the origin and M_2 at $\vec{r}$ then $\vec{r}_1 = 0$ and

$$\vec{F}_{\text{gravity on } M_2} = \frac{-GM_1M_2}{r^3}\vec{r}.$$

Example 8.1.1. *Suppose M_1 is at (x_1, y_1, z_1) and M_2 is at (x_2, y_2, z_2) then $\vec{r}_1 = \langle x_1, y_1, z_1\rangle$ and $\vec{r}_2 = \langle x_2, y_2, z_2\rangle$ and*

$$\|\vec{r}_2 - \vec{r}_1\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Therefore, the force of gravity on M_2 due to M_1 is given by

$$\vec{F} = \frac{-GM_1M_2}{[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2]^{3/2}}\langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle.$$

Notice the direction of the vector above points from $\vec{r}_2$ towards $\vec{r}_1$. Gravity is an **attractive force**, *we should check formulas to make certain that reality is reflected in our mathematics. I can also write the formula above by absorbing the minus sign into the vector part of the formula:*

$$\vec{F} = \frac{GM_1M_2}{[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2]^{3/2}}\langle x_1 - x_2, y_1 - y_2, z_1 - z_2\rangle.$$

Finally, notice the magnitude of the force depends on the inverse square of the distance between the masses:

$$F = \frac{GM_1M_2}{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Example 8.1.2. Problem: *How far apart would two $10^{17}\,kg$ masses be if the force of gravity produced a respective acceleration of $9.8m/s^2$ on each mass?*

Solution: *recall $G = 6.674 \times 10^{-11}\,\frac{Nm^2}{kg^2}$ we suppose the net-force on one of the masses is from gravity of the other mass hence, using $M = 10^{17}\ kg$,:*

$$F = Ma = Mg = \frac{GMM}{d^2} \quad \Rightarrow \quad d = \sqrt{\frac{GM}{g}} = \sqrt{\frac{(6.674\times 10^{-11}\frac{Nm^2}{kg^2})(10^{17}\ kg)}{9.8\frac{m}{s^2}}} = \boxed{825.24\,m.}$$

If $M = 10^9\,kg$ then $d = 0.08252\,m$.

If you think about it, if we have a mass $M_1 = 10^{17}\,kg$ which accelerates $M_2 = 10\,kg$ to an acceleration of $g = 9.8m/s^2$ then the mathematics of the above example is unaltered. It would still be the case that M_2 was $825.24\,m$ in order to experience such a gravitational acceleration.

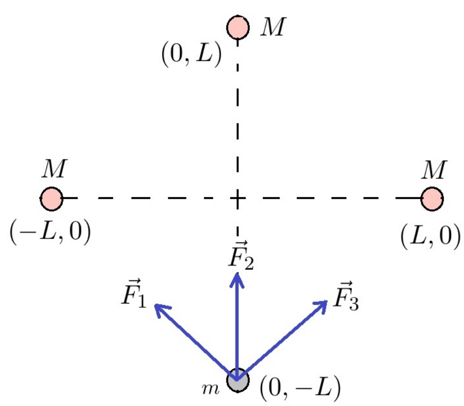
Example 8.1.3. Problem: *The Earth has mass $M_1 = 5.97 \times 10^{24}\,kg$ and produces an acceleration of $9.8m/s^2$ on M_2. How far is M_2 from the center of mass of M_1 ?*

Solution: *we use Newton's Second Law and his law of gravitation $M_2 g = \frac{GM_1M_2}{d^2}$ hence, cancelling M_2 and solving for d yields*[1]*:*

$$d = \sqrt{\frac{GM_1}{g}} = \sqrt{\frac{(6.674 \times 10^{-11}\frac{Nm^2}{kg^2})(5.97 \times 10^{24}\,kg)}{9.8\frac{m}{s^2}}} = 6.376 \times 10^6\,m = \boxed{6,376\,km.}$$

The average global radius of the earth is measured to be $6371\,km$. Technically, this is not quite correct since the earth is not technically a sphere. Because the earth is spinning the equatorial radius is larger than the polar radii. Roughly, the radius at the north and south poles is about $6,357\,km$ whereas the radii at the equator is something[2] *like $6,378\,km$.*

Example 8.1.4. Problem: *Suppose we have three identical masses with mass M placed at the positions $(0, L)$ and $(-L, 0)$ and $(L, 0)$. Find the net-force of gravity on a mass m placed at $(0, -L)$.*



Solution: *Let us set-up the forces of gravity for each mass pictured above:*

(i.) *The displacement vector from $(0, -L)$ to $(-L, 0)$ is $\vec{r}_1 = \langle -L, L\rangle$ this points in the direction of the force of gravity on m at $(0, -L)$ from the mass M at $(-L, 0)$. Then $r_1 = L\sqrt{2}$ and*

$$\vec{F}_1 = \frac{GmM}{(L\sqrt{2})^3}\langle -L, L\rangle = \frac{GmM}{L^2}\left\langle \frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right\rangle.$$

(ii.) *The displacement vector from $(0, -L)$ to $(0, L)$ is $\vec{r}_2 = \langle 0, 2L\rangle$ this points in the direction of the force of gravity on m at $(0, -L)$ from the mass M at $(0, L)$. Note $r_2 = 2L$ hence*

$$\vec{F}_2 = \frac{GmM}{(2L)^3}\langle 0, 2L\rangle = \frac{GmM}{L^2}\left\langle 0, \frac{1}{4}\right\rangle$$

[1] if you run the numbers with $9 = 9.81m/s^2$ then this works out to $6,373\,km$

[2] I found these figures at https://en.wikipedia.org/wiki/Earth_radius.

(iii.) *The displacement vector from* $(0,-L)$ *to* $(L,0)$ *is* $\vec{r}_3 = \langle L, L\rangle$ *this points in the direction of the force of gravity on* m *at* $(0,-L)$ *from the mass* M *at* $(L,0)$*. Then* $r_3 = L\sqrt{2}$ *and*

$$\vec{F}_3 = \frac{GmM}{(L\sqrt{2})^3}\langle L, L\rangle = \frac{GmM}{L^2}\left\langle \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right\rangle.$$

The total force on m *is found from the vector sum of* $\vec{F}_1, \vec{F}_2, \vec{F}_3$*,*

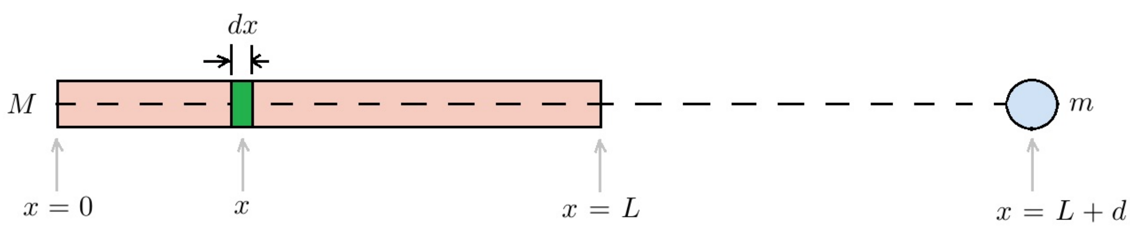
$$\vec{F}_{net} = \frac{GmM}{L^2}\left(\left\langle \frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right\rangle + \left\langle 0, \frac{1}{4}\right\rangle + \left\langle \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right\rangle\right) = \boxed{\frac{GmM}{L^2}\left\langle 0, \frac{1+2\sqrt{2}}{4}\right\rangle}.$$

Example 8.1.5. Problem: *Suppose a mass* M *is uniformly distributed along a rod of length* L*. Calculate the acceleration due to the gravitation of this rod a distance* d *from the rod's endpoint along the axis of the rod. Also, analyze the gravitation of the rod in the case* $d >> L$*.*

Solution: *we can think of this as a one-dimensional problem. Let us place the rod from* $x = 0$ *to* $x = L$ *and work towards calculating the acceleration due to gravity at* $x = L + d$*. We suppose a mass* m *is placed at* $L + d$*. The force of gravity from* M *cannot be directly calculated from Newton's Universal Law of Gravitation as one side of the rod is much closer than than the other side. We use the infinitesimal method to calculate the net-force. Consider a little bit of mass* dM *at* x *we find infinitesimal force*

$$dF = \frac{-Gm\,dM}{(L+d-x)^2} = \frac{-Gm\lambda\,dx}{(x-L-d)^2}$$

where $\lambda = \frac{M}{L} = \frac{dM}{dx}$ *by our assumption of uniform distribution of mass over the rod.*



The net-force on m *is found by integrating* x *from* $x = 0$ *to* $x = L$*,*

$$\begin{aligned} F_{net} &= \int_0^L \frac{-Gm\lambda\,dx}{(x-L-d)^2} \\ &= \frac{Gm\lambda}{x-L-d}\bigg|_0^L \\ &= Gm\lambda\left[\frac{1}{-d} - \frac{1}{-L-d}\right] \\ &= Gm\lambda\left[\frac{1}{L+d} - \frac{1}{d}\right] \\ &= Gm\frac{M}{L}\left[\frac{-L}{d(L+d)}\right] \\ &= \frac{-GmM}{d(L+d)} \end{aligned}$$

Then as $F_{net} = ma$ *we find* $\boxed{a = \frac{-GM}{d(L+d)}}$. *Notice that if* $L << d$ *then* $L+d \approx d$ *and* $\boxed{F_{net} \approx \frac{-GMm}{d^2}}$. *It is as if the rod is just a point-mass* M *if we are very far from the rod.*

Remark 8.1.6. *I might be tempted to think of the rod as a point particle at* $x = L/2$. *If I was to do that then the force of gravity would be calculated as*

$$F_{net} = \frac{-GmM}{(L+d-L/2)^2} = \frac{-GmM}{(d+L/2)^2} = \frac{-GmM}{d^2+Ld+L^2/4}$$

However, we calculated $F_{net} = \frac{-GmM}{d^2+dL}$. *We cannot approximate the rod as a particle if* m *and* M *are near each other. Of course, if* $d >> L$ *then the* $L^2/4$ *term in the denominator is very small in comparison to* d^2 *or even* Ld.

8.2 orbital motion

Newton's Universal Law of Gravitation explains most of the orbital motion of planets about the sun as well as the trajectories of asteroids, moons and satellites. All of this said, Newtonian gravity does not explain how the influence of gravity propagates.

The formula for the gravity force has no mechanism to explain how gravitational forces change with time. Gravitational waves are not naturally included in Newton's gravity, but with Einstein's General Relativity the propagation of gravity is seen through gravitational waves which travel at the speed of light. It turns out Newton was successful in ignoring the problem of gravity propagation because the problems we study with Newtonian gravity involve speeds which are very slow in comparison to the speed at which gravitational change is propagated. I only qualitatively describe General Relativity(GR) in these notes, the quantitative description requires the mathematics of curved space since GR views gravitational accelerations as manifestations of the curvature of the spacetime manifold.

In the examples which follow, we proceed as did Newton, and ignore the question of how gravity propagates. We assume gravity is instantaneous.

Example 8.2.1. *Suppose the sun has mass* M *and a planet has mass* m *as it orbits distance* R *from the sun. We place the sun at the origin and assume the planet orbits in a circle. The force of gravity is what causes the circular motion hence*

$$\frac{mv^2}{R} = \frac{GmM}{R^2} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{R}}$$

We find the motion has constant speed which is related to the mass M *of the sun and the radius* R *of the planet's orbit. If the time of the planets orbit is* T *then we note* $v = \frac{2\pi R}{T}$ *thus*

$$\frac{2\pi R}{T} = \sqrt{\frac{GM}{R}} \quad \Rightarrow \quad \frac{4\pi^2 R^2}{T^2} = \frac{GM}{R} \quad \Rightarrow \quad \boxed{\frac{R^3}{T^2} = \frac{GM}{4\pi^2}}.$$

The boxed equation above is Kepler's Third Law in its naive form. I say naive because a more sophisticated derivation reveals the motion of the planet is in an ellipse. We assume the sun is *fixed at the origin, but in reality the sun and planet both orbit their center of mass, but the orbit of the sun is very slight in comparison to the planets and the center of mass is actually within the sun itself*[3].

> *The ratio of the cube of the orbital radius and the square of the orbital period is constant for all planets orbiting the sun.*

Earth's orbit has $R = 1.496 \times 10^{11} m = 1.0\ AU$ *where we introduce the* **astronomical unit** (AU). *We know the period of the Earth's orbit is a year. It turns out there are about* 1.881 *years in every orbit of Mars. We can calculate the orbital radius of Mars using Kepler's Third Law:*

$$\frac{R_E^3}{T_E^2} = \frac{GM}{4\pi^2} = \frac{R_M^3}{T_M^2} \quad \Rightarrow \quad R_M^3 = \frac{R_E^3 T_M^2}{T_E^2} = R_E^3 (1.881)^2 \quad \Rightarrow \quad R_M = \sqrt[3]{(1.881)^2} R_E = 1.524\ AU.$$

That is, the orbital radius of Mars is $R_M = 2.28 \times 10^{11} m$. *If you are curious,* $1\ AU = 92.96$ *million miles so* $R_M = 141.67$ *million miles.*[4]

Example 8.2.2. *We can calculate the mass* M *of the Sun since we know* R_E, T_E *and* G, *continuing from the previous example,*

$$\frac{R_E^3}{T_E^2} = \frac{GM}{4\pi^2} \quad \Rightarrow \quad M = \frac{4\pi^2 R_E^3}{G T_E^2} = \frac{4\pi^2 (1.496 \times 10^{11} m)^3}{(6.674 \times 10^{-11} \frac{Nm^2}{kg^2})(365.25(24)(60)(60)s)^2} = 1.9887 \times 10^{30}\, kg.$$

There is nothing particularly special about the Sun being the center in the previous example. The same analysis holds if we consider the Earth and a satellite which orbits thanks to gravity. One particularly interesting case is when the orbital period of a satellite matches the time for an Earth day. This is known as **geosynchronous orbit**.

Example 8.2.3. Problem: *derive the altitude of a satellite in geosynchronous orbit.*

Solution: *let* $M = 5.97 \times 10^{24}\, kg$ *be the mass of the earth and* m *the mass of a satellite then if the satellite goes in circular orbit due to the earth's gravitation we have*

$$\frac{mv^2}{r} = \frac{GmM}{r^2} \quad \Rightarrow \quad v^2 = \frac{GM}{r}$$

where $r = R + h$ *is the distance from the center of the earth to the satellite and* $R = 6.371 \times 10^6 m$ *is the radius of the earth and* h *is the altitude of the satellite. Geosynchronous orbit has* $T = (24)(60)(60)s = 86,400s$ *and since* $v = 2\pi r/T$ *we find,*

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r} \quad \Rightarrow \quad r^3 = \frac{GMT^2}{4\pi^2} \quad \Rightarrow \quad r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} \quad \Rightarrow \quad h = \sqrt[3]{\frac{GMT^2}{4\pi^2}} - R.$$

[3] it's more complicated than this of course since there are multiple planets and a complete analysis must also address gravitational attraction amongst the planets and their moons, moreover, there are thousands of asteroids to consider as well. It took the collective effort of hundreds of mathematicians over the 19th century to come to a decisive understanding that a particular aspect of the motion of Mercury was not explained via Newtonian gravity. The so-called precession of the perihelion of mercury was not calculable from Newton's gravity. The success of General Relativity to calculate a correction to Newtonian gravity on this point is one of the major reasons we believe GR is a good theory of physics

[4] the eccentricity of the Martian orbit is about 9 times larger than that of Earth, it follows the orbit is more elliptical. The distance from the sun to Mars fluxuates about 10 % whereas the Earth's distance to the Sun ranges from about 0.983 AU to 1.017 AU. My point here is just that depending what time of the Martian year you read this the figure for R_M could be off about 14 million or so miles.

Thus,

$$h = \sqrt[3]{\frac{(6.674 \times 10^{-11} \frac{Nm^2}{kg^2})(5.97 \times 10^{24}\, kg)(86,400s)^2}{4\pi^2}} - 6.371 \times 10^6 m = 3.58643 \times 10^7\, m$$

Or, since $1.609\, km = 1.0$ *miles, we calculate*

$$h = (3.58643 \times 10^7\, m)\left(\frac{1.0\ miles}{1609\ m}\right) = 22,300\ miles.$$

A satellite placed at this altitude orbits around the earth with the same angular velocity that the earth itself spins. This means such a satellite can maintain communications with a given geographic region on the earth at all times of day and night.

Many communications satellites are in a much lower orbit which takes a considerably smaller amount of time. I believe one such network uses a collection of about 10 or 20 satellites with orbital periods of something like 90 minutes. Let's calculate the altitude for such a satellite.

Example 8.2.4. *Using the analysis of the previous example, with the modification that* $T = 90(60) = 5400\ s$ *we find*

$$h = \sqrt[3]{\frac{GMT^2}{4\pi^2}} - R = 2.807 \times 10^5\, m = 174\ miles.$$

Any satellite with an altitude of less than $2000\ km = 1243.7$ *miles is said to be in* **Low Earth Orbit**. *For example, the USSR's Sputnik-1 was in an orbit of* $133.6mi$ *and the International Space Station is in orbit at an altitude of* $211.3\ mi$. *Given the calculations in this example we can judge the orbit of Sputnik to be less than 90 minutes whereas the orbit of the space station is a bit more than 90 minutes. One last case to study, the moon. The lunar orbit takes* 27.322 *days thus* $T = (27.322)(86400)\ s$ *and*

$$h = \sqrt[3]{\frac{GMT^2}{4\pi^2}} - R = 3.7676 \times 10^8\, m = 234,158\ miles.$$

The moon is about 10 times further out from earth than the altitude for geosynchronous orbit.

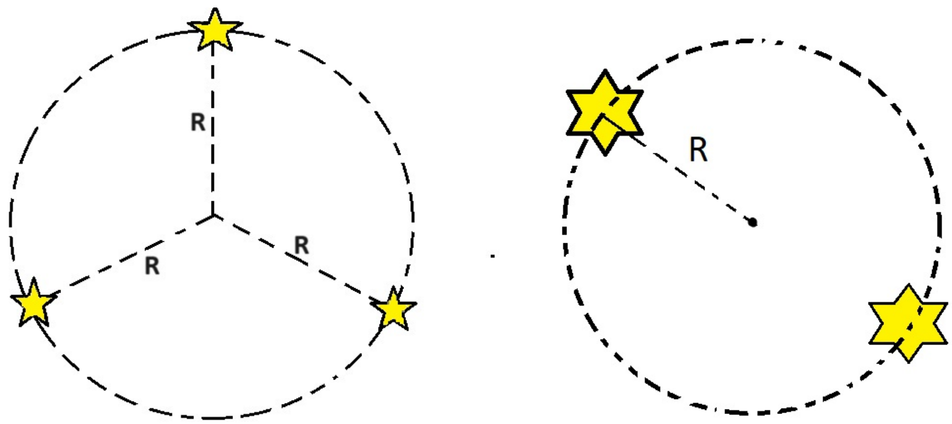
Example 8.2.5. *Suppose a planet* M *has two moons of equal mass* m *which orbit symmetrically in a circular orbit of radius* R *about the planet. The motion of a given moon is the result of the gravitational pull of both the other moon and the central planet. Notice the planet is distance* R *from the moon whereas the other moon is distance* $2R$ *from the given moon hence*

$$\frac{mv^2}{R} = \frac{GmM}{R^2} + \frac{Gmm}{(2R)^2} \quad \Rightarrow \quad v = \frac{\sqrt{G(M + m/4)}}{R}.$$

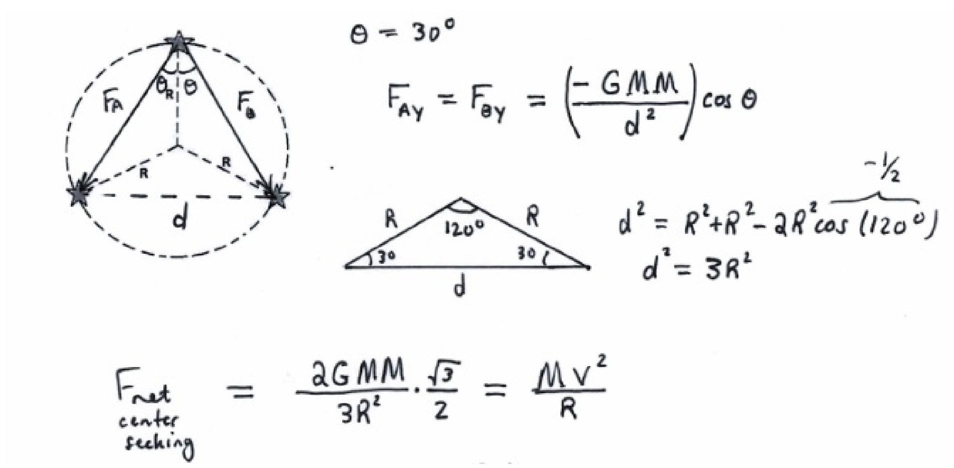
We can also calculate the orbital period of such a moon since $v = 2\pi R/T$ *we find*

$$\frac{2\pi R}{T} = \frac{\sqrt{G(M + m/4)}}{R} \quad \Rightarrow \quad T = \frac{2\pi R^2}{\sqrt{G(M + m/4)}}.$$

Example 8.2.6. *What if we had two or three stars which orbit symmetrically about a central point a distance R. These three stars are in circular motion due to the gravity of the pair which is opposite the given star.*



Can you set-up the equations of motion for such a star ? Some trigonometry is required. Here is how the 3 star problem goes:



Thus $v = \sqrt{\frac{GM}{R\sqrt{3}}}$.

I've asked this as a test question in several terms. Sometimes I ask it with two stars, sometimes three, sometimes four. Once I asked it with n-moons orbiting a planet, but it turns out that is ill-defined unless I modify the problem as to make the mass of the n-th star become suitably small as $n \to \infty$. See https://math.stackexchange.com/q/518390/36530[5]*.*

Remark 8.2.7. *All the examples in this section assume circular motion. Generally orbital motion need not be circular. It can be shown that orbits are either elliptical, parabolic or hyperbolic depending on the energy of the orbiting mass relative to the mass of the gravitating object. We will examine such energy analysis in the next section. To work out the equations of motion in the non-circular case requires analysis beyond the present course. If you are curious, feel free to peruse my derivation of the problem of http://www.supermath.info/centralforcesolution.pdf (on central force motion). This pdf contains a derivation of the claim made in this remark.*

[5]the student who solved part of this is Minh Nguyen who went on to earn a PhD in Mathematics from the University of Arkansas

8.3 gravity as a conservative force

Suppose mass M is at $\vec{r}_o$ then the force of gravity on m at $\vec{r}$ is given by

$$\vec{F}(\vec{r}) = \frac{-GmM}{\|\vec{r}-\vec{r}_o\|^3}(\vec{r}-\vec{r}_o).$$

Or, in Cartesian coordinates,

$$\vec{F}(x,y,z) = \frac{-GmM}{((x-x_o)^2+(y-y_o)^2+(z-z_o)^2)^{3/2}}\langle x-x_o, y-y_o, z-z_o\rangle.$$

Let $\mathscr{r} = \sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}$ and observe $\vec{F}(\vec{r}) = \frac{-GmM}{\mathscr{r}^3}\langle x-x_o, y-y_o, z-z_o\rangle$. Also,

$$\frac{\partial \mathscr{r}}{\partial x} = \frac{x-x_o}{\mathscr{r}}, \qquad \frac{\partial \mathscr{r}}{\partial y} = \frac{y-y_o}{\mathscr{r}}, \qquad \frac{\partial \mathscr{r}}{\partial z} = \frac{z-z_o}{\mathscr{r}}.$$

Likewise, notice:

$$\frac{\partial}{\partial x}\left[\frac{1}{\mathscr{r}}\right] = \frac{-1}{\mathscr{r}^2}\frac{\partial \mathscr{r}}{\partial x} = \frac{-(x-x_o)}{\mathscr{r}^3}, \quad \frac{\partial}{\partial y}\left[\frac{1}{\mathscr{r}}\right] = \frac{-1}{\mathscr{r}^2}\frac{\partial \mathscr{r}}{\partial y} = \frac{-(y-y_o)}{\mathscr{r}^3}, \quad \frac{\partial}{\partial z}\left[\frac{1}{\mathscr{r}}\right] = \frac{-1}{\mathscr{r}^2}\frac{\partial \mathscr{r}}{\partial z} = \frac{-(z-z_o)}{\mathscr{r}^3}$$

Consequently,

$$\nabla\left[\frac{GmM}{\mathscr{r}}\right] = \frac{-GmM}{\mathscr{r}^3}\langle x-x_o, y-y_o, z-z_o\rangle$$

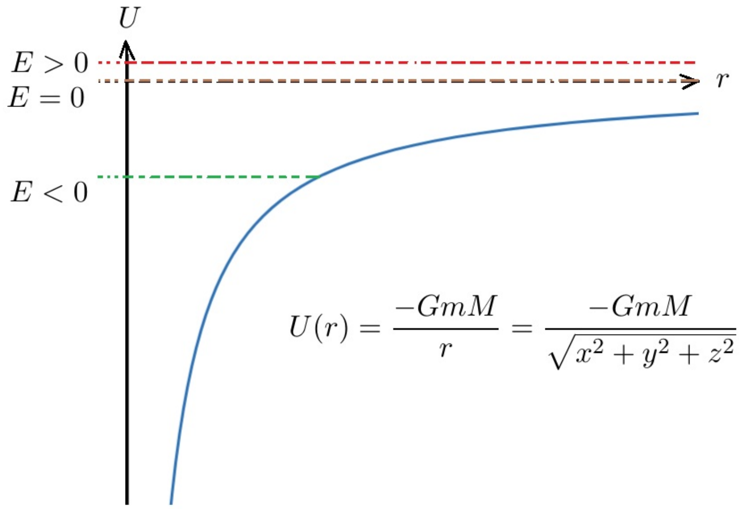
Therefore, $-\nabla\left[\frac{-GmM}{\mathscr{r}}\right] = \vec{F}(\vec{r})$. We find the potential energy due to gravity of M is given by

$$U(x,y,z) = \frac{-GmM}{\mathscr{r}} = \frac{-GmM}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}}.$$

If M is at the origin then the formula simplifies nicely to

$$U(r) = \frac{-GmM}{r} = \frac{-GmM}{\sqrt{x^2+y^2+z^2}}.$$

The convention we choose here is that $U(r) \to 0$ as $r \to \infty$. If gravity serves as the net-force for m then $E(r,v) = \frac{1}{2}mv^2 + U(r)$. If $E > 0$ then the motion is unbounded and such an orbit is known as a **hyperbolic motion**. If $E = 0$ then the motion is **parabolic motion**. Finally, if $E < 0$ then the motion is called an **elliptical orbit**. The diagram below is a little trickier than our earlier work on one-dimensional motion and energy diagrams. Notice, here we cannot see the full motion in the diagram. Notice, for circular motion the trajectory is simply a dot on this diagram. For an elliptical orbit we'd see the motion as traversing a horizontal line-segment which ranges from the orbital radii of closest and furthest approach to M. Likewise, for hyperbolic or parabolic motions, the closest approach corresponds to the highest speed in the motion, but then m never returns after this moment of closest approach.



Notice that for fixed E if we have $r \to 0$ then $U(r) \to -\infty$ and $\frac{1}{2}mv^2 \to \infty$. In other words, as we consider motion that approaches the origin we find trajectories with infinite speed. These are not quite physical because the gravitational force law only applies up to the surface of the gravitating object. Once we get inside M then the force of gravity will reduce since the mass outside that point and the mass inside that point pull in opposing directions. Let's focus on a particular case which is simple enough to calculate.

Consider a planet with constant mass density $\rho = \frac{dM}{dV} = \frac{M}{\frac{4\pi R^3}{3}}$ then $dM = \frac{3M}{4\pi R^3}dV$. If we consider a spherical shell at r with $0 \leq r \leq R$ with thickness dr then $dV = 4\pi r^2 dr$ and hence

$$dM = \frac{3M}{4\pi R^3}4\pi r^2 dr = \frac{3Mr^2 dr}{R^3}$$

The force of gravity from this shell of mass dM on a mass m at distance ℓ from the origin is given by

$$dF = -\frac{GmdM}{\ell^2} = \frac{-3GmMr^2 dr}{R^3 \ell^2}$$

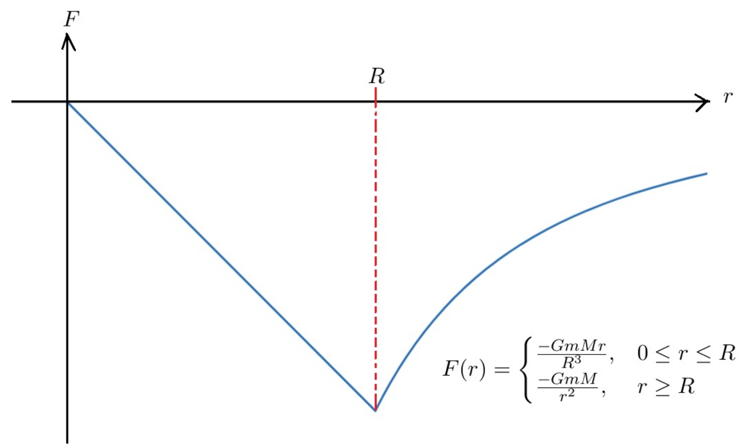
Suppose $\ell \leq R$ then only the mass which is closer to the origin than m gives a net gravitational pull because the mass outside pulls in all directions and it ends up cancelling out[6]. Thus,

$$F_{\text{on } m \text{ at } \ell} = \int dF = \int_0^\ell \frac{-3GmMr^2 dr}{R^3\ell^2} = \frac{-GmMr^3}{R^3\ell^2}\bigg|_0^\ell = \frac{-GmM\ell^3}{R^3\ell^2} = \frac{-GmM\ell}{R^3}.$$

Notice when $\ell = 0$ we get zero force whereas when $\ell = R$ we obtain $\frac{-GmM}{R^2}$. Let us collect these results using r rather than ℓ to characterize the force of gravity a distance r from the origin,

$$F(r) = \begin{cases} \frac{-GmMr}{R^3}, & 0 \leq r \leq R \\ \frac{-GmM}{r^2}, & r \geq R \end{cases} \tag{8.1}$$

[6] I will not prove that given our current technology, there is an easier way once we use superposition and potential theory properly



The potential energy function for the uniform density planet is also a natural extension of the potential energy function for a point mass M. Since the force is in the radial direction we can use integration to derive the potential energy functions from:

$$\frac{dU}{dr} = \frac{GmM}{r^2} \text{ for } r \geq R \qquad \& \qquad \frac{dU}{dr} = \frac{GmMr}{R^3} \text{ for } 0 \leq r \leq R.$$

Integrating,

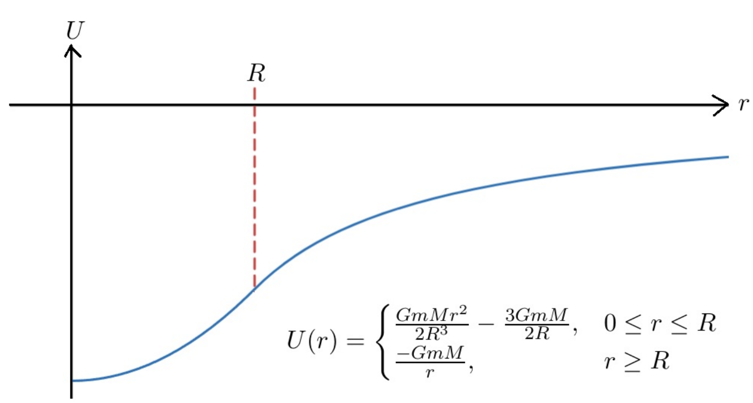
$$U(r) = \frac{-GmM}{r} + C_1 \text{ for } r \geq R \qquad \& \qquad U(r) = \frac{GmMr^2}{2R^3} + C_2 \text{ for } 0 \leq r \leq R.$$

Recall, $U(r) = \frac{-GmM}{r}$ for the potential energy due to the gravitation of M at $r = 0$ acting on the mass m at r hence we should set $C_1 = 0$ in the interest of the formulas being compatible[7]. Having chosen $C_1 = 0$ we use continuity[8] of the potential energy at $r = R$ to select the value for C_2:

$$U(R) = \frac{-GmM}{R} = \frac{GmMR^2}{2R^3} + C_2 = \frac{GmM}{2R} + C_2 \quad \Rightarrow \quad C_2 = \frac{-3GmM}{2R}.$$

We find potential energy for a uniform mass, assuming $U(r) \to 0$ as $r \to \infty$ is given by

$$U(r) = \begin{cases} \frac{GmMr^2}{2R^3} - \frac{3GmM}{2R}, & 0 \leq r \leq R \\ \frac{-GmM}{r}, & r \geq R \end{cases}.$$



[7] we can choose the zero of potential energy however we like, but I prefer this choice for the sake of naturalness

[8] discontinuity would imply a spike in the force at $r = R$, but we know there is no such spike in force at $r = R$

It turns out, the speed of a mass which fell into the center of the earth would not got to infinity since the gravitational force reduces to zero as $r \to 0$. In fact, we could calculate the speed from formulas above, but I will leave that for homework or some other activity.

Let us revisit an example from earlier in the course. In particular, let us study the inclined plane without friction. When we analyzed the motion before we used $U = mgh$ for potential energy due to gravity. That formula is not technically correct in the same way that $F = mg$ is not quite right. Both are approximate formulas based on working in the limit that $r \approx R$ which is to say $h \approx 0$ since the altitude h and the radius of the earth R are related by $r = R + h$. Now that we have the more accurate versions of force and potential energy we may return to our earlier studies and quantify the error in our naive formulas.

Example 8.3.1. Problem: *Suppose a mass m slides without friction down an inclined plane of height H. If the object begins at rest and slides to the base of the incline then what is the final speed v_f of the mass ? Use the unapproximated form of potential energy due to Newtonian gravity.*

Solution: *the given problem has two forces acting on m. Those forces are the normal force and gravity. Since the normal force is perpendicular to the motion it does no work. Thus the work done by the net-force is the work done by gravity and hence energy is conserved. Here*

$$E(y, v) = \frac{1}{2}mv^2 - \frac{GmM}{R+y}$$

where $M = 5.97 \times 10^{24}\,kg$ is the mass of the earth and $R = 6.371 \times 10^6 m$ is the radius of the earth. Then $E_o = E_f$ with $v_o = 0$ and $y_o = H$ whereas $y_f = 0$

$$-\frac{GmM}{R+H} = \frac{1}{2}mv_f^2 - \frac{GmM}{R}$$

Divide by m and multiply by two to obtain:

$$-\frac{2GM}{R+H} = v_f^2 - \frac{2GM}{R} \quad \Rightarrow \quad v_f = \sqrt{2GM\left(\frac{1}{R} - \frac{1}{R+H}\right)} \quad \Rightarrow \quad \boxed{v_f = \sqrt{\frac{2GMH}{R(R+H)}}}.$$

In the case $H << R$ we have that $H/R << 1$ hence series in H/R are very good approximations even just with the first two terms. Notice, if we set $u = H/R$ then

$$v_f^2 = \frac{2GMH}{R(R+H)} = \frac{2GM(H/R)}{R(1+H/R)} = \frac{2GM}{R}\frac{u}{1+u} = \frac{2GM}{R}\left(u - u^2 + u^3 + \cdots\right)$$

using the geometric series with $a = u$ and $r = -u$ where $a + ar + ar^2 + \cdots = \frac{a}{1-r}$ is known from calculus. This is an alternating series hence we find by the alternating series estimation theorem the error is at most as big as the next term not used in the approximation. In particular,

$$v_f^2 = \frac{2GMu}{R} \pm \frac{2GMu^2}{R} = \frac{2GMH}{R^2} \pm \frac{2GMH^2}{R^3}$$

For comparison with our previous work note $GM/R^2 = 9.816 m/s^2 = g$ (up to the crudeness of my typical calculation in this course). In Chapter 5, we would have solved this via $E(v, y) = \frac{1}{2}mv^2 + mgy$ which gives $mgH = \frac{1}{2}mv_f^2$ hence $v_f^2 = 2gH$. What was the error in that solution ? Apparently, *it was at most* $\frac{2GMH^2}{R^3}$. *Admittedly, approximating the square of the velocity is a bit strange, let's work out the velocity directly. Continuing to use* $u = H/R$ *to emphasize this quantity is that which we wish to expand in (since it is tiny). I use the binomial series* $(1+u)^k = 1 + ku + \cdots$ *with* $k = -1/2$,

$$v_f = \sqrt{\frac{2GM}{R}\frac{u}{1+u}} = \sqrt{\frac{2GMu}{R}}(1+u)^{-1/2} = \sqrt{\frac{2GMu}{R}}\left(1 - \frac{1}{2}u + \cdots\right)$$

Therefore,

$$v_f = \sqrt{\frac{2GMH}{R^2}}\left(1 \pm \frac{H}{2R}\right)$$

If we are content with 1% *accuracy then* $H/2R = 0.05$ *is the maximum we can allow. This means* $H = 0.01R = 0.01(6.371 \times 10^6 m) = 6.371 \times 10^4 m = 63.71 km$. *So, our use of* $PE = mgh$ *isn't too bad an approximation as long as we're not studying something like orbital motion with altitudes surpassing say* $100km$.

Let's analyze the usual formulas $PE = mgy$ and $F = mg$ directly. I subtract $\frac{GmM}{R}$ so that $U(0) = 0$ which makes direct comparison to $PE = mgy$ reasonable.

$$U(y) = \frac{GmM}{R+y} - \frac{GmM}{R} = \frac{GmM}{R}\left(\frac{1}{1+y/R}\right) - \frac{GmM}{R} = \frac{GmM}{R}\left(1 + \frac{y}{R} - \frac{y^2}{R^2} + \cdots\right) - \frac{GmM}{R}$$

Therefore, using $g = \frac{GM}{R^2}$,

$$\boxed{U(y) = \frac{GmM}{R}\left(\frac{y}{R} - \frac{y^2}{R^2} + \cdots\right) = m\left(\frac{GM}{R^2}\right)y - \frac{GmMy^2}{R^3} + \cdots = mg\left(y - \frac{y^2}{R} + \cdots\right)}.$$

Using y and ignoring the higher terms is only as reasonable as y^2/R is small. Next, consider force of gravity on m due to the mass of the earth M:

$$F = \frac{-GmM}{(R+y)^2} = \frac{-GmM}{R^2(1+y/R)^2} = -\frac{GmM}{R^2}\left(\frac{1}{1+y/R}\right)^2$$

Notice $\dfrac{1}{1+y/R} = 1 - \dfrac{y}{R} + \cdots$ by the geometric series assuming $|y/R| < 1$. Thus,

$$\begin{aligned} F &= -\frac{GmM}{R^2}\left(1 - \frac{y}{R} + \cdots\right)^2 \\ &= -\frac{GmM}{R^2}\left(1 - \frac{2y}{R} + \cdots\right) \\ &= -m\frac{GM}{R^2}\left(1 - \frac{2y}{R} + \cdots\right) \end{aligned}$$

Thus $\boxed{F = -mg\left(1 - \frac{2y}{R} + \cdots\right)}$ and we find $F = -mg$ is reasonable approximation in as much as $|2y/R|$ is small. We could further improve this analysis by using the more nuanced formula for the force in the case $y < 0$ as we discussed in Equation 8.1.