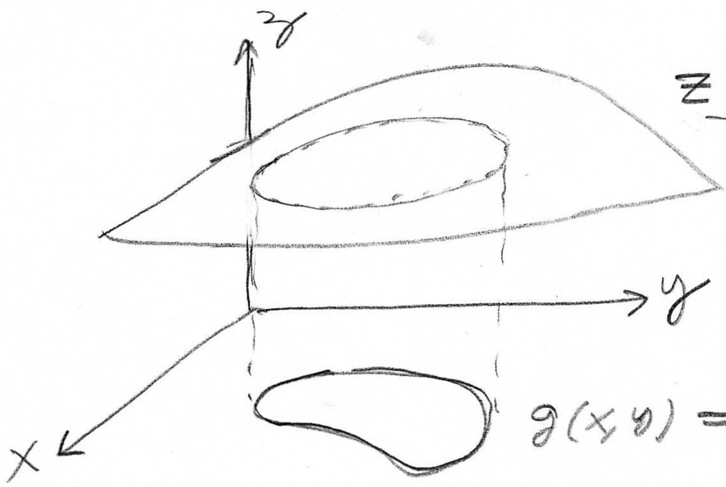
Lagrange Multipliers

① Maximize or Minimize $f(x,y,z)$ subject to $g(x,y,z) = 0$.

② Max/Minimize $f(x,y)$ subject to $g(x,y) = 0$.



$$z = f(x,y)$$

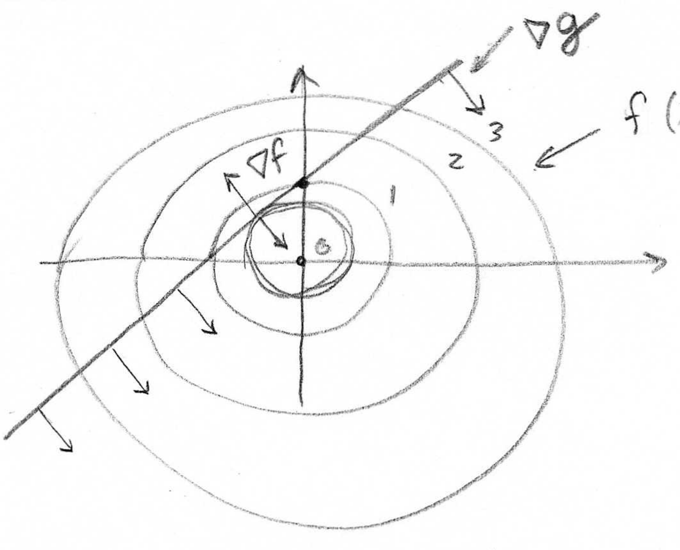
$$f(x,y) - z = 0$$

$$\nabla f - \hat{k} = 0$$

$$\nabla f = \hat{k} \; ?$$

$g(x,y) = 0$

How are ∇g and ∇f related?



$$f(x,y) = \sqrt{x^2+y^2}$$

$$g(x,y) = \underbrace{x - y + 1 = 0}_{y = 1 + x}$$

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle$$

$$\nabla g = \langle 1, -1 \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle = \lambda \langle 1, -1 \rangle$$

$$x = \lambda\sqrt{x^2+y^2}$$
$$y = -\lambda\sqrt{x^2+y^2}$$

$$\frac{x}{y} = -1$$

$$y = -x$$

$$x^2 = \lambda^2(x^2+y^2)$$
$$y^2 = \lambda^2(x^2+y^2)$$
$$x^2+y^2 = 2\lambda^2(x^2+y^2) \Rightarrow 2\lambda^2 = 1$$

$$\lambda = \pm\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} =$$

Eliminate λ

$f(x,y) = x^2 + y^2$ $\qquad$ $\nabla f = \langle 2x, 2y \rangle$

$g(x,y) = x - y + 1$ $\qquad$ $\nabla g = \langle 1, -1 \rangle$

$\nabla f = \lambda \nabla g$

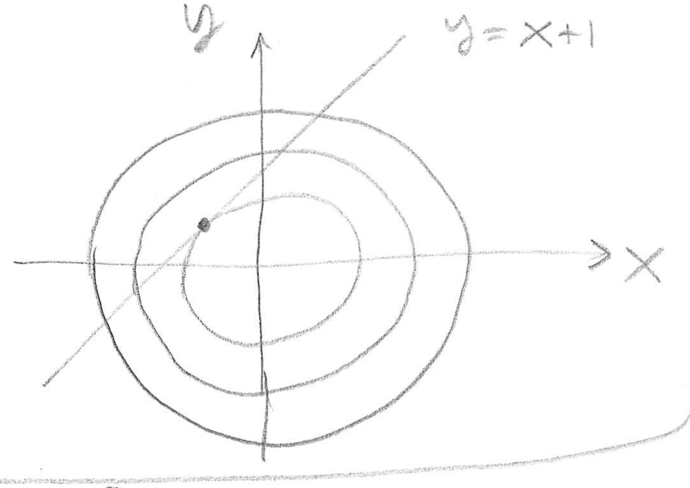
$\langle 2x, 2y \rangle = \lambda \langle 1, -1 \rangle$

$\left.\begin{array}{l} 2x = \lambda \\ +2y = -\lambda \end{array}\right\} \Rightarrow \lambda = 2x = -2y$

$\Rightarrow y = -x$

and $g = 0 = x - y + 1 = 2x + 1 = 0$

$\therefore \underline{x = -1/2}\,,\ \underline{y = 1/2}$



$g(x,y) = x - 1 = 0$

$\nabla g = \langle 1, 0, 0 \rangle$

$z = f(x,y)$ has tangent plane

$z = \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) + f(a,b).$

normal $= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \rangle$

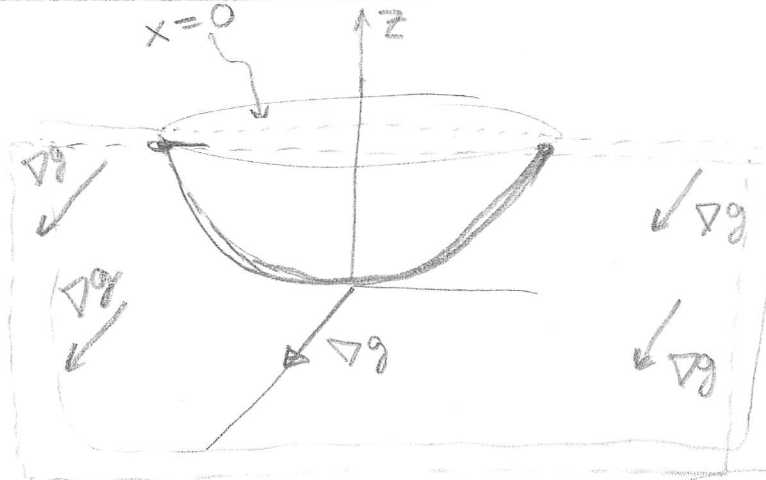
$= \nabla f - \hat{k}.$

$\nabla f = \langle 2x, 2y \rangle$

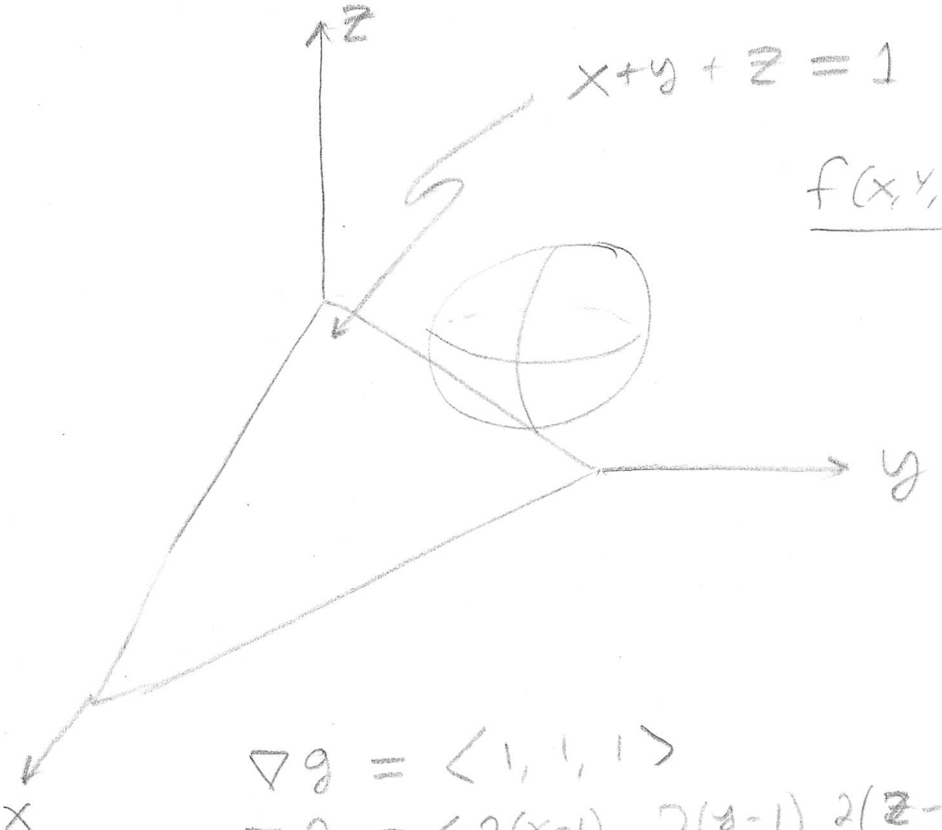
$\nabla f = \lambda \langle 1, 0, 0 \rangle$

$\Rightarrow \lambda = 2x \Rightarrow$

$\therefore 2y = 0 \Rightarrow y = 0$



$\nabla f = \lambda \nabla g \longrightarrow \begin{array}{l} 2x = \lambda \Rightarrow \lambda = 2 \\ 2y = 0 \Rightarrow y = 0 \end{array}$



$x+y+z=1 \rightarrow \underline{g(x,y,z)=x+y+z-1}$

$\underline{f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2}$

Level Surfaces are spheres, centered about $(1,1,1)$

$\nabla g = \langle 1,1,1 \rangle$

$\nabla f = \langle 2(x-1), 2(y-1), 2(z-1) \rangle$

$\nabla f = \lambda \nabla g$

$2(x-1)=\lambda$

$2(y-1)=\lambda$

$2(z-1)=\lambda$

$x-1=y-1 \rightarrow x=y$

$y-1=z-1 \rightarrow y=z$

$x+y+z-1=0 \Rightarrow 3x=1 \therefore \underline{x=y=z=1/3}$.

Thus $(1/3, 1/3, 1/3)$ is the point on the plane $x+z+y=1$ for which $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$ is maximized.