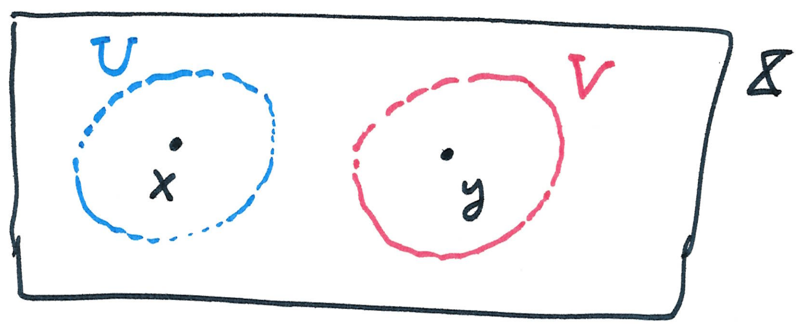
# Lecture 10 : Hausdorff Spaces:

**Def^n/** A topological space is called a Hausdorff, or $T_2$, space if any two distinct points admit disjoint nbhds.

Equivalently, a space is Hausdorff if given $x, y \in X$ with $x \neq y$ we may provide $U, V$ open with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.



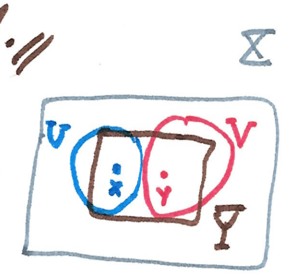

**Remark:** we may separate $x$ & $y$ with open sets $U$ & $V$.

$\boxed{E1}$ If $X \neq \emptyset$ and $\mathcal{T} = \{\emptyset, X\}$ defines the topology on $X$ then if $X$ has 2 or more points then it is not Hausdorff.

$\boxed{E2}$ If $(X, d)$ is metric space then the metric topology on $X$ is Hausdorff. If $d(x,y) > 0$ then when $0 < r < \frac{1}{2} d(x,y)$ we may argue $B(x,r) \cap B(y,r) = \emptyset$. Towards $\rightarrow\leftarrow$ suppose $\exists z \in B(x,r) \cap B(y,r)$ then $d(x,y) \leq d(x,z) + d(z,y) < 2r < d(x,y)$ oops! So no such $z$ exists.

**Lemma**: In a Hausdorff space finite subsets are closed

Proof: we show singletons are closed since that suffices to establish the Lemma. Let $X$ be Hausdorff and suppose $x \in X$. Let $y \in X - \{x\}$ thus $\exists\, U, V$ open with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Observe $x \notin V$ thus $V \subseteq X - \{x\}$ and as $y$ was arbitrary this shows each point in $X - \{x\}$ is an interior point $\Rightarrow X - \{x\}$ is open $\therefore \{x\}$ is closed. //




**Proposition**: Subspaces and products of Hausdorff spaces are Hausdorff

Proof: Suppose $X$ is Hausdorff and let $Y \subseteq X$ be a subspace and $x, y \in Y$ distinct points. Since $X$ Hausdorff, $\exists\, U, V \subseteq X$ open with $x \in U$ and $y \in V$ with $U \cap V = \emptyset$. Observe $U \cap Y$, $V \cap Y$ are open sets in subspace topology of $Y$ with $x \in U \cap Y$, $y \in V \cap Y$ with $(U \cap Y) \cap (V \cap Y) = \emptyset$. Thus $Y$ Hausdorff.

---

Suppose $X, Y$ are Hausdorff with distinct points $(x,y), (z,w) \in X \times Y$ wlog, assume $x \neq z$. Since $X$ Hausdorff, $\exists$ disjoint open $U, V \subseteq X$ s.t. $x \in U$ and $z \in V$. Then $(x,y) \in \underbrace{U \times Y}_{\text{open}}$ & $(z,w) \in \underbrace{V \times Y}_{\text{open}}$ and $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \emptyset \times Y = \emptyset$. //

**Th$^{m}$ (3.69):**

$\boxed{\text{A topological space is Hausdorff iff the diagonal is closed in the product}}$

Proof: The diagonal of $X$ is $\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$.

Suppose $X$ is Hausdorff. Let $(x,y) \in X \times X - \Delta$ then $x \neq y$

$\therefore\ \exists$ open $U, V \subseteq X$ with $x \in U$ and $y \in V$ with $U \cap V = \emptyset$.

Consequently, $(x,y) \in U \times V \subseteq X \times X - \Delta \Rightarrow X \times X - \Delta$ is open

thus $\Delta$ is closed.//

Conversely, if $\Delta$ is closed in $X \times X$ and suppose $x \neq y$ for $x, y \in X$

then as $X \times X - \Delta$ is open and $(x,y) \in X \times X - \Delta$ we find $\exists$ open set $U \times V$ in $X \times X - \Delta$ with

$$(x,y) \in U \times V \subseteq X \times X - \Delta$$

$$\Rightarrow x \in U,\ y \in V \text{ and } U \cap V = \emptyset.$$

$$\therefore\ X \text{ Hausdorff.//}$$

**Corollary:** Let $f, g : X \longrightarrow Y$ be continuous with $Y$ Hausdorff. The coincidence set $C = \{x \in X \mid f(x) = g(x)\}$ is closed in $X$.

**Proof:** consider the map

$$(f, g) : X \longrightarrow Y \times Y$$
$$(f, g)(x) = (f(x), g(x)) \quad \forall x \in X.$$

this map is continuous and $C = (f, g)^{-1}(\Delta)$ where $\Delta = \{(y, y) \mid y \in Y\}$ is thus closed by previous Th. //

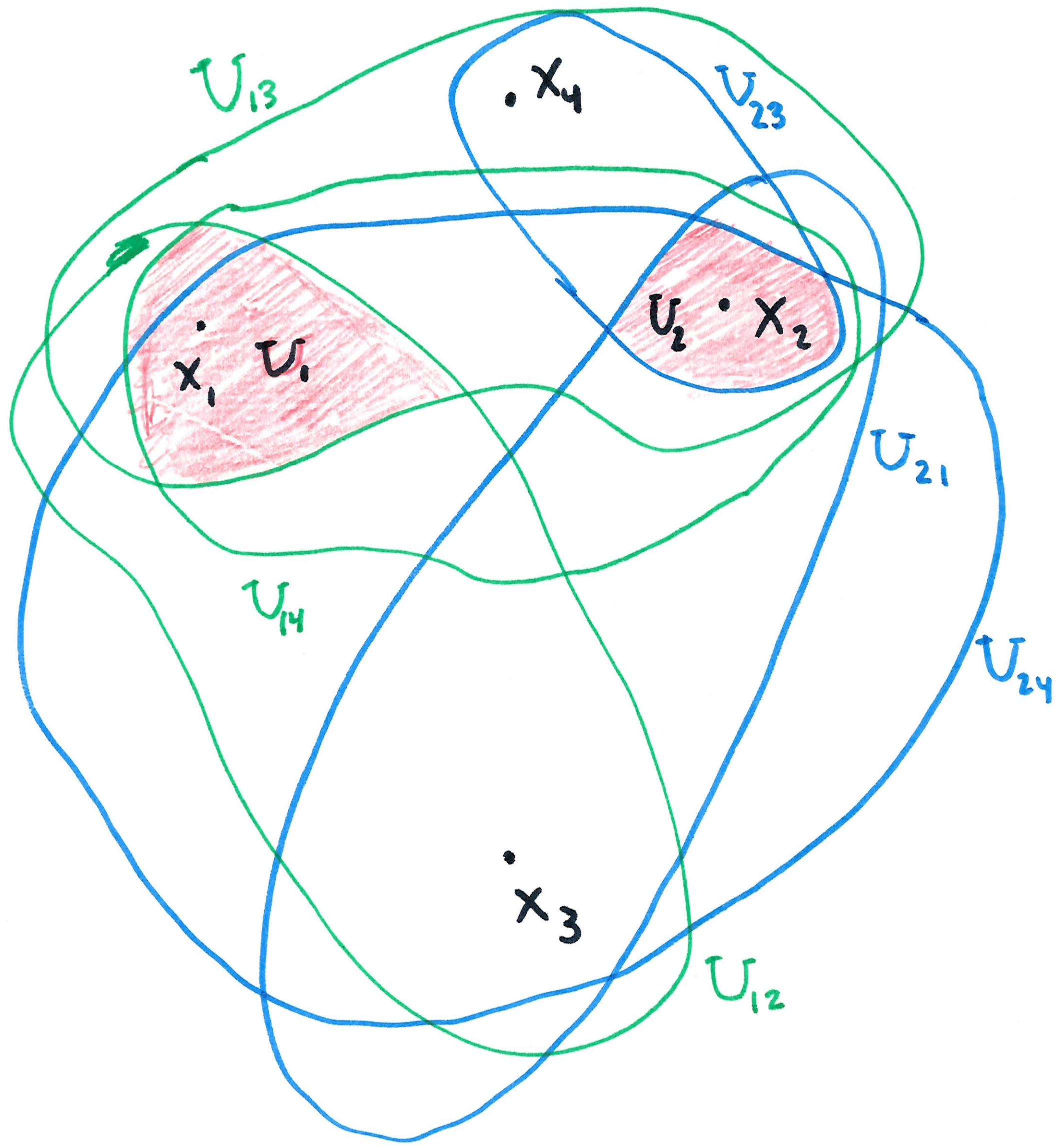
$\boxed{E3}$ Let $f : X \longrightarrow X$ be a continuous map on $X$ then $x \in X$ for which $f(x) = x$ is called a <u>fixed point</u> of $f$.

Notice for $X$ Hausdorff the set of fixed points for a continuous map is $C = \{x \in X \mid f(x) = x = g(x)\}$ $\quad g(x) = Id(x)$ is a coincidence set of $f$ and $Id$ both maps from $X \longrightarrow X$ thus the set of fixed points is <u>closed</u>.

**Lemma:** Let $X$ be Hausdorff space and $x_1, x_2, \dots, x_n$ distinct points in $X$. Then $\exists$ open sets $U_1, \dots, U_n$ in $X$ such that $x_j \in U_j$ $\forall j = 1, 2, \dots, n$ and $U_i \cap U_j = \emptyset$ for every $i \neq j$


$U_i = \bigcap_{i \neq j} U_{ij}$

5




$$U_i = \bigcap_{i \neq j} U_{ij}$$

$$U_1 = U_{12} \cap U_{13} \cap U_{14}$$

$$U_2 = U_{21} \cap U_{23} \cap U_{24}$$

$$U_1 \cap U_2 = \emptyset$$