Lecture 10: Laplace's Equation in Non-Cartesian Coordinates

Let us examine the form of $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ in cylindrical or spherical coordinates

$$\boxed{\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial V}{\partial s}\right] + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}}$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}}$$

I'll begin with spherical case and follow Griffith's advice to presuppose $\frac{\partial V}{\partial \phi} = 0$ (azimuthal symmetry)

problems invariant w.r.t. rotations about the z-axis by ϕ.

We thus face a slightly easier 2D problem of solving: (multiply by r^2 to clean it up)

$$\boxed{\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] = 0}$$

Separation Ansatz: $V(r,\theta) = R(r)\,\Theta(\theta)$

Plug in the $V = R\Theta$, notice $\frac{\partial V}{\partial r} = R'\Theta$ and $\frac{\partial V}{\partial \theta} = R\Theta'$

therefore, we find,

$$\frac{\partial}{\partial r}\left[r^2 R'\Theta\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\, R\,\Theta'\right] = 0$$

$$\frac{\partial}{\partial r}\left[r^2 R'\right]\Theta + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\,\Theta'\right]R = 0$$

$$\therefore \quad \boxed{\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial R}\right] + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial \Theta}{\partial \theta}\right] = 0}$$

We find $V(r,\theta) = R(r)\,\Theta(\theta)$ must solve

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right]}_{\text{function of } r} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right]}_{\text{function of } \theta} = 0$$

$\therefore$ must be <u>constant</u>

We follow Griffiths and set $\frac{1}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] = \ell(\ell+1)$

$$\boxed{\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] = \ell(\ell+1)R} \quad \text{I}$$

CLAIM: $R(r) = Ar^{\ell} + \dfrac{B}{r^{\ell+1}}$ solves I

The above is easy to check, notice it then forces

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] = -\ell(\ell+1) \text{ hence we face,}$$

$$\boxed{\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] = -\ell(\ell+1)\sin\theta\,\Theta} \quad \text{II}$$

The solution to II is less obvious, but it is known,

CLAIM: $\Theta(\theta) = P_\ell(\cos\theta)$ where P_ℓ is the ℓ^{th} order Legendre polynomial defined iteratively by Rodrigues formula

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

Solves II.

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$
$P_3(x) = \frac{1}{2}(5x^3-3x)$
etc.
(see Table 3.1 Griffiths)

Remark: $\exists$ other solutions to II but those blow-up at $\theta = \pi$ etc hence we ignore on physical grounds

Continuing, the potential potential in spherical coordinates for problems with azumthal symmetry is given by

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)}$$

Remark: next time we apply the general spherical solution (assuming azmuthal sym.) to several problems.

CYLINDRICAL COORDINATES

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial V}{\partial s}\right] + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Suppose $V = S\Phi Z$ where $S = S(s)$, $\Phi = \Phi(\phi)$, $Z = Z(z)$ then substituting to the Laplacian gives us

$$\frac{1}{s}\frac{d}{ds}\left[s\frac{dS}{ds}\right]\Phi Z + \frac{1}{s^2}\frac{d^2\Phi}{d\phi^2}SZ + \frac{d^2 Z}{dz^2}S\Phi = 0$$

Then divide by $V = S\Phi Z$ to obtain

$$\boxed{\underbrace{\frac{1}{sS}\frac{d}{ds}\left[s\frac{dS}{ds}\right] + \frac{1}{s^2\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{function of } s \text{ and } \phi \text{ alone}} + \underbrace{\frac{d^2 Z}{dz^2}}_{\text{function of } z \text{ alone}} = 0}$$

$$\therefore \quad \frac{Z''}{Z} = -C \quad \text{and} \quad \underbrace{\frac{1}{sS}\frac{d}{ds}\left[s\frac{dS}{ds}\right] + \frac{1}{s^2\Phi}\frac{d^2\Phi}{d\phi^2} = C}_{\text{to be continued in a future episode}}$$

☺.