LECTURE 11 : EQUIVALENCE PRINCIPLE & MANIFOLD CONCEPT

$\underbrace{\text{WEAK EQUIVALENCE PRINCIPLE}}_{\text{WEP}}$: gravitational mass = inertial mass

$\vec{F} = m_i \vec{a}$: m_i = inertial mass

$\vec{F}_g = -m_g \nabla \Phi$: m_g = gravitational mass

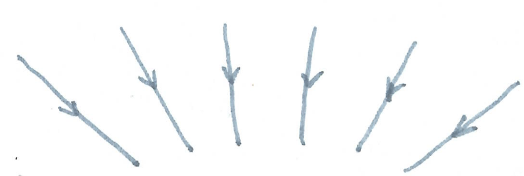
$$\boxed{m_i = m_g}.$$

$$\underbrace{\vec{a} = -\nabla \Phi}$$

acceleration independent of mass of "test" particles, no ± gravitational charge unlike Electrostatics.

WEP REFORMULATED : The motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, in small enough regions of space time.

(too large a region gives tidal forces which correspond to inhomogeneities in the gravitational field)



<u>Einstein Equivalence Principle (EEP)</u> (think about Physicist in a box)

In small enough regions of spacetime, the laws of physics reduce to special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.

EEP $\Rightarrow$ attribute action of gravity to curvature of spacetime

- <u>Special Relativity</u>: measure acceleration of charged particle w.r.t family of inertially related frames of reference.
- EEP: no such thing as "gravitationally neutral object" to judge as benchmark against grav. accel. Hence acceleration due to gravity cannot be reliably defined!
- Def^n/ "unaccelerated" or "freely falling" can be made and particles only under gravity's influence alone will be unaccelerated in this understanding.

  (so gravity is not a force which causes acceleration)

SPECIAL RELATIVITY: can construct inertial frame by arranging clocks and rigid rods in principle...

no longer possible in GR because gravity interferes with the construction of the inertial frame... at best we can hope to construct local coordinates tied to idea of $\underbrace{\text{locally inertial frames}}$

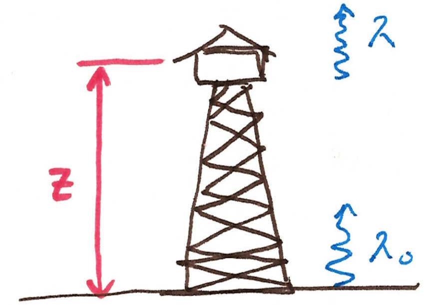
follow motion of individual freely falling particles in small enough regions of spacetime

- relative velocity of far-away objects no longer meaningful (we will not share the same inertial ref. frame most likely)

- All of this leads us to suggest that we should think of spacetime as a differentiable manifold where curvature of the manifold is what gives gravitational physics.

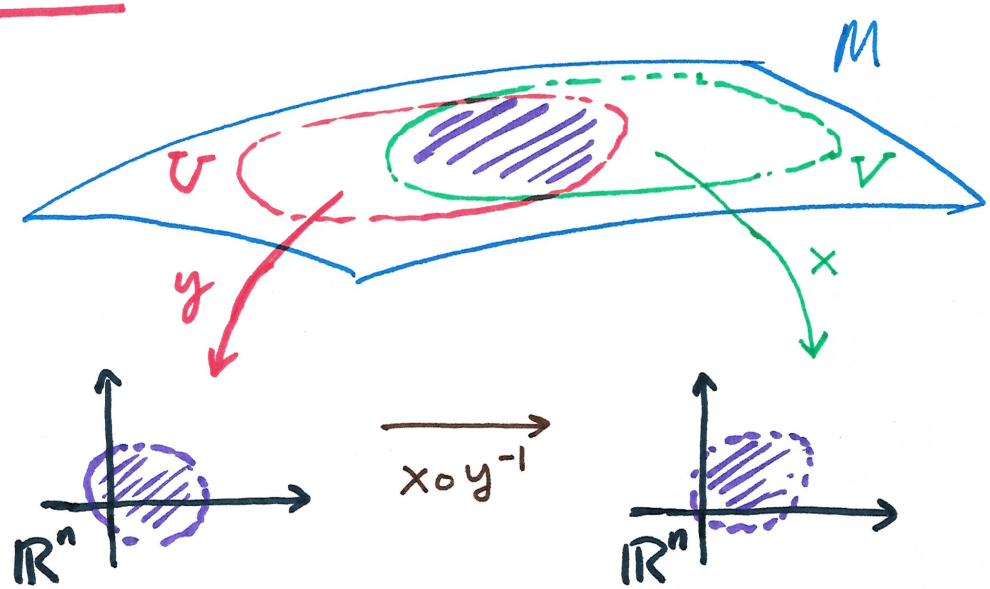
- EEP $\Rightarrow$ GRAVITATIONAL REDSHIFT : $\dfrac{\Delta\lambda}{\lambda_0} = \dfrac{a_g z}{c^2}$

$\Rightarrow$ "clock on the tower appears to run more quickly" $\qquad \dfrac{\Delta\lambda}{\lambda_0} = \Delta\Phi$



MANIFOLDS

• John Lee's books helpful for background.



coordinate charts

$x : V \longrightarrow \mathbb{R}^n$

$y : U \longrightarrow \mathbb{R}^n$

are bijections from open subsets of M to their image in $\mathbb{R}^n$.

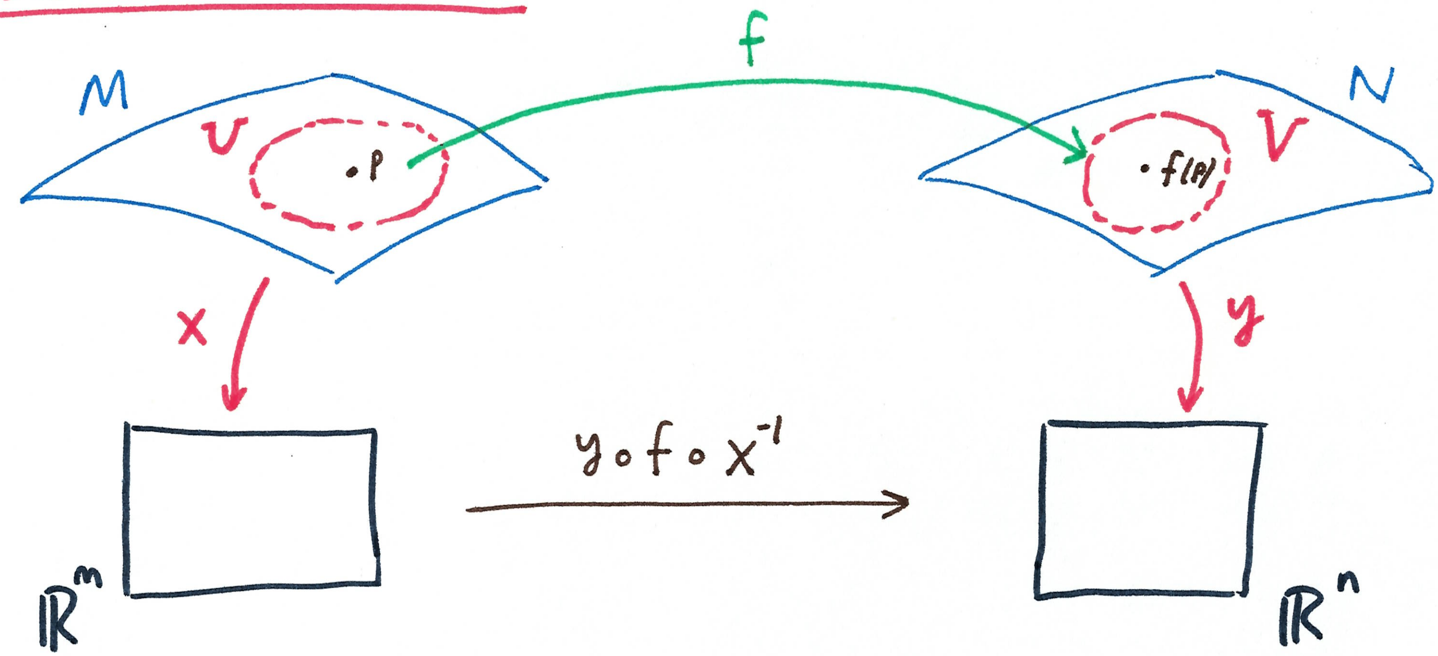
$$x \circ y^{-1} : y(U \cap V) \longrightarrow x(U \cap V)$$

Def^n/ charts (x, V) and (y, U) are compatible if $U \cap V = \emptyset$ or if $U \cap V \neq \emptyset$ and $x \circ y^{-1}$ is a smooth map on $\mathbb{R}^n$.

Def^n/ If M is a set for which $\bigcup_\alpha U_\alpha = M$ and $\{(U_\alpha, x_\alpha) \mid \alpha \in I\} = \mathcal{A}$ is a family of compatible charts then we call $\mathcal{A}$ an atlas for M. The set M paired with an atlas is a manifold.

• Given an atlas on M we can adjoin all possible compatible charts and so obtain a MAXIMAL ATLAS. A set paired with a maximal atlas gives a differentiable structure. For a given set there may exist more than one such structure.

CALCULUS ON MANIFOLDS



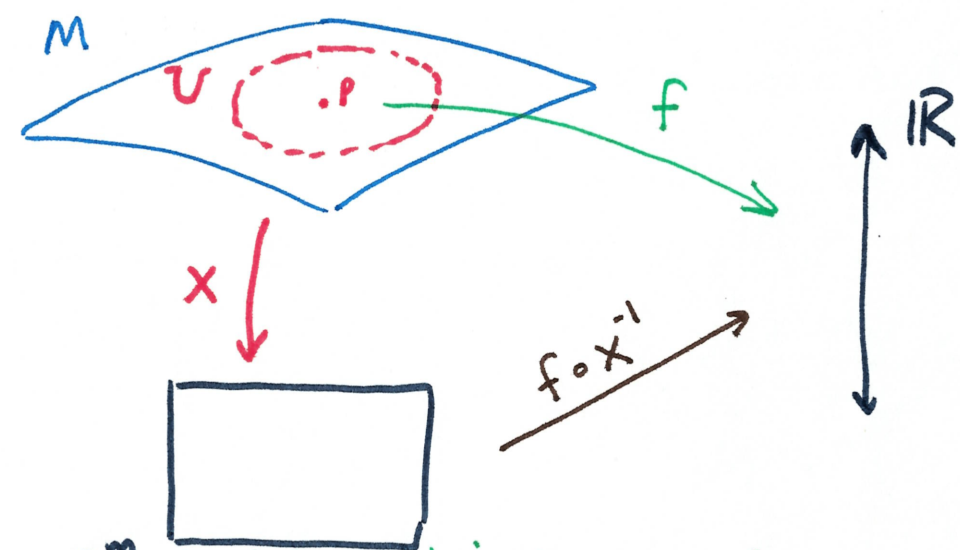
Def^n/ $f: M \longrightarrow N$ is smooth at $p \in M$ if it has a smooth local coordinate representative at p. If f is smooth at each $p \in M$ then f is smooth on M.

- The local coordinate rep. $F = y \circ f \circ x^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is smooth if $J_F = \left[\frac{\partial F}{\partial u_1} \middle| \frac{\partial F}{\partial u_2} \middle| \cdots \middle| \frac{\partial F}{\partial u_m} \right]$ is a matrix of smooth functions from $\mathbb{R}^m \longrightarrow \mathbb{R}$.

- If one local coord. rep. is smooth then any other is likewise smooth at p. Suppose $U \cap \bar{U} \ni p$ and $f(p) \in V \cap \bar{V}$ then

$$\bar{y} \circ f \circ \bar{x}^{-1} = \underbrace{\bar{y} \circ y^{-1}}_{\text{smooth}} \circ \underbrace{y \circ f \circ x^{-1}}_{\text{smooth}} \circ \underbrace{x \circ \bar{x}^{-1}}_{\text{smooth}}$$

CALCULUS ON MANIFOLDS CONTINUED



$\mathbb{R}^m = \{(u^1, u^2, \dots, u^m) \mid u^i \in \mathbb{R}, i = 1, 2, \dots, m\}$

$$\boxed{\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}\left[(f \circ x^{-1})(u^1, \dots, u^m)\right]\Big|_{x(p)}}$$

- The idea is that differentiation on M is implemented via derivatives of the local coord. rep. of the given function. But, $\mathbb{R}$ has global coord. chart so only x^{-1} is needed.

- LIFT CALCULUS on $\mathbb{R}^m$ to CALCULUS ON CURVED space M.

$\boxed{E1}$ $M = \mathbb{R}^{2\times 2}$ and $f : M \longrightarrow \mathbb{R}$ defined by det; $f(A) = \det(A)$.

$$x(A) = (A_{11}, A_{21}, A_{12}, A_{22})$$

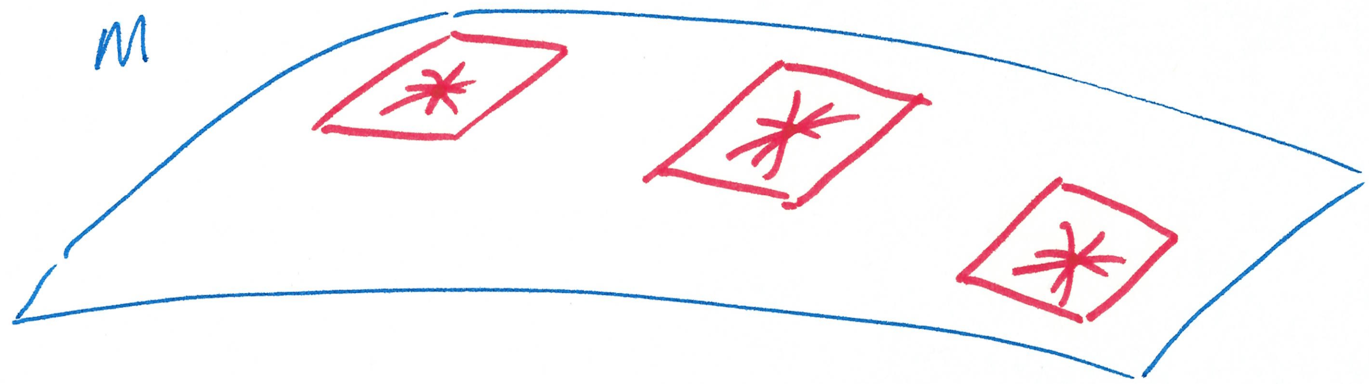
global chart on $\mathbb{R}^{2\times 2}$ given by concatenating columns... well, stacking columns...

$$x^{-1}(u^1, u^2, u^3, u^4) = \begin{bmatrix} u^1 & u^3 \\ u^2 & u^4 \end{bmatrix}$$

$$(f \circ x^{-1})(\vec{u}) = f\begin{bmatrix} u^1 & u^3 \\ u^2 & u^4 \end{bmatrix} = u^1u^4 - u^2u^3$$

$$\frac{\partial f}{\partial x^1}(A) = \frac{\partial}{\partial u^1}\left[u^1u^4 - u^2u^3\right]\Big|_{x(A)} = u^4\Big|_{(A_{11}, A_{21}, A_{12}, A_{22})} = A_{22}$$

CALCULUS ON MANIFOLDS CONTINUED



- we can imagine the tangent space to M at p as a flat-space which approximates M near to p. However, because M is abstract and visualization fails, we instead take a formal approach where T_pM is a set of <u>derivations</u>

$$T_pM = \text{span}\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^{m} = \left\{ \sum_{i=1}^{m} v^i \frac{\partial}{\partial x^i}\Big|_p \;\Big|\; v^i \in \mathbb{R} \right\}$$

If (U, x) and $(\bar{U}, \bar{x})$ are compatible charts at p then the coordinate derivatives are related by the chain-rule,

$$\boxed{\frac{\partial}{\partial x^i}\Big|_p = \sum_{j=1}^{m} \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial}{\partial \bar{x}^j}\Big|_p}$$

Consider for $V \in T_pM$ we may compare components w.r.t. x and $\bar{x}$

$$V = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p = \sum_k \bar{v}^k \frac{\partial}{\partial \bar{x}^k}\Big|_p \neq$$

$$\underbrace{\sum_i \sum_j v^i \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial}{\partial \bar{x}^j}\Big|_p = \sum_k \bar{v}^k \frac{\partial}{\partial \bar{x}^k}\Big|_p}_{\text{act on } \bar{x}^\alpha \text{ to see}}$$

$$\boxed{\bar{v}^\alpha = \sum_i v^i \frac{\partial \bar{x}^\alpha}{\partial x^i}}$$

basis and coord. of vectors transform inversely.

CALCULUS ON MANIFOLDS CONTINUED

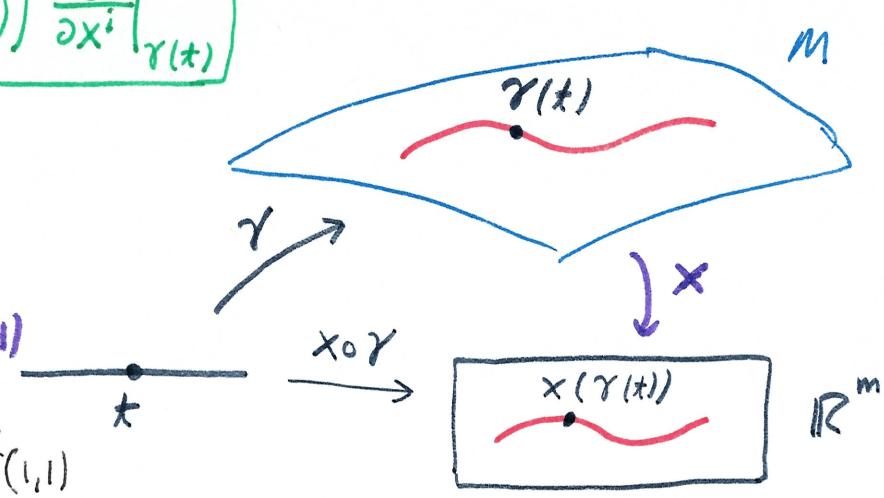
Remark: coordinates on manifolds often resemble coord. on $\mathbb{R}^n$.
For example, $\frac{\partial u^i}{\partial u^j} = \delta_{ij}$ and $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$. Also,

$$\sum_i \frac{\partial \bar{u}^i}{\partial u^j}\frac{\partial u^k}{\partial \bar{u}^i} = \delta_{jk} \quad \& \quad \sum_i \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial x^k}{\partial \bar{x}^i} = \delta_{jk}$$

Def^n/ a vector field on M is a smooth assignment of a vector in T_pM at each $p \in M$. The set of all vector fields on M is denoted $\mathfrak{X}(M)$ then $V \in \mathfrak{X}(M) \Rightarrow V_p \in T_pM$ and if $f : M \to \mathbb{R}$ is smooth function then $V[f]$ is smooth fnct. on M.

We may also consider vector fields along a subset of a manifold. For example, $\gamma : I \subseteq \mathbb{R} \to M$ a path has a tangent vector field defined as follows (suppose $\gamma(I) \subset U$ where (U, x) a chart on M)

$$\boxed{\gamma'(t) = \sum_{i=1}^{m} \frac{d}{dt}\left(x^i(\gamma(t))\right) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}}$$



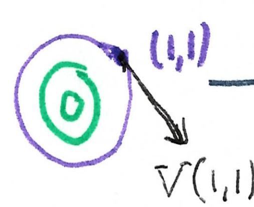
$\boxed{E2}$ Consider $V = x\partial_x - y^2\partial_y \in \mathfrak{X}(\mathbb{R}^3)$

$V[x] = (x\partial_x - y^2\partial_y)(x) = x$

$V[y] = (x\partial_x - y^2\partial_y)(y) = -y^2$

$V[z] = (x\partial_x - y^2\partial_y)(z) = 0$

$V(x^2+y^2) = 2x^2 - 2y^3$



CALCULUS ON MANIFOLDS CONTINUED

$\boxed{\text{Def}^{n}/\ T_pM^* = \{\alpha : T_pM \to \mathbb{R} \mid \alpha \text{ is } \mathbb{R}\text{-linear}\} \text{ is known as the } \underline{\text{cotangent}}\ \underline{\text{space}} \text{ to } M \text{ at } p}$

We define the differential of a function $f : M \to \mathbb{R}$ such that it acts on vectors at a point to give #, or vector fields to yield fnct.

$$\boxed{\text{Def}^{n}/\ df(V) = V[f]}$$

In particular we may study $f = x^i : U \to \mathbb{R}$

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j}[x^i] = \delta_{ij}$$

thus $\underbrace{\{dx^1_p, dx^2_p, \dots, dx^m_p\}}_{\text{basis for } T_pM^*}$ is <u>DUAL</u> <u>BASIS</u> to $\underbrace{\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^m}\Big|_p\right\}}_{\text{basis for } T_pM}$

type (k, l) tensor built from tensor product of $\partial/\partial x^i = \partial_i$ and dx^i

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l}$$

we can look at <u>tensor at a point</u>, or assign a tensor to each point and thus produce a <u>tensor</u> <u>field</u>. Notice ∂_μ identified with double dual,

$$T(dx^{\alpha_1}, \dots, dx^{\alpha_k}, \partial_{\beta_1}, \dots, \partial_{\beta_l}) = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \underbrace{\partial_{\mu_1}(dx^{\alpha_1})}_{dx^{\alpha_1}(\partial_{\mu_1}) = \partial_{\mu_1} x^{\alpha_1} = \delta_{\mu_1 \alpha_1}} \cdots \partial_{\mu_k}(dx^{\alpha_k})\, dx^{\nu_1}(\partial_{\beta_1}) \cdots \underbrace{dx^{\nu_l}(\partial_{\beta_l})}_{\delta_{\nu_l \beta_l}}$$

$$= T^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_l}$$

CALCULUS ON MANIFOLDS CONTINUED

Def^{n}/ A <u>completely</u> <u>antisymmetric</u> <u>tensor</u> is known as a <u>differential form</u>. On an m-dim'l manifold there are nontrivial differential forms of degree $0, 1, 2, \ldots, m$ which we refer to as p-forms:

$$0\text{-form}: \quad f : M \longrightarrow \mathbb{R}$$

$$1\text{-form}: \quad \alpha = \alpha_\mu \, dx^\mu$$

$$2\text{-form}: \quad \beta = \beta_{\mu\nu} \, dx^\mu \otimes dx^\nu = \sum_{\mu<\nu} \beta_{\mu\nu} \, dx^\mu \otimes dx^\nu + \sum_{\mu>\nu} \beta_{\mu\nu} \, dx^\mu \otimes dx^\nu$$

$$= \sum_{\mu<\nu} \beta_{\mu\nu} \, dx^\mu \otimes dx^\nu + \beta_{\nu\mu} \, dx^\nu \otimes dx^\mu$$

$$= \sum_{\mu<\nu} \beta_{\mu\nu} \left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \right)$$

$$= \sum_{\mu<\nu} \beta_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

$$p\text{-form}: \quad \gamma = \gamma_{\mu_1 \cdots \mu_p} \, dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p} = \sum_{I \in \mathcal{I}_p} \gamma_I \, dx^I$$

where I is a multindex
$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$
$I = (i_1, i_2, \ldots, i_p)$ where
$i_1 < i_2 < \cdots < i_p$

$\Omega(M) = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_m$ is the <u>exterior</u> <u>algebra</u> on M and we recall for $\alpha_p \in \Lambda_p$, $\beta_q \in \Lambda_q$

$$\boxed{\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p}$$

Also $d\alpha = \sum_I d\alpha_I \wedge dx^I$ is a $(p+1)$-form if α is p-form. Moreover,

$$\boxed{d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q} \quad \text{and} \quad \boxed{d(d\gamma) = 0}$$

CALCULUS ON MANIFOLDS CONTINUED

Carroll discusses TENSOR DENSITIES on p. 82-84, this is fairly important for formulas which are central to GR. Let's follow along,

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} 1 & \text{if } \mu_1\mu_2,\cdots,\mu_n \text{ is even permutation of } 1,2,\ldots,n \\ -1 & \text{if } \mu_1,\mu_2,\ldots,\mu_n \text{ is odd permutation of } 1,2,\ldots,n \\ 0 & \text{if } \exists i\neq j \text{ for which } \mu_i=\mu_j \text{ (a repeat)} \end{cases}$$

This is a "symbol" and not a tensor, it's a non-tensorial object according to Carroll, However, if we transform it as if it was a tensor, then a determinant appears.

$$\tilde{\epsilon}_{\mu_1'\mu_2'\cdots\mu_n'}\det(M) = \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} M^{\mu_1}{}_{\mu_1'} M^{\mu_2}{}_{\mu_2'}\cdots M^{\mu_n}{}_{\mu_n'} \qquad \text{(Eq. 2.66)}$$

Ok, well, I usually define the determinant by:

$$\det(M) = \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} M^{\mu_1}{}_1 M^{\mu_2}{}_2 \cdots M^{\mu_n}{}_n$$

(completely antisymmetric multilinear function of columns of M for which $\det(I_n)=1$)

I can see how 2.66 follows from by def$^{\underline{n}}$ above. Next Carroll looks at $M = \left[\dfrac{\partial x^\mu}{\partial x^{\mu'}}\right]$ $\quad \det(M^{-1}) = \dfrac{1}{\det M}$

Levi-Civita symbol is tensor density of weight one.

$$\boxed{\tilde{\epsilon}_{\mu_1'\mu_2'\cdots\mu_n'} = \det\left[\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right]\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}}$$

$$\bar{\epsilon}_{\alpha_1\alpha_2\cdots\alpha_n} = \det\left[\frac{\partial \bar{x}}{\partial x}\right]\epsilon_{\mu_1\mu_2\cdots\mu_n}\frac{\partial x^{\mu_1}}{\partial \bar{x}^{\alpha_1}}\frac{\partial x^{\mu_2}}{\partial \bar{x}^{\alpha_2}}\cdots\frac{\partial x^{\mu_n}}{\partial \bar{x}^{\alpha_n}}$$

– (I sometimes like to use — to indicate different frame choice, physicists encode such choice into the primed indices) –

CALCULUS ON MANIFOLDS CONTINUED

We'll discuss the metric g for M in the next lecture.

For now just know $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$ which gives

$$\bar{g}_{\mu\nu} = \underbrace{\frac{\partial x^\alpha}{\partial \bar{x}^\mu}}_{P_{\mu\alpha}} \underbrace{\frac{\partial x^\beta}{\partial \bar{x}^\nu}}_{P^T_{\beta\nu}} g_{\alpha\beta}$$

row column

$$P_{\mu\alpha}\, P^T_{\beta\nu}\, g_{\alpha\beta} = P_{\mu\alpha}\, g_{\alpha\beta}\, P^T_{\beta\nu}$$

$$\bar{g} = P g P^T$$

$$\det(\bar{g}) = \det(P g P^T) = \det(P)\det(g)\det(P^T) = (\det(P))^2 \det(g)$$

$P = \left[\dfrac{\partial x^\alpha}{\partial \bar{x}^\mu}\right]$ then $P^{-1} = \left[\dfrac{\partial \bar{x}^\mu}{\partial x^\alpha}\right]$ and so Carroll writes

$$\det[g_{\mu\nu}] = g(x^\mu)$$

$$\det[g_{\mu'\nu'}] = g(x^{\mu'})$$

$$\det\left[\frac{\partial x^{\mu'}}{\partial x^\mu}\right] = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right|$$

$$(\det(P))^2 = \frac{1}{(\det P^{-1})^2}$$

$$\det P^{-1} = \frac{1}{\det P}$$

$$\boxed{g(x^{\mu'}) = \left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right|^{-2} g(x^\mu)}$$

- determinant of metric is a tensor density of weight -2
- we can use g to make $\tilde{\epsilon}$ an honest tensor

CALCULUS ON MANIFOLDS CONTINUED

Def^n/ The <u>Levi-Civita tensor</u> is $\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$ where $\tilde{\epsilon}_{\mu_1\cdots\mu_n} = \operatorname{sgn}(\mu_1\cdots\mu_n)$ and $|g| = |\det[g_{\mu\nu}]|$ and $g = g_{\mu\nu}\,dx^\mu \otimes dx^\nu$ is the metric tensor on M

Likewise, <u>define</u> (I think what follows is Carroll's intention)

$$\epsilon^{\mu_1\mu_2\cdots\mu_n} = \frac{1}{\sqrt{|g|}}\,\tilde{\epsilon}^{\mu_1\cdots\mu_n}$$

where $\tilde{\epsilon}^{\mu_1\cdots\mu_n} = \operatorname{sgn}(g)\,\tilde{\epsilon}_{\mu_1\cdots\mu_n}$ and $\operatorname{sgn}(g)$ = sign of the determinant of $[g_{\mu\nu}]$

<u>FUN FACT</u>:

$$\epsilon^{\mu_1\mu_2\cdots\mu_p\alpha_1\cdots\alpha_{n-p}}\,\epsilon_{\mu_1\mu_2\cdots\mu_p\beta_1\cdots\beta_{n-p}} = (-1)^s p!\,(n-p)!\,\underbrace{\delta^{[\alpha_1}_{\beta_1}\cdots\delta^{\alpha_{n-p}]}_{\beta_{n-p}}}_{\text{antisymmetrized product of } \delta\text{'s.}}$$

s = # of negative eigenvalues for your metric (we have $s=1$) (Riemannian Geometry, $s=0$)

I've mentioned all of this primarily to include the following:

Def^n/ If $\alpha \in \Lambda_p(M)$ then $*\alpha \in \Lambda_{n-p}(M)$ for M of dimension n. In particular the <u>Hodge</u> <u>Dual</u> of α is $*\alpha$ where

$$(*\alpha)_{\mu_1\mu_2\cdots\mu_{n-p}} = \frac{1}{p!}\,\epsilon^{\nu_1\nu_2\cdots\nu_p}{}_{\mu_1\mu_2\cdots\mu_{n-p}}\,\alpha_{\nu_1\nu_2\cdots\nu_p}$$

$$**\alpha = (-1)^{s+p(n-p)}\alpha$$

CALCULUS ON MANIFOLDS CONCLUSION

Bleeker, Gauge Theory

has Hodge dual

$$d^n x = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1}$$

$$\sqrt{|g|}\, d^n x = \sqrt{|g|} \left(\tilde{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} \right) \frac{1}{n!}$$

$$= \frac{1}{n!}\, \epsilon_{\mu_1 \mu_2 \cdots \mu_n}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}$$

$$= \underbrace{\epsilon_{\mu_1 \mu_2 \cdots \mu_n}\, dx^{\mu_1} \otimes dx^{\mu_2} \otimes \cdots \otimes dx^{\mu_n}}_{\text{invariant volume tensor}} = \epsilon$$

to integrate a scalar function ϕ over manifold we calculate

$$I = \int \phi(x) \sqrt{|g|}\, d^n x$$