LECTURE 11 : Legendre Polynomial Solutions

Begin by recalling the general solution for Laplace's Equation in spherical coordinates with ϕ-invariance,

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)}$$

where P_l is the l-th order Legendre polynomial which we should list a few for reference,

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Legendre polynomials obey an orthonormality-type result,

$$\boxed{\text{Th}^{m}/ \quad \int_0^{\pi} P_l(\cos\theta)\, P_k(\cos\theta) \sin\theta \, d\theta = \frac{2}{2l+1}\,\delta_{l,k}}$$

we can use this to solve for the coefficients A_l or B_l appropriate to a given problem.

E1 Suppose $V_0(\theta)$ is specified at $r = R$. Find the potential in the hollow sphere

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)} \quad \text{I}$$

- We set $B_l = 0$ since $\frac{1}{r^{l+1}}$ blows up at $r = 0$

Apply the BC at $r = R$,

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta) \quad \text{II}$$

$$\int_0^{\pi} \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta = \int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

$$LHS = \sum_{l=0}^{\infty} A_l R^l \underbrace{\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta}_{\frac{2}{2m+1}\delta_{l,m}} = \frac{2A_m R^m}{2m+1}$$

$$\therefore \quad A_m = \frac{2m+1}{2R^m}\int_0^{\pi} V_0(\theta) P_m(\cos\theta)\sin\theta\, d\theta$$

Now choose A_l in I according to above formula and we've solved the Laplace Eq = with give spherical BC $V_0(\theta) = V(R,\theta)$.

Follow-up: we can avoid the integration for some problems if we assume properties of $P_l(\cos\theta)$.

$$V_0(\theta) = k\sin^2(\theta/2) = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$$

Examining II we find $A_0 = \frac{k}{2}$ and $A_1 R = -\frac{k}{2}$

$$\therefore \quad \boxed{V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R}P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)}$$

$\boxed{E2}$ Once more $V_0(\theta)$ specified on $r = R$ but now find potential for $r > R$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

($A_l = 0$ since $r^l \to \infty$ as $r \to \infty$ for $r > R$)

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$$

$$\Rightarrow \quad B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

$\boxed{E3}$ An uncharged metal sphere of radius R placed in uniform (otherwise) field $\vec{E} = E_0 \hat{z}$

$V \sim -E_0 z + C$ for points far away from sphere

BC 1| $V \approx -E_0 r\cos\theta$ for $r >> R$

BC 2| $V = 0$ for $r = R$ (set $V = 0$ on sphere, it's an equipotential since it is a conductor)

We seek A_l, B_l such that,

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)} \quad - (*)$$

fits $\boxed{\text{BC 1}}$ and $\boxed{\text{BC 2}}$

BC 2 $\Rightarrow$ $V(R,\theta) = 0$

$$\Rightarrow \sum_{l=0}^{\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta) = 0$$

$$\Rightarrow A_l R^l + B_l / R^{l+1} = 0$$

E3 continued

From $V=0$ at $r=R$ we obtain $B A_l R^l + \frac{B_l}{R^{l+1}} = 0$ thus

$$B_l = -A_l R^{2l+1} \quad \text{I}$$

Next, study BC1 with $r >> R$ we find $\frac{B_l}{r^{l+1}} \approx 0$ hence we face: (using *)

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \overset{\text{BC 1}}{\approx} -E_0 r \cos\theta \quad \text{II}$$

Recall $P_1(\cos\theta) = \cos\theta$ and then $-E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$ and by LI of Legendre polynomials we equate coefficients in II to deduce,

$rA_1 = -E_0 r$ and $A_l = 0$ for $l \neq 1$

$\underline{A_1 = -E_0}$. $\Rightarrow$ $B_1 = -(-E_0)R^{2(1)+1} = E_0 R^3$ (from I)

$B_l = 0$ for $l \neq 1$.

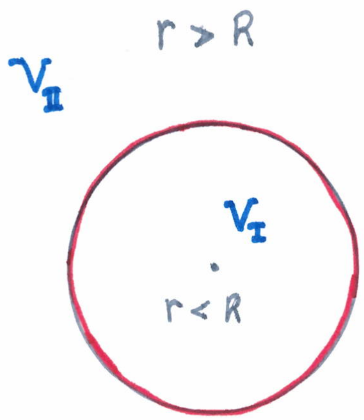
Thus the infinite series reduces to just the $l=1$ term,

$$V(r,\theta) = \left(A_1 r^1 + \frac{B_1}{r^2}\right) P_1(\cos\theta)$$

$$= \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta$$

$$\therefore \boxed{V(r,\theta) = E_0 \cos\theta\left(\frac{R^3}{r^2} - r\right)}$$

E4 glue charge density $\sigma_0(\theta)$ over $r = R$
find potential inside and out. (Example 3.9)



$$\underbrace{V_I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)}_{}$$

$r \leq R$ set $B_l = 0$ since $\frac{1}{r^{l+1}}$ blows-up at $r = 0$

$$\underbrace{V_{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)}_{}$$

$r \geq R$, set $A_l = 0$ since $r^l \longrightarrow \infty$ as $r \longrightarrow \infty$

By continuity of potential at $r = R$ we obtain that

$$A_l R^l = \frac{B_l}{R^{l+1}}$$

$$\therefore \quad \underline{B_l = A_l R^{2l+1}} \quad (*)$$

The localized charge $\sigma_0(\theta)$ at $r = R$ implies a discontinuity in the normal derivative to $r = R$,

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Bigg|_{r=R} = -\frac{1}{\varepsilon_0}\sigma_0(\theta)$$

$V_{out} = V_{II}$

$V_{in} = V_I$

$$\sum_{l=0}^{\infty} \left(\frac{-(l+1)B_l}{R^{l+2}} - l R^{l-1} A_l\right) P_l(\cos\theta) = \frac{-1}{\varepsilon_0}\sigma_0(\theta)$$

$$(*) \Rightarrow \underline{\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\varepsilon_0}\sigma_0(\theta)} \quad **$$

E4 continued

$$\int_0 \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta)\, P_m(\cos\theta) \sin\theta\, d\theta = \int_0 \frac{\sigma_0(\theta)}{\varepsilon_0} P_m(\cos\theta) \sin\theta\, d\theta$$

$$\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} \frac{2}{2\ell+1} \delta_{\ell,m} = \int_0 \frac{\sigma_0}{\varepsilon_0} P_m(\cos\theta) \sin\theta\, d\theta$$

$$2m\ 2A_m R^{m-1} = \int_0 \frac{\sigma_0(\theta)}{\varepsilon_0} P_m(\cos\theta) \sin\theta\, d\theta$$

Therefore, we find solution,

$$\boxed{\begin{gathered} \underline{A_m = \frac{1}{2\varepsilon_0 R^{m-1}} \int_0^{\pi} \sigma_0(\theta) P_m(\cos\theta) \sin\theta\, d\theta} \\[1em] V(r,\theta) = \begin{cases} \displaystyle\sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta) & : r \le R \\ \displaystyle\sum_{m=0}^{\infty} \frac{A_m R^{2m+1}}{r^{m+1}} P_m(\cos\theta) & : r \ge R \end{cases} \end{gathered}}$$

If $\sigma_0(\theta) = k\cos\theta = kP_1(\cos\theta)$ then we can calculate,

$$A_m = \frac{1}{2\varepsilon_0 R^{m-1}} \int_0^{\pi} k\, P_1(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = \frac{2k\,\delta_{m,1}}{2\varepsilon_0 R^{m-1}(2m+1)}$$

$$A_m = \frac{k\,\delta_{m,1}}{\varepsilon_0 R^{m-1}(2m+1)} = \begin{cases} \frac{k}{3\varepsilon_0} & : m = 1 \\ 0 & : m \ne 1 \end{cases}$$

$$\boxed{V(r,\theta) = \begin{cases} \frac{k}{3\varepsilon_0} r\cos\theta & : r \le R \\ \frac{kR^3}{3\varepsilon_0} \frac{1}{r^2} \cos\theta & : r \ge R \end{cases}}$$