LECTURE 12: METRIC & COSMOLOGICAL EXAMPLE

Def^n/ Given a manifold M we say g is a <u>metric</u> on M if g is a smooth assignment of a symmetric $(0,2)$-tensor which is non-degenerate at each $p \in M$.

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

where $g_{\mu\nu} = g_{\nu\mu}$ and $\det[g_{\mu\nu}] \neq 0$. The <u>inverse</u> of the matrix $[g_{\mu\nu}]$ is typically denoted $g^{\mu\nu}$ where $g_{\mu\nu}\, g^{\nu\alpha} = \delta_\mu^\alpha$

The pair (M, g) is a semi-Riemannian manifold

Let $G = [g_{\mu\nu}]$ for discussion's sake. Linear algebra tells us that $\det G = \lambda_1 \lambda_2 \cdots \lambda_n$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are non zero since G^{-1} exists eigenvalues of G. Moreover, since $G^T = G$ the <u>real</u> <u>spectral</u> <u>Th^m</u> provides orthonormal eigenbasis $V_1, V_2, \dots, V_n$ for which $GV_i = \lambda_i V_i$ and $V_i^T V_j = \delta_{ij}$. Then if we rescale the eigenbasis by $1/\sqrt{|\lambda_i|}$ then setting $W_i = \frac{1}{\sqrt{|\lambda_i|}} V_i$

Sylverster's Law of Inertia

we calculate

$$W_i^T G W_j = \frac{1}{\sqrt{|\lambda_i||\lambda_j|}} V_i^T G V_j = \frac{\lambda_i \lambda_j \delta_{ij}}{\sqrt{|\lambda_i||\lambda_j|}} = \begin{cases} 0 & i \neq j \\ \frac{\lambda_i}{|\lambda_i|} & i = j \end{cases}$$

Supposing $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \cdots \leq \lambda_n$ we find

$$W^T G W = \mathrm{Diag}(-1, -1, \dots, -1, 1, \dots, 1)$$

s-fold copies of -1
$(n-s)$-copies of 1

Remark: in our application of semi-Riemannian geometry to GR we study a 4-dim'l manifold with metric of signature $(-+++)$ by which I mean $\exists$ coordinates at each point of the spacetime manifold for which $G_p = \eta = [^{-1}\ {}_1\ {}_1\ {}_1]$. But, generally, we don't expect $G_p = \eta$, and ~~we'll soon see it's not possible to restrict g s.t.~~ ~~$p \mapsto G_p = \eta$ for all p is some nbhd of p. On the other hand,~~ it is possible to select a non-coordinate basis $e_1, e_2, \dots, e_n$ near p for which $g(e_i, e_j) = \eta_{ij}$

Def^n/ If V is a vector field on M defined on $U \subseteq M$ then V is a coordinate vector field on U if $\exists$ chart (U, x) for which $V = \frac{\partial}{\partial x^i}$ for some choice of $i \in \{1, 2, \dots, n\}$. To say $e_1, e_2, \dots, e_n$ is a non-coordinate basis for M on U is to say $\nexists$ chart (U, x) for which $e_i = \frac{\partial}{\partial x^i}$ for $i = 1, 2, \dots, n$. Sometimes a non-coordinate basis is called anholonomic basis.

Induced Metrics on Submanifolds

- And now for a few examples from Riemannian Geometry inside $\mathbb{R}^3$. Notice $g = dx \otimes dx + dy \otimes dy + dz \otimes dz = dx^2 + dy^2 + dz^2$ is the standard Euclidean metric on $\mathbb{R}^3$. It can be converted to cylindrical or spherical coordinates by substituting the coordinate change formulas and taking appropriate differentials.

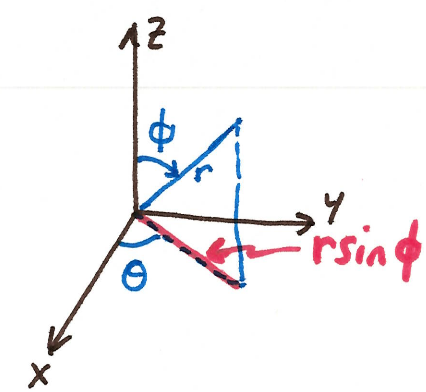
$$\left.\begin{aligned} x &= s\cos\theta \\ y &= s\sin\theta \\ z &= z \end{aligned}\right\} \quad \begin{aligned} dx &= \cos\theta\, ds - s\sin\theta\, d\theta \\ dy &= \sin\theta\, ds + s\cos\theta\, d\theta \end{aligned}$$

$$\begin{aligned} dx \otimes dx + dy \otimes dy &= (\cos\theta\, ds - s\sin\theta\, d\theta) \otimes (\cos\theta\, ds - s\sin\theta\, d\theta) \\ &\quad + (\sin\theta\, ds + s\cos\theta\, d\theta) \otimes (\sin\theta\, ds + s\cos\theta\, d\theta) \\ &= (\cos^2\theta + \sin^2\theta)\, ds \otimes ds + s^2(\sin^2\theta + \cos^2\theta)\, d\theta \otimes d\theta \\ &= ds \otimes ds + s^2\, d\theta \otimes d\theta \end{aligned}$$

$$\begin{aligned} g &= dx \otimes dx + dy \otimes dy + dz \otimes dz \\ &= ds \otimes ds + s^2\, d\theta \otimes d\theta + dz \otimes dz \\ &= \underbrace{dr \otimes dr + r^2\, d\phi \otimes d\phi + r^2 \sin^2\phi\, d\theta \otimes d\theta} \end{aligned}$$

$$\left.\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned}\right\}$$

I guess this is "math" with $\rho = r$



$\boxed{E1}$ Consider $M \subset \mathbb{R}^3$ given by $z = z_o$ then (x, y) serve as coordinate chart on M and $dz = dz_o = 0$ thus,

$$h = g/_M = \underline{dx \otimes dx + dy \otimes dy}.$$

thus $h = dx^2 + dy^2$ serves as the <u>induced</u> metric on M.

$\boxed{E2}$ Consider $M \subset \mathbb{R}^3$ given by $a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$ then $a\,dx + b\,dy + c\,dz = 0$ on M and if $a \neq 0$ then $dx = \frac{-b}{a}dy - \frac{c}{a}dz$ and we may substitute into g for $\mathbb{R}^3$

$$h = g/_M = \left(\frac{-b}{a}dy - \frac{c}{a}dz\right) \otimes \left(\frac{-b}{a}dy - \frac{c}{a}dz\right) + dy \otimes dy + dz \otimes dz$$

$$= \underline{\left(\frac{b^2}{a^2} + 1\right) dy \otimes dy + \left(\frac{c^2}{a^2} + 1\right) dz \otimes dz - \frac{bc}{a^2}\left(dy \otimes dz + dz \otimes dy\right)}.$$

$\boxed{E3}$ Consider $M \subset \mathbb{R}^3$ given by $s = s_o$, this is a cylinder $ds = ds_o = 0$

$$h = g/_M = \underline{s_o^2\, d\theta \otimes d\theta + dz \otimes dz}.$$

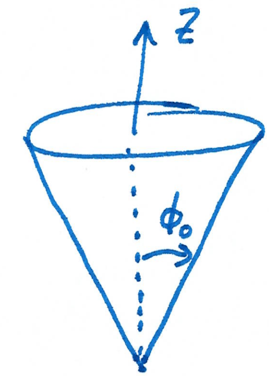
$\boxed{M = C_{s_o}}$

Cylinder of radius $s_o > 0$.

E4 cone described by $\phi = \phi_o$ is almost a submanifold M

Since $d\phi = d\phi_o = 0$ we substitute into g in sphericals to find

$$h = g|_M = dr \otimes dr + r^2 \sin^2\phi_o \, d\theta \otimes d\theta$$



$r = 0$ is singular point for M

Notice in the $\{\partial/\partial r, \partial/\partial\theta\}$ coordinate frame we find

$$[h] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \sin^2\phi_o \end{bmatrix}$$

Note $\det[h] = r^2 \sin^2\phi_o = 0$ for $r = 0$

Def^{n}/ a space which is a manifold, except at a finite collection of singular points is known as an orbifold.

E5 The sphere of radius $r = R_o$ has $dr = dR_o = 0$ thus from the g written in spherical coordinates,

$$h = g|_M = R_o^2 \, d\phi \otimes d\phi + R_o^2 \sin^2\phi \, d\theta \otimes d\theta = R_o^2 \, d^2\Omega$$

CONSTRUCTING NEW MANIFOLDS FROM OLD

Def^n/ Suppose (M, g_1) and (N, g_2) are semi-Riemannian manifolds of dimensions m and n respective with metrics g_1 and g_2 respective then $M \times N$ is a manifold with metric $g = g_1 \oplus g_2$ meaning

$$g\big((V_1, V_2), (W_1, W_2)\big) = g_1(V_1, W_1) + g_2(V_2, W_2)$$

If (U_1, x_1) is chart on M and (U_2, x_2) is chart on N then $z : U_1 \times U_2 \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$ is given by $z(p_1, p_2) = (x_1(p_1), x_2(p_2))$

The construction above allows us to combine two 2-dim'l manifolds to create a 4-dim'l manifold. Let me introduce 2-dim'l spacetime

$\boxed{E6}$ $M_2 = \mathbb{R}^2$ with global chart $\varphi = (t, x)$ and the Minkowski metric

$$g(V^0\partial_t + V^1\partial_x, W^0\partial_t + W^1\partial_x) = -V^0W^0 + V^1W^1. \qquad g = -dt \otimes dt + dx \otimes dx.$$

the basis $\{\partial_t, \partial_x\}$ is a coordinate basis and $[g] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ in this basis.

$\boxed{E7}$ $M_2 \times S_{R_0} \subset \mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^5$ however $M_2 \times S_{R_0}$ is 4-dim'l with

$$g = -dt \otimes dt + dx \otimes dx + R_0^2\, d\phi \otimes d\phi + R_0^2 \sin^2\phi\, d\theta \otimes d\theta. \qquad G = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & R_0^2 & 0 \\ 0 & 0 & 0 & R_0^2\sin^2\phi \end{array}\right]$$

$\boxed{E8}$ $\underbrace{M_2 \times C_{S_0}}_{4\text{-dim'l}} \subset \mathbb{R}^5$ with $g = -dt \otimes dt + dx \otimes dx + S_0^2\, d\theta \otimes d\theta + dz \otimes dz.$

Expanding Universe Toy Models

If we imagine a universe which is Euclidean spatially for a moment in time, but expands then given scale factor $a(t)$ we study metric:

$$\boxed{ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]}$$

special case of Robertson Walker metric $0 < t < \infty$

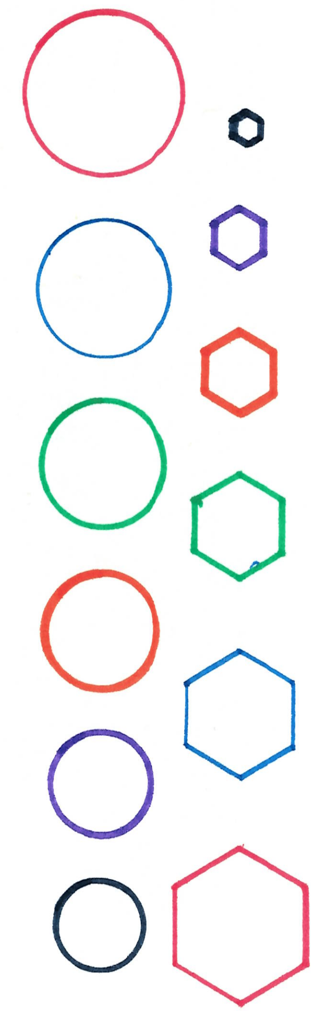
Less concisely,

$$\boxed{g = -dt \otimes dt + a^2(t)\left[dx \otimes dx + dy \otimes dy + dz \otimes dz\right]}$$

The submanifold of spacetime $t = t_0$ has a Euclidean metric $g|_{t=t_0} = a^2(t_0)\left[dx^2 + dy^2 + dz^2\right]$. A path $\varphi(\lambda)$ for which $(t, x, y, z)(\varphi(\lambda)) = (\lambda, x_0, y_0, z_0)$ is said to be <u>comoving</u>. If we have two comoving worldlines $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$ with comoving coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) then

$$d_t = \left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right) a(t)$$

describes the distance between $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$ at time t. I think in this context we can set $\lambda = t$.



Typical scale factor has form $a(t) = t^q$ for some $0 < q < 1$

matter dominated $q = 2/3$

radiation dominated $q = 1/2$

$0 < t < \infty$

"BIG BANG"

Studying Light Cones in Expanding Universe Model

We consider $ds^2 = -dt^2 + t^{2q}(dx^2+dy^2+dz^2)$ and search for null-paths where $ds^2 = 0$. Focus attention on case $y = y_0$, $z = z_0$ hence $dy = 0$ and $dz = 0$ hence,

$$0 = -dt^2 + t^{2q}dx^2$$

$$\Rightarrow \frac{dx^2}{dt^2} = \frac{-1}{t^{2q}}$$

$$\Rightarrow \frac{dx}{dt} = \pm t^{-q}$$

$$\therefore \int dx = \pm \int t^{-q}dt$$

$$\Rightarrow \pm \int_{x_0}^{x_1} dx = \int_{t_0}^{t_1} t^{-q}dt$$

$$\pm(x_1 - x_0) = \left.\frac{t^{1-q}}{1-q}\right|_{t_0}^{t_1} = \frac{t_1^{1-q} - t_0^{1-q}}{1-q}$$

Consider path $x = x(\lambda)$ has tangent field

$$V = \left(\frac{dx^\mu}{d\lambda}\right)\frac{\partial}{\partial x^\mu}$$

$$dt^2(V,V) = (dt \otimes dt)(V,V) = dt(V)\,dt(V)$$

Where we have

$$dt(V) = dt\left(\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}\right) = \frac{dx^\mu}{d\lambda}\frac{\partial t}{\partial x^\mu} = \frac{dt}{d\lambda}$$

Likewise

$$dx^2(V,V) = dx(V)\,dx(V) = \left(\frac{dx}{d\lambda}\right)^2$$

Thus

$$0 = -\left(\frac{dt}{d\lambda}\right)^2 + t^{2q}\left(\frac{dx}{d\lambda}\right)^2$$

Then $\frac{dx}{dt} = \frac{dx}{d\lambda}\frac{d\lambda}{dt}$

$$\frac{dx}{d\lambda} = \frac{dx}{dt}\frac{1}{d\lambda/dt}$$

$$\Rightarrow \frac{dx}{dt} = \pm t^{-q}$$

Setting $x_1 = x$ and $t_1 = t$ and... gulp... $t_0 = 0$

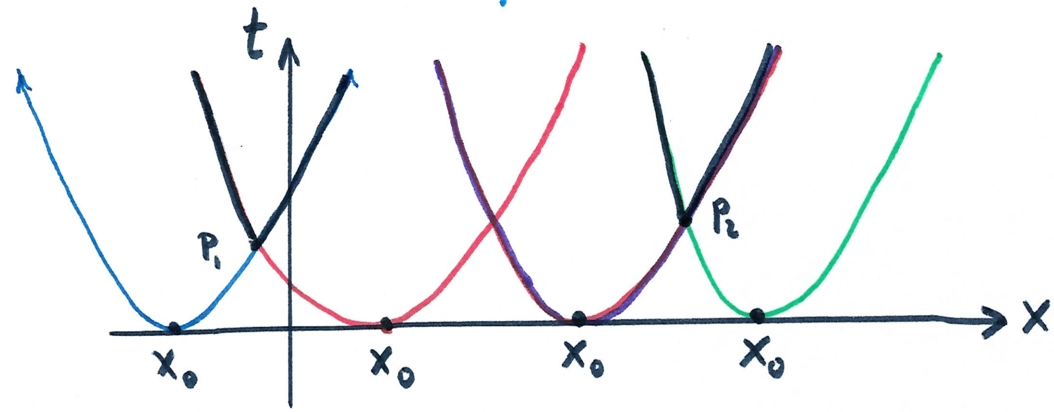
$$\pm(x - x_0) = \frac{1}{1-q}\left(t^{1-q}\right)$$

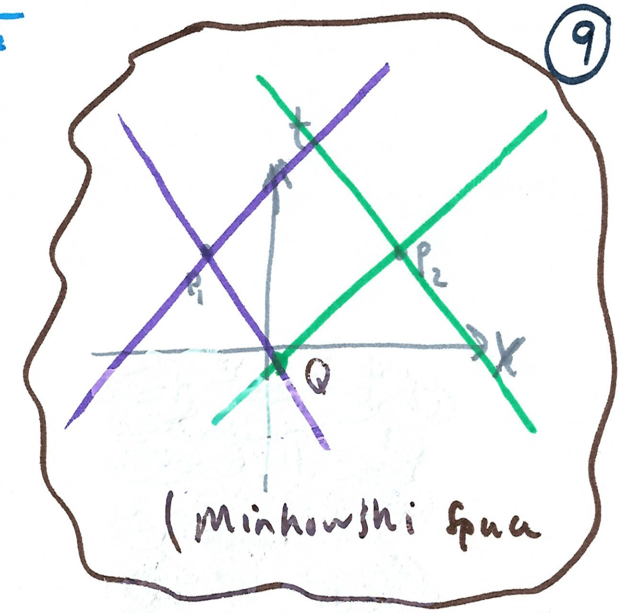
$$t^{1-q} = \pm(1-q)(x - x_0) \quad \therefore \quad \boxed{t = \pm(1-q)^{\frac{1}{1-q}}(x - x_0)^{\frac{1}{1-q}}}$$

Set $q = 1/2$ then $t = \pm \left(\frac{1}{2}\right)^{\frac{1}{1-1/2}} (x - x_0)^{\frac{1}{1-1/2}}$

$$t = \pm \frac{1}{4}(x - x_0)^2$$

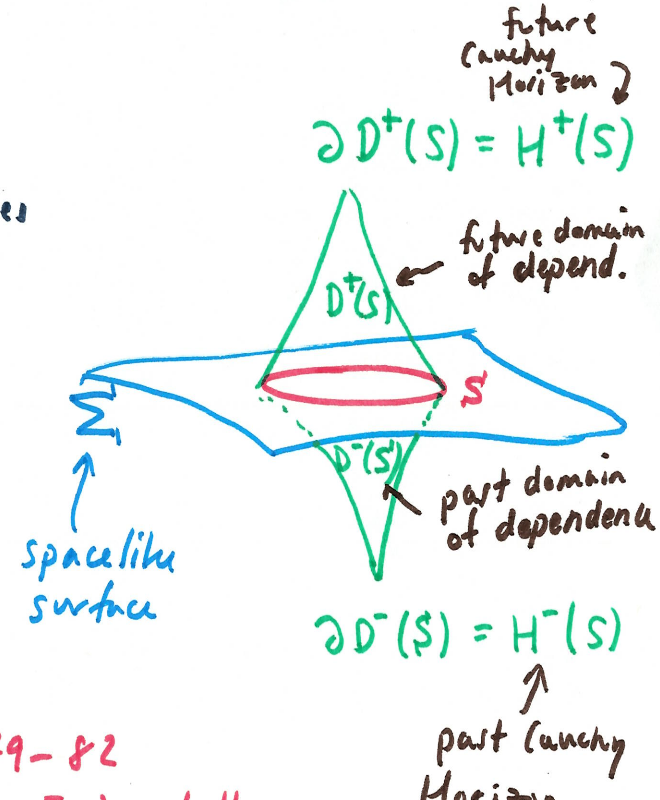
$$t = \frac{1}{4}(x - x_0)^2$$





Light cones of P_1 & P_2 do not meet in the past. This does not happen in Minkowski space, light cones of separate events will intersect...

Def^n/ events such as P_1 & P_2 are out of CAUSAL CONTACT because their past light cones do not intersect.



Remark: Eq^{n}s 2.63 & 2.64 provide example spacetime exhibiting closed timelike curves. See pgs. 79–82 of Carroll for some discussion or watch Interstellar.

DEGREES OF FREEDOM PLAUSIBILITY ARGUMENT FOR LOCALLY INERTIAL COORD.

Locally inertial coordinates at p have $g_{\hat{\mu}\hat{\nu}}(P) = \eta_{\hat{\mu}\hat{\nu}}$ & $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(P) = 0$ these look like flat space to first order, Chapter 3 explains how to construct such coord., for now we argue they're plausible, assume $\bar{x}^{\mu}(P) = 0 = x^{\mu}(P)$ in what follows to reduce some clutter

$$\bar{g}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} g_{\alpha\beta}$$

$$x^{\mu} = \left(\frac{\partial x^{\mu}}{\partial \bar{x}^{\alpha}}\right)_p \bar{x}^{\alpha} + \frac{1}{2}\left(\frac{\partial^2 x^{\mu}}{\partial \bar{x}^{\alpha} \partial \bar{x}^{\rho}}\right)_p \bar{x}^{\alpha}\bar{x}^{\rho} + \frac{1}{3!}\left(\frac{\partial^3 x^{\mu}}{\partial \bar{x}^{\alpha}\partial \bar{x}^{\rho}\partial \bar{x}^{\gamma}}\right)_p \bar{x}^{\alpha}\bar{x}^{\rho}\bar{x}^{\gamma} + \dots$$

"Schematically" to 2nd order,

$$\bar{g}_p + (\bar{\partial}\bar{g}_p)\bar{x} + (\bar{\partial}\bar{\partial}\bar{g})_p \bar{x}\bar{x} = \left(\frac{\partial x}{\partial \bar{x}}\frac{\partial x}{\partial \bar{x}} g\right)_p + \left(\frac{\partial x}{\partial \bar{x}}\frac{\partial^2 x}{\partial \bar{x}\partial \bar{x}} g + \frac{\partial x}{\partial \bar{x}}\frac{\partial x}{\partial \bar{x}} \bar{\partial} g\right)_p \bar{x}$$
$$+ \left(\frac{\partial x}{\partial \bar{x}}\frac{\partial^3 x}{\partial \bar{x}\partial \bar{x}\partial \bar{x}} g + \frac{\partial^2 x}{\partial \bar{x}\partial \bar{x}}\frac{\partial^2 x}{\partial \bar{x}\partial \bar{x}} g + \frac{\partial x}{\partial \bar{x}}\frac{\partial^2 x}{\partial \bar{x}\partial \bar{x}} \bar{\partial} g + \frac{\partial x}{\partial \bar{x}}\frac{\partial x}{\partial \bar{x}} \bar{\partial}\bar{\partial} g\right)_p \bar{x}\bar{x}$$

Then we can equate coefficients of $\bar{x}$ of equal order

ORDER 0 : $\bar{g}_{\mu\nu} = \bar{g}_{\nu\mu} \rightarrow 10$ components, matches $\frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}}$: 4x4 matrix, no constraints

So, enough to put $\bar{g}_{\mu\nu}$ into order with 6 d.o.f. left over, these are the d.o.f. for the Lorentz Group (these leave $\bar{g}_{\mu\nu}$ unchanged)...

ORDER 1 : $\bar{\partial}_{\alpha} \bar{g}_{\mu\nu} \rightarrow 4(10) = 40$ components, but from RHS get to chose $\frac{\partial^2 x^{\mu}}{\partial \bar{x}^{\alpha}\partial \bar{x}^{\rho}}$ which also gives 40 d.o.f.

$\Rightarrow$ can set $\bar{\partial}\bar{g} = 0$ at P.

ORDER 2 : $\bar{\partial}_{\alpha}\bar{\partial}_{\rho}\bar{g}_{\mu\nu} \rightarrow 10(10) = 100$ d.o.f,

RHS, what's new?, well $\left(\frac{\partial^3 x^{\mu}}{\partial \bar{x}^{\alpha}\partial \bar{x}^{\rho}\partial \bar{x}^{\gamma}}\right)_p \Rightarrow 4(20) = 80$ d.o.f.

$\therefore$ cannot set $\bar{\partial}\bar{\partial}\bar{g} = 0$

Remark : the missing 20 d.o.f which prevent setting $\bar{\partial}\bar{\partial}\bar{g} = 0$ are accounted for in Riemann tensor