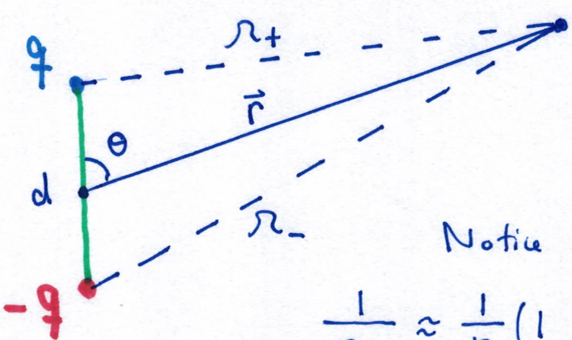
Lecture 12: Multipole Expansion

In mechanics we learn a collection of masses can be regarded as a single mass at the center of mass and instead of thinking about many m_i's we instead just think of M. In some sense, perhaps the multipole expansion is like that. We're ultimately able to replace an extended charge distribution over space with a single charge at the origin if we think of the far field observation... however, this isn't quite it, we also need to throw in dipole, octopole and higher multipoles at the origin. Of these the dipole is most important (past the of course far more central monopole which we have already studied in some depth; monopole is Coulomb field)

$\boxed{\text{E1}}$ Physical electric dipole consists of q and $-q$ separated by distance d. Let's examine the potential for these if we are far away.



$$\mathscr{r}_{\pm}^2 = r^2 + (d/2)^2 \mp rd\cos\theta$$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right)$$

$r \gg d$,

Notice $\mathscr{r}_{\pm}^2 = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right) \approx r^2\left(1 \mp \frac{d}{r}\cos\theta\right)$

$$\frac{1}{\mathscr{r}_{\pm}} \approx \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} = \frac{1}{r}(1+u)^{-1/2} \qquad u = \mp\frac{d}{r}\cos\theta$$

Binomial Series, $(1+u)^{-1/2} \approx 1 - \frac{1}{2}u$ for $u \approx 0$. Then,

$$\frac{1}{\mathscr{r}_{\pm}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \Rightarrow \frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \approx \frac{d}{r^2}\cos\theta$$

E1 continued

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-}\right) \approx \frac{q\, d \cos\theta}{4\pi\varepsilon_0 r^2}$$

Compare to Coulomb (monopole) potential $V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}$ and see the dipole drops off like $\frac{1}{r^2}$ and there is a directionality as evidenced by $\cos\theta$

monopole $V \sim \frac{1}{r}$

dipole $V \sim \frac{1}{r^2}$

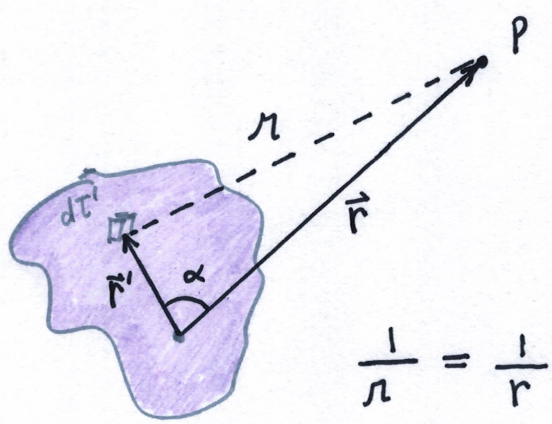
quadropole $V \sim \frac{1}{r^3}$

octopole $V \sim \frac{1}{r^4}$

Let's work on deriving the multipole expansion. We begin with the potential derived via integrating the density $\rho = \frac{dq}{d\tau}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')\, d\tau'$$

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$



$$\mathscr{r} = r\sqrt{1+\varepsilon}$$

$$\varepsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2}$$ binomial series

$$= \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right)$$

Continuing, use the binomial expansion for $\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \cdots\right)$ to obtain using $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$ that,

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \cdots\right]$$

$$= \frac{1}{r}\left[P_0(\cos\alpha) + \left(\frac{r'}{r}\right)P_1(\cos\alpha) + \left(\frac{r'}{r}\right)^2 P_2(\cos\alpha) + \left(\frac{r'}{r}\right)^3 P_3(\cos\alpha) + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

Consequently,

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\,\rho(\vec{r}')\,d\tau'}$$

Expanding this explicitly to name names,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\Bigg[\frac{1}{r}\underbrace{\int \rho(\vec{r}')\,d\tau'}_{\text{monopole}} + \frac{1}{r^2}\underbrace{\int r'\cos\alpha\,\rho(\vec{r}')\,d\tau'}_{\text{dipole}}$$

$$+ \frac{1}{r^3}\underbrace{\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\vec{r}')\,d\tau'}_{\text{quadrapole}} + \cdots\Bigg]$$

(this is the multipole expansion of the potential)

$$V_{mon}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \text{where } \boxed{Q = \int \rho\, d\tau}$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{where } \underbrace{\boxed{\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'}}_{\substack{\text{dipole moment} \\ \text{(continuous case)}}}$$

For collection of point charges, $q_1, .., q_n$ at $\vec{r}_1\,', .., \vec{r}_n\,'$

$$\boxed{\vec{p} = \sum_{i=1}^{n} q_i\,\vec{r}_i\,'}$$

Remark: physical dipole has q and $-q$ separated by $\vec{d}$

then $\boxed{\vec{p} = q\vec{r}_+\,' - q\vec{r}_-\,' = q(\vec{r}_+\,' - \vec{r}_-) = q\vec{d}}$

However, the "ideal dipole" has $d \to 0$ and $q \to \infty$ s.t. $qd = p$ held fixed.

MULTIPOLE EXPANSION AND ORIGIN OF COORDINATES

Point charge q at origin is a pure monopole, however, if q moves away from origin then its multipole expansion includes dipoles etc.

- MONOPOLE MOMENT Q IS INDEPENDENT OF COORDINATES
- DIPOLE MOMENT IS SOMETIMES INDEPENDENT OF CHOICE OF ORIGIN

$$\vec{\bar{p}} = \int \vec{\bar{r}}' \rho(\vec{r}')\,d\tau'$$

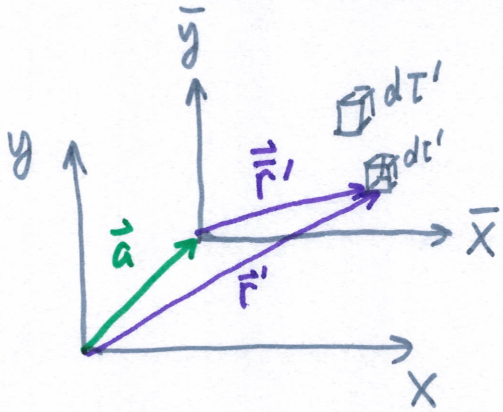
$$= \int (\vec{r}' - \vec{a})\,\rho(\vec{r}')\,d\tau'$$

$$= \int \vec{r}' \rho(\vec{r}')\,d\tau' - \vec{a}\int \rho(\vec{r}')\,d\tau'$$

$$= \vec{p} - Q\vec{a}$$

$\longrightarrow$ $Q = 0$ gives $\vec{\bar{p}} = \vec{p}$

dipole moment in $(\bar{x},\bar{y},\bar{z})$ — dipole moment in (x,y,z)

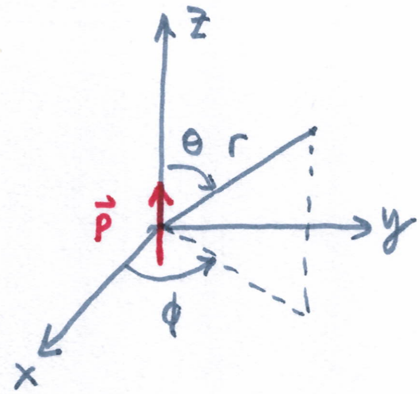


$$\vec{r}' = \vec{a} + \vec{\bar{r}}'$$

Electric Field of a Dipole

$$V_{dip}(r,\theta) = \frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \qquad (\vec{p} = p\hat{z})$$

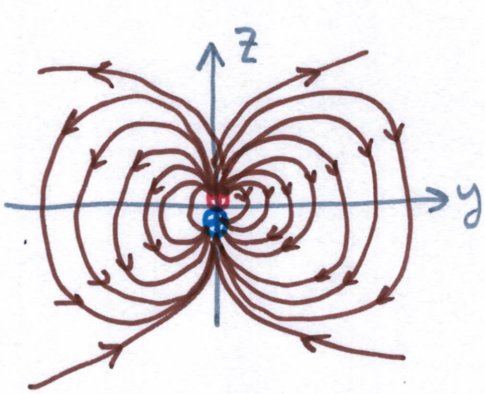
$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

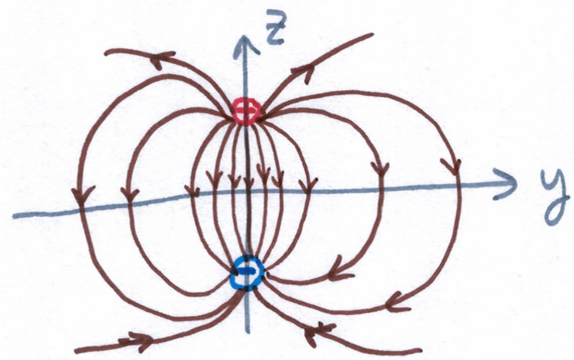


$$E_\theta = \frac{-1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\phi = \frac{-1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\boxed{\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$





Remark: far away they look the same.

$$\boxed{\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

coordinate invariant expression for $\vec{E}$ due to dipole $\vec{p}$ at origin.
(perfect)