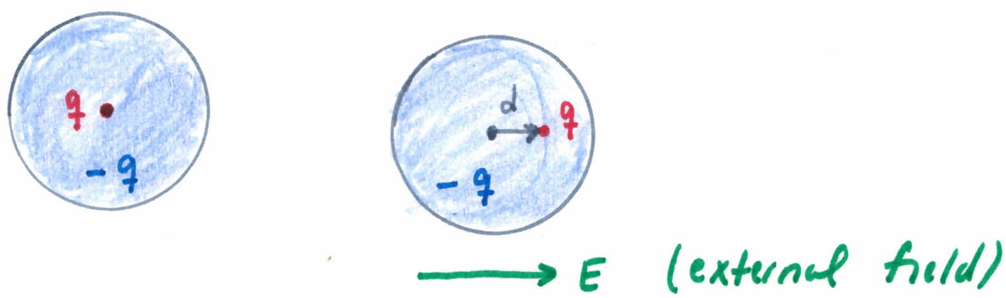
LECTURE 13: ELECTROSTATICS IN MATTER

Insulators have positive and negative charge. When subjected to E-field these charges are drawn apart and the result is the creation of a <u>dipole</u>

Def^{n}/ $\vec{p} = \alpha \vec{E}$, α = atomic polarizability

E1 Atom crudely consists of nucleus with positive charge q and an electron cloud with charge $-q$. For E not too big the e-cloud remains spherical



$$E_e = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3}(-\hat{x})$$

(electric field due to uniform e-cloud at point where q is positioned distance d from cloud center)

$$E = \frac{qd}{4\pi\varepsilon_0 a^3}$$

$$p = qd = (4\pi\varepsilon_0 a^3)E$$

$$\underbrace{\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0 \mathcal{V}}_{\text{GRIFFITHS CLAIMS ACCURATE TO WITHIN FACTOR OF 4 FOR MANY SIMPLE ATOMS.}}$$

$$\underbrace{\mathcal{V}}_{\text{volume of atom}} = \frac{4}{3}\pi a^3$$

What about a material comprised of a collection of polarized atoms or molecules? We can consider these aligned by some external field $\vec{E}$ and so define

$$\boxed{\underbrace{\vec{P}}_{\text{polarization}} \overset{\text{def}^n}{=\!=} \text{dipole moment per unit-volume}}$$

We should be careful, there are different fields to consider

(1.) the field which caused the polarization of the material; that is the field created $\vec{P}$.

(2.) the field created by the polarized material, that is the field due to $\vec{P}$.

Recall $V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}\cdot\hat{\imath}}{\imath^2}$ is potential for dipole.

Consider $\vec{P} = \dfrac{d\vec{p}}{d\tau} \Rightarrow d\vec{p} = \vec{P}\,d\tau$ and thus,

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\vec{P}(\vec{r}\,')\cdot\hat{\imath}}{\imath^2}\,d\tau'}$$

<u>CLAIM</u>: If $\nabla' = \hat{x}'\dfrac{\partial}{\partial x'} + \hat{y}'\dfrac{\partial}{\partial y'} + \hat{z}'\dfrac{\partial}{\partial z'}$ and $\vec{\imath} = \vec{r} - \vec{r}\,'$

then $\nabla'\left(\dfrac{1}{\imath}\right) = \dfrac{\hat{\imath}}{\imath^2}$

<u>Proof sketch</u>:

$$\imath^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 = \sum_{i=1}^{3}(x_i'-x_i)^2$$

$$2\imath\frac{\partial \imath}{\partial x_j'} = 2(x_j'-x_j) \quad \therefore \quad \frac{\partial \imath}{\partial x_j'} = \frac{x_j'-x_j}{\imath}$$

$$\nabla'\left(\frac{1}{\imath}\right) = \frac{-1}{\imath^2}\nabla'(\imath) \Rightarrow \nabla'\left(\frac{1}{\imath}\right) = \frac{\hat{\imath}}{\imath^2}$$

Continuing to study potential for $\vec{P}$ given over some volume $\mathcal{V}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \vec{P}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

$\nabla'\cdot(f\vec{A}) = \nabla' f\cdot\vec{A} + f(\nabla'\cdot\vec{A})$

Product rule

$f = \frac{1}{\mathscr{r}}$

$\vec{A} = \vec{P}$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_{\mathcal{V}}\left[\nabla'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}(\nabla'\cdot\vec{P})\right]d\tau'\right.$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\nabla'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right)d\tau' - \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{\nabla'\cdot\vec{P}}{\mathscr{r}}\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint_{\partial\mathcal{V}}\frac{1}{\mathscr{r}}\vec{P}\cdot d\vec{a}\,' - \int_{\mathcal{V}}\frac{1}{\mathscr{r}}(\nabla'\cdot\vec{P})\,d\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\oint_{\partial\mathcal{V}}\frac{1}{\mathscr{r}}\underbrace{(\vec{P}\cdot\hat{n})}_{\sigma_{bound}=\sigma_b}da' - \int_{\mathcal{V}}\frac{1}{\mathscr{r}}\underbrace{(\nabla'\cdot\vec{P})}_{\rho_{bound}=\rho_b}d\tau'\right]$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0}\left[\oint_{\partial\mathcal{V}}\frac{\sigma_b}{\mathscr{r}}\,da'\right] - \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}}\frac{\rho_b}{\mathscr{r}}\,d\tau'}$$

Def^n/ $\sigma_b = \vec{P}\cdot\hat{n}$ and $\rho_b = \nabla\cdot\vec{P}$ give the bound charge on the surface and in the volume with polarization $\vec{P}$

Thm/ to find potential due to given polarization $\vec{P}$ over volume $\mathcal{V}$ we can calculate the bound surface charge $\sigma_b = \vec{P} \cdot \hat{n}$ and the bound volume charge ρ_b given by $\nabla \cdot \vec{P} = \rho_b$ then proceed as before to write potential due to given charge distribution.

E2 Find E-field produced by uniformly polarized sphere of radius R. Let's set $\vec{P} = P\hat{z}$ for convenience where P is a <u>constant</u> then,

$$\sigma_b = (P\hat{z}) \cdot \hat{n} = P\hat{z} \cdot \hat{r} = P\cos\theta$$

$$\rho_b = \nabla \cdot \vec{P} = \nabla \cdot (P\hat{z}) = 0$$

Recall Lecture 11, **E4** we found potential for $\sigma_0(\theta) = k\cos\theta$ hence setting $k = P$ we find,

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\varepsilon_0} r\cos\theta & \text{for } r \leq R \\ \dfrac{P}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta & \text{for } r \geq R \end{cases}$$

① For $r \leq R$ we have $V = \frac{P}{3\varepsilon_0} z$ thus $\vec{E} = -\nabla V$ gives

$$\boxed{\vec{E} = \frac{-P}{3\varepsilon_0}\hat{z} \quad \text{for } r < R}$$

② For $r \geq R$ we have $V = \frac{PR^3}{3\varepsilon_0}\frac{\cos\theta}{r^2} = \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$

it is as if there was a single perfect dipole at origin with dipole moment $\rightarrow \vec{p} = \frac{4}{3}\pi R^3 \vec{P} = \frac{4}{3}\pi R^3 P\hat{z}$

<u>Remark</u>: see pg. 177–178 for a heuristic method to derive the results here. It's pretty slick.