Lecture 14: Electric Displacement & Linear Dielectrics

In matter we generally expect there are two types of charge; bound charge which is tied to the polarization and free charge which is not tied to a particular location in the matter. The total charge density

$$\boxed{\rho = \rho_{bound} + \rho_{free}} \qquad (\rho = \rho_b + \rho_f)$$

The total charge density relates to the $\vec{E}$-field via Gauss' Law,

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$

polarization

$$\Rightarrow \varepsilon_0(\nabla \cdot \vec{E}) = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f$$

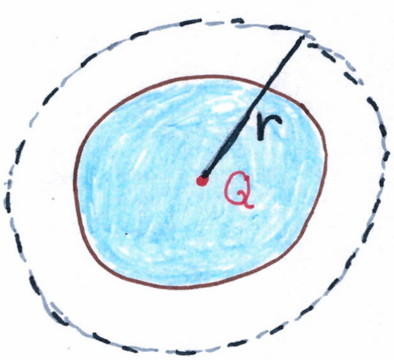
$$\therefore \quad \nabla \cdot (\underbrace{\varepsilon_0 \vec{E} + \vec{P}}) = \rho_f$$

$$\boxed{\text{Def}^{n}/ \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}} \quad \text{(Electric Displacement)}$$

Gauss' Law for Electric Displacement

$$\boxed{\nabla \cdot \vec{D} = \rho_f}$$

$\boxed{E1}$ Consider point charge at center of rubber sphere radius R. The only free charge is Q at origin. By spherical symmetry



$$\int_{B_r} (\nabla \cdot \vec{D})\, d\tau = \int_{B_r} \rho_f \, d\tau$$

$$\int_{\partial B_r} \vec{D} \cdot d\vec{a} = 4\pi r^2 D = Q$$

$$\vec{D} = \left(\frac{Q}{4\pi r^2}\right)\hat{r} \Rightarrow \overbrace{\varepsilon_0 \vec{E} = \left(\frac{Q}{4\pi r^2}\right)\hat{r}}^{\vec{P} = 0 \text{ for } r > R} \quad \text{for } r > R$$

$$\therefore \quad \vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \quad \text{outside sphere}$$

LINEAR DIELECTRICS

For many materials the polarization is proportional to the field $\vec{E}$, we define:

$$\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}}$$

Constitutive relation for LINEAR DIELECTRIC

χ_e is the electric susceptibility

The electric displacement $\vec{D}$ is then proportional to $\vec{E}$,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$$

$$\boxed{\text{Def}^{\underline{n}}/ \; \underbrace{\varepsilon = \varepsilon_0 (1 + \chi_e)}_{\text{permitivity}}}$$

Remark: empty space has $\varepsilon = \varepsilon_0$ and $\chi_e = 0$. This is why we call ε_0 the permitivity of free space.

$$\nabla \cdot \vec{D} = \rho_f$$

$$\int_{\mathcal{V}} (\nabla \cdot \vec{D})\, d\tau = \int_{\mathcal{V}} \rho_f \, d\tau$$

$$\parallel$$

$$\int_{\partial \mathcal{V}} \vec{D} \cdot d\vec{a} = Q_f \text{ (enclosed by } \mathcal{V})$$

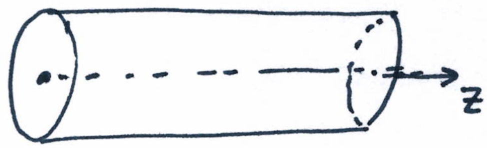
$$\therefore \boxed{\int_{\partial \mathcal{V}} \vec{D} \cdot d\vec{a} = Q_{f,enc}}$$

$$\nabla \times \vec{D} = \nabla \times (\varepsilon \vec{E}) = \varepsilon \nabla \times \vec{E} = 0$$

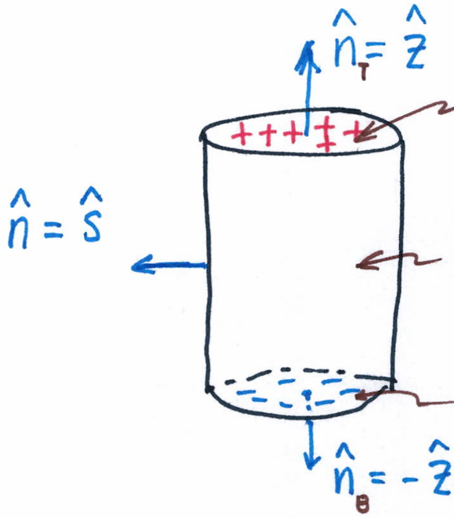
$\therefore$ $\vec{D}$ is conservative vector field in a linear dielectric

BAR ELECTRET (PROBLEM 4.11)

$\boxed{E2}$



$\vec{P} = P_0 \hat{z}$ for $0 \le z \le L$
$0 \le s \le R$



$\sigma_b^T = \vec{P} \cdot \vec{A}_T = P_0 \pi R^2$

$\sigma_b^{side} = \int \vec{P} \cdot d\vec{a} = 0$ since $\hat{s} \cdot \hat{z} = 0.$

$\sigma_b^B = \vec{P} \cdot \vec{A}_B = -P_0 \pi R^2$

Notice $\rho_b = -\nabla \cdot \vec{P} = 0$ so there is only bound surface charge, no bound volume charge for the bar electret.

Remark: apparently $\nabla \times \vec{P} \neq 0$ somewhere for the bar electret, I suspect it has to do with the hidden step function,

$$\vec{P} = P_0 \left(\Theta(z) - L \right)$$

$$\vec{P} = \underbrace{P_0 \left(\Theta(z) - \Theta(z-L) \right) \left(\Theta(s) - \Theta(s-R) \right)}_{P_z} \hat{z}$$

$$\nabla \times \vec{P} = -\frac{\partial P_z}{\partial s} \hat{\phi} = -P_0 \left(\Theta(z) - \Theta(z-L) \right) \left[\frac{\partial \Theta}{\partial s} - \frac{\partial}{\partial \theta} \Theta(s-R) \right] \hat{\phi}$$

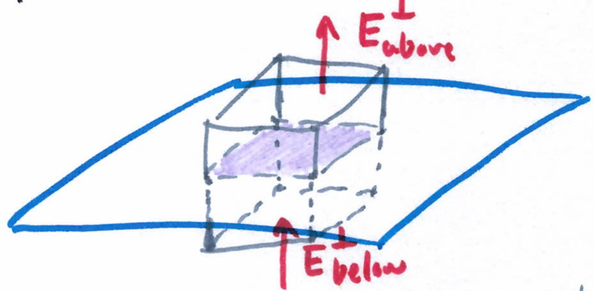
$$\nabla \times \vec{P} = -P_0 \left(\Theta(z) - \Theta(z-L) \right) \left(\delta(s) - \delta(s-R) \right) \hat{\phi}$$

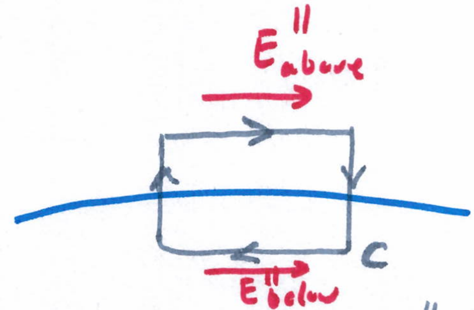
Comment:
GRIFFITHS WARNS WE CANNOT MAKE A STRICT ANALOGY OF $\vec{E}$ and $\vec{D}$ BECAUSE $\nabla \times \vec{D} = \nabla \times (\epsilon \vec{E} + \vec{P}) = \nabla \times \vec{P}$ and $\nabla \times \vec{P} \neq 0$ is possible... however many nice examples have $\nabla \times \vec{P} = 0$.

Boundary Conditions in Matter

Recall we previously found $\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$

we derived this from examining little box and loop about the surface





$$\frac{\sigma A}{\varepsilon_0} = AE^{\perp}_{above} - AE^{\perp}_{below}$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$$

$$\oint_C \vec{E}\cdot d\vec{l} = E^{\parallel}_{above} - E^{\parallel}_{below}$$

$$\nabla\times\vec{E} = 0 \Rightarrow \oint_C \vec{E}\cdot d\vec{l} = 0.$$

Since $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ and $\nabla\cdot\vec{D} = \rho_f$ we can repeat the little box argument,

$$\int_{Box} (\nabla\cdot\vec{D})\, d\tau = \int_{Box} \rho_f\, d\tau$$

$$\int_{\partial(Box)} \vec{D}\cdot d\vec{a} = Q_{enc\,(free)}$$

$$A D^{\perp}_{above} - A D^{\perp}_{below} = A\sigma_f$$

$$\boxed{D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f}$$

For the loop, $\oint_C \vec{D}\cdot d\vec{l} = D^{\parallel}_{above} - D^{\parallel}_{below}$

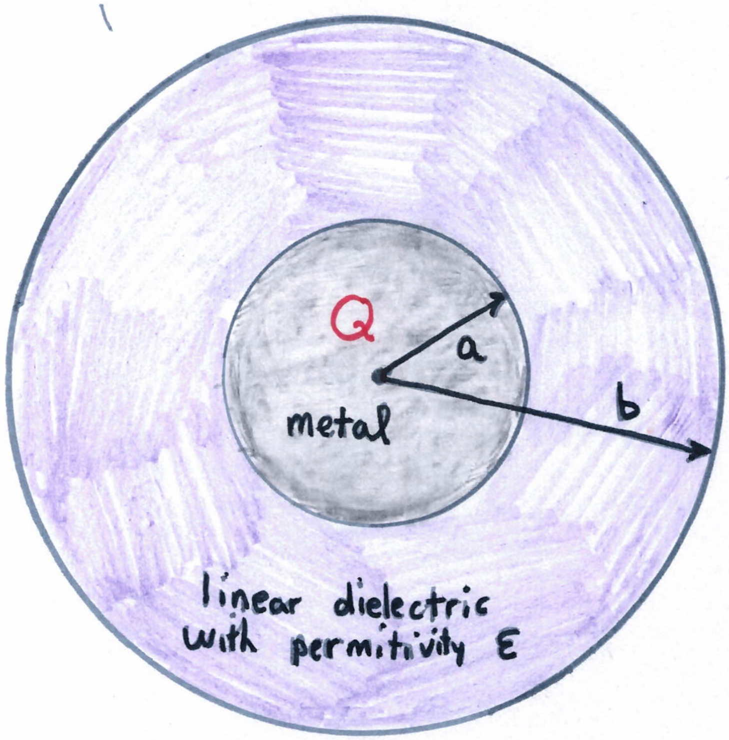
$$= \left(\varepsilon_0 E^{\parallel}_{above} + P^{\parallel}_{above}\right) - \left(\varepsilon_0 E^{\parallel}_{below} + P^{\parallel}_{below}\right)$$

$$= P^{\parallel}_{above} - P^{\parallel}_{below}$$

$$\therefore \boxed{D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}}$$

E3 Metal sphere of radius a has charge Q is surrounded by linear dielectric with permitivity ε. Find the potential

We know free charge Q so we can find $\vec{D}$.



For any sphere $r > a$

$$\int_{S_r} \vec{D} \cdot d\vec{a} = Q$$

hence by spherical symmetry

$$4\pi r^2 D = Q$$

$$\Rightarrow \boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}}$$

$(r > a)$

(1.) $0 \le r < a$ have $\vec{E} = \vec{D} = \vec{P} = 0$.

(2.) For $a < r < b$ we have $\vec{D} = \varepsilon \vec{E}$ $\therefore$ $\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon}\frac{\hat{r}}{r^2}}$ $(a < r < b)$

(3.) For $r > b$ we have $\vec{D} = \varepsilon_0 \vec{E}$ since $\vec{P} = 0$ outside the sphere. Thus $\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}}$ $(r > b)$

The potential is thus calculated, setting $V(\infty) = 0$,

$$V_0 = -\int_{\infty}^{\vec{r}_0} \vec{E} \cdot d\vec{l} = \int_{r_0}^{\infty} \vec{E} \cdot d\vec{l} = \int_{r_0}^{a} \vec{0} \cdot d\vec{l} + \int_a^b \frac{Q\,dr}{4\pi\varepsilon r^2} + \int_b^{\infty} \frac{Q\,dr}{4\pi\varepsilon_0 r^2}$$

$$V(\vec{r}_0) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q}{4\pi\varepsilon_0}\left(\frac{-1}{\infty} + \frac{1}{b}\right)$$

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi}\left(\frac{1}{a\varepsilon} + \frac{1}{b\varepsilon_0} - \frac{1}{b\varepsilon}\right)}$$

potential within the metal sphere

E3 continued

$$\vec{E}(\vec{r}) = \begin{cases} 0 & : \text{for } 0 \le r < a \\ \dfrac{Q}{4\pi\varepsilon}\dfrac{\hat{r}}{r^2} & : \text{for } a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{r}}{r^2} & : \text{for } r > b \end{cases}$$

Then the potential with $V(\infty) = 0$ is,

$$V(r) = \begin{cases} \dfrac{Q}{4\pi}\left(\dfrac{1}{\varepsilon a} - \dfrac{1}{\varepsilon b} + \dfrac{1}{\varepsilon_0 b}\right) & : \text{for } 0 \le r \le a \\ \dfrac{Q}{4\pi}\left(\dfrac{1}{\varepsilon r} - \dfrac{1}{\varepsilon b} + \dfrac{1}{\varepsilon_0 b}\right) & : \text{for } a \le r \le b \\ \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r} & : \text{for } r \ge b \end{cases}$$

Differentiate to find,

$$\frac{\partial V}{\partial r} = \begin{cases} 0 & : \text{for } 0 \le r < a \\ \dfrac{-Q}{4\pi\varepsilon r^2} & : \text{for } a < r < b \\ \dfrac{-Q}{4\pi\varepsilon_0 r^2} & : \text{for } r > b \end{cases}$$

Recall boundary conditions, since $\vec{E} = -\nabla V = -\dfrac{\partial V}{\partial r}\hat{r}$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \sigma_a = \sigma_f + \sigma_b$$

total charge density at $r=a$.

$r=a$: $\quad E^{\perp}_{above} = \dfrac{\sigma_a}{\varepsilon_0}\hat{r} = \dfrac{Q}{4\pi\varepsilon a^2}\hat{r} \Rightarrow \sigma_a = \dfrac{Q\varepsilon_0}{4\pi\varepsilon a^2}$

$$\sigma_f = \frac{Q}{4\pi a^2} \Rightarrow \sigma_b = \sigma_a - \sigma_f$$

$$= \frac{Q}{4\pi a^2}\left(\frac{\varepsilon_0}{\varepsilon} - 1\right)$$

$$= \frac{Q}{4\pi a^2}\left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon}\right) = \frac{-\varepsilon_0 \chi_e Q}{4\pi a^2 \varepsilon}$$

Reminder

$\varepsilon = \varepsilon_0(1 + \chi_e)$

$\varepsilon_0 - \varepsilon = -\varepsilon_0\chi_e$

E3 continued

$r = b$ $\quad E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma_2}{\varepsilon_0}\hat{r}$

$$\frac{Q}{4\pi\varepsilon_0 b^2} - \frac{Q}{4\pi\varepsilon a^2} = \frac{\sigma_2}{\varepsilon_0}$$

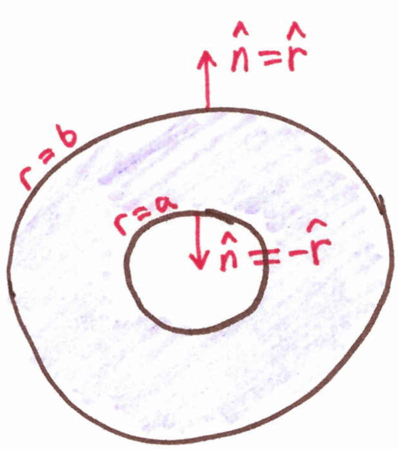
$$\sigma_2 = \sigma_{\text{bound at } r=b} + \sigma_{\text{free at } r=b}$$

$$\sigma_2 = \frac{Q}{4\pi}\left(\frac{1}{b^2} - \frac{\varepsilon_0}{\varepsilon a^2}\right) = \sigma_{\substack{\text{BOUND at}\\ r=b}} + \sigma_{\substack{\text{free at}\\ r=b}} \qquad \text{I}$$

However, there is another path to calculate σ_{BOUND}.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\hat{r} \quad (a < r < b)$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} \dfrac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2} & \text{at } r = b \qquad \text{II} \\ \dfrac{-\varepsilon_0 \chi_e Q}{4\pi\varepsilon a^2} & \text{at } r = a \end{cases}$$



Now we could calculate σ_{free} at $r = b$ from I and II

E3 continued

$$\frac{Q}{4\pi}\left(\frac{1}{b^2} - \frac{\varepsilon_0}{\varepsilon a^2}\right) = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon b^2} + \sigma_{f \text{ at } r=b}$$

$$\sigma_{f \text{ at } r=b} = \frac{Q}{4\pi}\left[\frac{1}{b^2} - \frac{\varepsilon_0}{\varepsilon a^2} - \frac{\varepsilon_0 \chi_e}{\varepsilon b^2}\right]$$

$$= \frac{Q}{4\pi}\left[\frac{1}{b^2}\left(1 - \frac{\varepsilon_0 \chi_e}{\varepsilon}\right) - \frac{\varepsilon_0}{\varepsilon a^2}\right]$$

$$= \frac{Q}{4\pi}\left[\frac{1}{b^2}\left(\frac{\varepsilon - \varepsilon_0 \chi_e}{\varepsilon}\right) - \frac{\varepsilon_0}{\varepsilon a^2}\right]$$

$$= \frac{Q}{4\pi}\left[\frac{\varepsilon_0}{\varepsilon b^2} - \frac{\varepsilon_0}{\varepsilon a^2}\right]$$

$$= \underline{\frac{Q\varepsilon_0}{4\pi\varepsilon}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)}.$$

<u>Remark</u>: I'm not 100% sure my analysis is correct here. My results at $r = a$ did agree with GRIFFITHS.