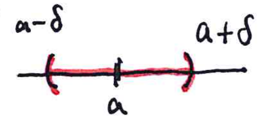
LECTURE 14: TOPOLOGY OF $\mathbb{R}$ (metric)

2.6 OPEN SETS, CLOSED SETS, COMPACT SETS, AND LIMIT POINTS



The *open ball* in $\mathbb{R}$ with center $a \in \mathbb{R}$ and radius $\delta > 0$ is the set

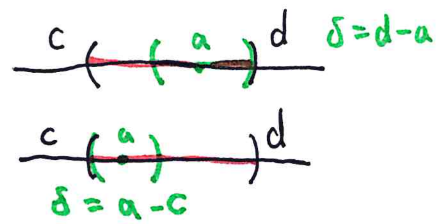
$$B(a;\delta) = (a-\delta, a+\delta).$$

Definition 2.6.1 A subset A of $\mathbb{R}$ is said to be *open* if for each $a \in A$, there exists $\delta > 0$ such that

$$B(a;\delta) \subset A.$$

■ **Example 2.6.1** (1) Any open interval $A = (c,d)$ is open. Indeed, for each $a \in A$, one has $c < a < d$. Let

$$\delta = \min\{a-c, d-a\}.$$



Then

$$B(a;\delta) = (a-\delta, a+\delta) \subset A.$$

Therefore, A is open.

(2) The sets $A = (-\infty, c)$ and $B = (c, \infty)$ are open, but the set $C = [c, \infty)$ is not open. The reader can easily verify that A and B are open. Let us show that C is not open. Assume by contradiction that C is open. Then, for the element $c \in C$, there exists $\delta > 0$ such that

$$B(c;\delta) = (c-\delta, c+\delta) \subset C.$$

However, this is a contradiction because $c - \delta/2 \in B(c;\delta)$, but $c - \delta/2 \notin C$.

$c \in [c,d]$

$\forall \delta > 0 \ (c-\delta, c+\delta) \not\subseteq [c,d]$

$\therefore \nexists\, \delta > 0$ s.t. $B_\delta(c) \subseteq [c,d]$

$\therefore [c,d]$ not open.

Theorem 2.6.1 The following hold:

(a) The subsets $\emptyset$ and $\mathbb{R}$ are open.
(b) The union of any collection of open subsets of $\mathbb{R}$ is open.
(c) The intersection of a finite number of open subsets of $\mathbb{R}$ is open.

Proof: The proof of (a) is straightforward.

(b) Suppose $\{G_\alpha : \alpha \in I\}$ is an arbitrary collection of open subsets of $\mathbb{R}$. That means G_α is open for every $\alpha \in I$. Let us show that the set

$$G = \bigcup_{\alpha \in I} G_\alpha$$

is open. Take any $a \in G$. Then there exists $\alpha_0 \in I$ such that

$$a \in G_{\alpha_0}.$$

Since G_{α_0} is open, there exists $\delta > 0$ such that

$$B(a;\delta) \subset G_{\alpha_0}. \subseteq \bigcup_{\alpha \in I} G_\alpha = G$$

This implies

$$B(a;\delta) \subset G$$

because $G_{\alpha_0} \subset G$. Thus, G is open.

Lemma: if $\delta_1 < \delta_2$ then $B_{\delta_1}(a) \subset B_{\delta_2}(a)$

(c) Suppose $G_i, i = 1, \ldots, n$, are open subsets of $\mathbb{R}$. Let us show that the set

$$G = \bigcap_{i=1}^{n} G_i$$

is also open. Take any $a \in G$. Then $a \in G_i$ for $i = 1, \ldots, n$. Since each G_i is open, there exists $\delta_i > 0$ such that

$$B(a; \delta_i) \subset G_i.$$

Let $\delta = \min\{\delta_i : i = 1, \ldots, n\}$. Then $\delta > 0$ and

$$B(a; \delta) \subset G.$$

$\mathbb{R} - [c,d] = \underbrace{(-\infty, c)}_{\text{open}} \cup \underbrace{(d, \infty)}_{\text{open}}$

Thus, G is open. □

Definition 2.6.2 A subset S of $\mathbb{R}$ is called *closed* if its complement, $S^c = \mathbb{R} \setminus S$, is open.

■ **Example 2.6.2** The sets $[a,b]$, $(-\infty, a]$, and $[a, \infty)$ are closed. Indeed, $(-\infty, a]^c = (a, \infty)$ and $[a, \infty)^c = (-\infty, a)$ which are open by Example 2.6.1. Since $[a,b]^c = (-\infty, a) \cup (b, \infty)$, $[a,b]^c$ is open by Theorem 2.6.1. Also, single element sets are closed since, say, $\{b\}^c = (-\infty, b) \cup (b, \infty)$.

$\mathbb{R} - \{b\} = (-\infty, b) \cup (b, \infty).$

Theorem 2.6.2 The following hold:

(a) The sets $\emptyset$ and $\mathbb{R}$ are closed.
(b) The intersection of any collection of closed subsets of $\mathbb{R}$ is closed.
(c) The union of a finite number of closed subsets of $\mathbb{R}$ is closed.

Proof: The proofs for these are simple using the De Morgan's law. Let us prove, for instance, (b). Let $\{S_\alpha : \alpha \in I\}$ be a collection of closed sets. We will prove that the set

$$S = \bigcap_{\alpha \in I} S_\alpha$$

$S^c = \mathbb{R} - S$

is also closed. We have

$$S^c = \left(\bigcap_{\alpha \in I} S_\alpha \right)^c = \bigcup_{\alpha \in I} S_\alpha^c.$$

Thus, S^c is open because it is a union of opens sets in $\mathbb{R}$ (Theorem 2.6.1(b)). Therefore, S is closed. □

■ **Example 2.6.3** It follows from part (c) and Example 2.6.2 that any finite set is closed. $\{a_1, a_2, \ldots, a_n\} = \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_n\}$

Theorem 2.6.3 A subset A of $\mathbb{R}$ is closed if and only if for any sequence $\{a_n\}$ in A that converges to a point $a \in \mathbb{R}$, it follows that $a \in A$.

Proof: Suppose A is a closed subset of $\mathbb{R}$ and $\{a_n\}$ is a sequence in A that converges to a. Suppose by contradiction that $a \notin A$. Since A is closed, there exists $\varepsilon > 0$ such that $B(a; \varepsilon) = (a - \varepsilon, a + \varepsilon) \subset A^c$. Since $\{a_n\}$ converges to a, there exists $N \in \mathbb{N}$ such that

$B_\delta(a) = B(a; \delta)$

$$a - \varepsilon < a_N < a + \varepsilon. \leftrightarrow (|a_N - a| < \varepsilon)$$

This implies $a_N \in A^c$, a contradiction.

Let us now prove the converse. Suppose by contradiction that A is not closed. Then A^c is not open. Since A^c is not open, there exists $a \in A^c$ such that for any $\varepsilon > 0$, one has $B(a;\varepsilon) \cap A \neq \emptyset$. In particular, for such an a and for each $n \in \mathbb{N}$, there exists $a_n \in B(a;\frac{1}{n}) \cap A$. It is clear that the sequence $\{a_n\}$ is in A and it is convergent to a (because $|a_n - a| < \frac{1}{n}$, for all $n \in \mathbb{N}$). This is a contradiction since $a \notin A$. Therefore, A is closed. □

max(0,1) d.n.e.

Theorem 2.6.4 If A is a nonempty subset of $\mathbb{R}$ that is closed and bounded above, then $\max A$ exists. Similarly, if A is a nonempty subset of $\mathbb{R}$ that is closed and bounded below, then $\min A$ exists

Proof: Let A be a nonempty closed set that is bounded above. Then $\sup A$ exists. Let $m = \sup A$. To complete the proof, we will show that $m \in A$. Assume by contradiction that $m \notin A$. Then $m \in A^c$, which is an open set. So there exists $\delta > 0$ such that

$$(m-\delta, m+\delta) \subset A^c.$$

This means there exists no $a \in A$ with $a \in (m-\delta, m+\delta) \rightarrow \nexists a$ with $m-\delta < a < m+\delta$

$$m - \delta < a \leq m.$$

This contradicts the fact that m is the least upper bound of A (see Proposition 1.5.1). Therefore, $\max A$ exists. □

Definition 2.6.3 A subset A of $\mathbb{R}$ is called *compact* if for every sequence $\{a_n\}$ in A, there exists a subsequence $\{a_{n_k}\}$ that converges to a point $a \in A$.[1]

■ **Example 2.6.4** Let $a, b \in \mathbb{R}$, $a \leq b$. We show that the set $A = [a,b]$ is compact. Let $\{a_n\}$ be a sequence in A. Since $a \leq a_n \leq b$ for all n, then the sequence is bounded. By the Bolzano-Weierstrass theorem (Theorem 2.4.1), we can obtain a convergent subsequence $\{a_{n_k}\}$. Say, $\lim_{k\to\infty} a_{n_k} = s$. We now must show that $s \in A$. Since $a \leq a_{n_k} \leq b$ for all k, it follows from Theorem 2.1.5, that $a \leq s \leq b$ and, hence, $s \in A$ as desired. We conclude that A is compact.

Theorem 2.6.5 A subset A of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof: Suppose A is a compact subset of $\mathbb{R}$. Let us first show that A is bounded. Suppose, by contradiction, that A is not bounded. Then for every $n \in \mathbb{N}$, there exists $a_n \in A$ such that

$$|a_n| \geq n.$$

Since A is compact, there exists a subsequence $\{a_{n_k}\}$ that converges to some $a \in A$. Then

$$|a_{n_k}| \geq n_k \geq k \quad \text{for all } k.$$

Therefore, $\lim_{k\to\infty} |a_{n_k}| = \infty$. This is a contradiction because $\{|a_{n_k}|\}$ converges to $|a|$. Thus A is bounded.

Let us now show that A is closed. Let $\{a_n\}$ be a sequence in A that converges to a point $a \in \mathbb{R}$. By the definition of compactness, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to $b \in A$. Then $a = b \in A$ and, hence, A is closed by Theorem 2.6.3.

For the converse, suppose A is closed and bounded and let $\{a_n\}$ be a sequence in A. Since A is bounded, the sequence is bounded and, by the Bolzano-Weierstrass theorem (Theorem 2.4.1), it

[1] This definition of compactness is more commonly referred to as *sequential compactness*.

has a convergent subsequence, $\{a_{n_k}\}$. Say, $\lim_{k\to\infty} a_{n_k} = a$. It now follows from Theorem 2.6.3 that $a \in A$. This shows that A is compact as desired. $\square$

Definition 2.6.4 (cluster/limit/accumulation point). Let A be a subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ (not necessarily in A) is called a *limit point* of A if for any $\delta > 0$, the open ball $B(a;\delta)$ contains an infinite number of points of A.

A point $a \in A$ which is not an accumulation point of A is called an *isolated point of A*.

Example 2.6.5 (1) Let $A = [0,1)$. Then $a = 0$ is a limit point of A and $b = 1$ is also a limit point of A. In fact, any point of the interval $[0,1]$ is a limit point of A. The set $[0,1)$ has no isolated points.

0 1

(2) Let $A = \mathbb{Z}$. Then A does not have any limit points. Every element of $\mathbb{Z}$ is an isolated point of $\mathbb{Z}$.

$z \in \mathbb{Z}$

$B_\delta(z) \cap \mathbb{Z} = \{z\}$ for $\delta < 1$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

(3) Let $A = \{1/n : n \in \mathbb{N}\}$. Then $a = 0$ is the only limit point of A. All elements of A are isolated points.

Example 2.6.6 If G is an open subset of $\mathbb{R}$ then every point of G is a limit point of G. In fact, more is true. If G is open and $a \in G$, then a is a limit point of $G \setminus \{a\}$. Indeed, let $\delta > 0$ be such that $B(a;\delta) \subset G$. Then $(G \setminus \{a\}) \cap B(a;\delta) = (a-\delta, a) \cup (a, a+\delta)$ and, thus $B(a;\delta)$ contains an infinite number of points of $G \setminus \{a\}$.

The following theorem is a variation of the Bolzano-Weierstrass theorem.

Theorem 2.6.6 Any infinite bounded subset of $\mathbb{R}$ has at least one limit point.

Proof: Let A be an infinite subset of $\mathbb{R}$ and let $\{a_n\}$ be a sequence of A such that

$$a_m \neq a_n \text{ for } m \neq n$$

(see Theorem 1.2.7). Since $\{a_n\}$ is bounded, by the Bolzano-Weierstrass theorem (Theorem 2.4.1), it has a convergent subsequence $\{a_{n_k}\}$. Set $b = \lim_{k\to\infty} a_{n_k}$. Given $\delta > 0$, there exists $K \in \mathbb{N}$ such that $a_{n_k} \in B(b;\delta)$ for $k \geq K$. Since the set $\{a_{n_k} : k \geq K\}$ is infinite, it follows that b is a limit point of A. $\square$

The following definitions and results provide the framework for discussing convergence within subsets of $\mathbb{R}$.

– Relative Topology –

Definition 2.6.5 Let D be a subset of $\mathbb{R}$. We say that a subset V of D is *open in D* if for every $a \in V$, there exists $\delta > 0$ such that

$$B(a;\delta) \cap D \subset V.$$

Theorem 2.6.7 Let D be a subset of $\mathbb{R}$. A subset V of D is open in D if and only if there exists an open subset G of $\mathbb{R}$ such that

$$V = D \cap G.$$

Proof: Suppose V is open in D. By definition, for every $a \in V$, there exists $\delta_a > 0$ such that

$$B(a;\delta_a) \cap D \subset V.$$

Define

$$G = \cup_{a\in V} B(a;\delta_a)$$

Then G is a union of open subsets of $\mathbb{R}$, so G is open. Moreover,

$$V \subset G \cap D = \cup_{a\in V}[B(a;\delta_a)\cap D] \subset V.$$

Therefore, $V = G\cap D$.

Let us now prove the converse. Suppose $V = G\cap D$, where G is an open set. For any $a \in V$, we have $a \in G$, so there exists $\delta > 0$ such that

$$B(a;\delta) \subset G.$$

It follows that

$$B(a;\delta)\cap D \subset G\cap D = V.$$

The proof is now complete. □

■ **Example 2.6.7** Let $D = [0,1)$ and $V = [0,\frac{1}{2})$. We can write $V = D\cap(-1,\frac{1}{2})$. Since $(-1,\frac{1}{2})$ is open in $\mathbb{R}$, we conclude from Theorem 2.6.7 that V is open in D. Notice that V itself is not an open subset of $\mathbb{R}$.

The following theorem is now a direct consequence of Theorems 2.6.7 and 2.6.1.

Theorem 2.6.8 Let D be a subset of $\mathbb{R}$. The following hold:

(a) The subsets $\emptyset$ and D are open in D.
(b) The union of any collection of open sets in D is open in D.
(c) The intersection of a finite number of open sets in D is open in D.

Definition 2.6.6 Let D be a subset of $\mathbb{R}$. We say that a subset A of D is *closed in* D if $D\setminus A$ is open in D.

Theorem 2.6.9 Let D be a subset of $\mathbb{R}$. A subset K of D is closed in D if and only if there exists a closed subset F of $\mathbb{R}$ such that

$$K = D\cap F.$$

Proof: Suppose K is a closed set in D. Then $D\setminus K$ is open in D. By Theorem 2.6.7, there exists an open set G such that

$$D\setminus K = D\cap G.$$

It follows that

$$K = D\setminus(D\setminus K) = D\setminus(D\cap G) = D\setminus G = D\cap G^c.$$

Let $F = G^c$. Then F is a closed subset of $\mathbb{R}$ and $K = D\cap F$.

Conversely, suppose that there exists a closed subset F of $\mathbb{R}$ such that $K = D\cap F$. Then

$$D\setminus K = D\setminus(D\cap F) = D\setminus F = D\cap F^c.$$

Since F^c is an open subset of $\mathbb{R}$, applying Theorem 2.6.7 again, one has that $D\setminus K$ is open in D. Therefore, K is closed in D by definition. □

■ **Example 2.6.8** Let $D = [0,1)$ and $K = [\frac{1}{2},1)$. We can write $K = D \cap [\frac{1}{2},2]$. Since $[\frac{1}{2},2]$ is closed in $\mathbb{R}$, we conclude from Theorem 2.6.9 that K is closed in D. Notice that K itself is not a closed subset of $\mathbb{R}$.

Corollary 2.6.10 Let D be a subset of $\mathbb{R}$. A subset K of D is closed in D if and only if for every sequence $\{x_k\}$ in K that converges to a point $\bar{x} \in D$ it follows that $\bar{x} \in K$.

Proof: Let D be a subset of $\mathbb{R}$. Suppose K is closed in D. By Theorem 2.6.9, there exists a closed subset F of $\mathbb{R}$ such that

$$K = D \cap F.$$

Let $\{x_k\}$ be a sequence in K that converges to a point $\bar{x} \in D$. Since $\{x_k\}$ is also a sequence in F and F is a closed subset of $\mathbb{R}$, $\bar{x} \in F$. Thus, $\bar{x} \in D \cap F = K$.

Let us prove the converse. Suppose by contradiction that K is not closed in D or $D \setminus K$ is not open in D. Then there exists $\bar{x} \in D \setminus K$ such that for every $\delta > 0$, one has

$$B(\bar{x};\delta) \cap D \not\subseteq D \setminus K.$$

In particular, for every $k \in \mathbb{N}$,

$$B\left(\bar{x};\frac{1}{k}\right) \cap D \not\subseteq D \setminus K.$$

For each $k \in \mathbb{N}$, choose $x_k \in B(\bar{x};\frac{1}{k}) \cap D$ such that $x_k \notin D \setminus K$. Then $\{x_k\}$ is a sequence in K and, moreover, $\{x_k\}$ converges to $\bar{x} \in D$. Then $\bar{x} \in K$. This is a contradiction. We conclude that K is closed in D. $\square$

The following theorem is a direct consequence of Theorems 2.6.9 and 2.6.2.

Theorem 2.6.11 Let D be a subset of $\mathbb{R}$. The following hold:

(a) The subsets $\emptyset$ and D are closed in D.

(b) The intersection of any collection of closed sets in D is closed in D.

(c) The union of a finite number of closed sets in D is closed in D.

■ **Example 2.6.9** Consider the set $D = [0,1)$ and the subset $A = [\frac{1}{2},1)$. Clearly, A is bounded. We showed in Example 2.6.8 that A is closed in D. However, A is not compact. We show this by finding a sequence $\{a_n\}$ in A for which no subsequence converges to a point in A.

Indeed, consider the sequence $a_n = 1 - \frac{1}{2n}$ for $n \in \mathbb{N}$. Then $a_n \in A$ for all n. Moreover, $\{a_n\}$ converges to 1 and, hence, every subsequence also converges to 1. Since $1 \notin A$, it follows that A is not compact.

Exercises

2.6.1 Prove that a subset A of $\mathbb{R}$ is open if and only if for any $x \in A$, there exists $n \in \mathbb{N}$ such that $(x - 1/n, x + 1/n) \subset A$.

2.6.2 Prove that the interval $[0,1)$ is neither open nor closed.

2.6.3 ▶ Prove that if A and B are compact subsets of $\mathbb{R}$, then $A \cup B$ is a compact set.

2.6.4 Prove that the intersection of any collection of compact subsets of $\mathbb{R}$ is compact.

2.6.5 Find all limit points and all isolated points of each of the following sets:

(a) $A = (0,1)$.
(b) $B = [0,1)$.
(c) $C = \mathbb{Q}$.
(d) $D = \{m + 1/n \, : \, m,n \in \mathbb{N}\}$.

2.6.6 Let $D = [0,\infty)$. Classify each subset of D below as open in D, closed in D, neither or both. Justify your answers.

(a) $A = (0,1)$.
(b) $B = \mathbb{N}$.
(c) $C = \mathbb{Q} \cap D$.
(d) $D = (-1,1]$.
(e) $E = (-2,\infty)$.