Lecture 15: Crystals, Capacitors & Energy

A **crystal** may be easier to polarize in some directions than others. The simple linear dielectric had $\vec{P} = \epsilon_0 \chi_e \vec{E}$ where as generally we need a tensor for susceptibility

$$\underbrace{\begin{aligned} P_x &= \epsilon_0 \left(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z \right) \\ P_y &= \epsilon_0 \left(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z \right) \\ P_z &= \epsilon_0 \left(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z \right) \end{aligned}}$$

$$\vec{P} = \epsilon_0 \, \underline{\chi}_e \vec{E} \quad , \quad (\underline{\chi}_e)_{ij} = \chi_{e_{x_i x_j}}$$

susceptibility tensor

$\boxed{E1}$ Example 4.7

$\boxed{E2}$ Example 4.8

Energy in Dielectric Systems

Recall the work to charge up capacitor

$$W = \frac{1}{2} C V^2$$

If capacitor filled with linear dielectric then

$$\underbrace{C = \epsilon_r C_{vac}}$$

you have to accumulate more free charge to achieve a given potential if we add a dielectric

Recall, energy stored in electrostatic system,

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau$$

We suspect, from the dielectric filled capacitor

$$W = \frac{\epsilon_0}{2} \int \epsilon_r E^2 d\tau = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau$$

Next we derive the result above

Bring in free charge a bit at a time

$$\Delta\rho_f \longrightarrow \Delta W = \int (\Delta\rho_f) V \, d\tau$$

Recall $\nabla\cdot\vec{D} = \rho_f$ hence $\Delta\rho_f = \nabla\cdot(\Delta\vec{D})$

$$\Delta W = \int [\nabla\cdot(\Delta\vec{D})] V \, d\tau$$

$$= \int \nabla\cdot[(\Delta\vec{D})V] d\tau - \int (\Delta\vec{D})\cdot\nabla V \, d\tau$$

$$= \int \nabla\cdot[\Delta\vec{D} V] d\tau + \int (\Delta\vec{D})\cdot\vec{E} \, d\tau$$

Taking this integral over all space the

$$\int_{\mathcal{V}} \nabla\cdot[\Delta\vec{D} V] \, d\tau = \int_{\partial\mathcal{V}} (\Delta\vec{D} V)\cdot d\vec{a} \longrightarrow 0$$

Hence,

$$\boxed{\Delta W = \int (\Delta\vec{D})\cdot\vec{E} \, d\tau}$$

for any material we can make the argument above.

For <u>linear</u> <u>dielectric</u> $\vec{D} = \epsilon\vec{E}$ then, infinitesimally,

$$\frac{1}{2}\Delta(\vec{D}\cdot\vec{E}) = \frac{1}{2}\Delta(\epsilon\vec{E}\cdot\vec{E})$$

$$= \frac{1}{2}\Delta(\epsilon E^2)$$

$$= \epsilon(\Delta\vec{E})\cdot\vec{E}$$

$$= \Delta(\epsilon\vec{E})\cdot\vec{E}$$

$$= (\Delta\vec{D})\cdot\vec{E}$$

$$\Delta W = \Delta\left(\frac{1}{2}\int (\vec{D}\cdot\vec{E}) d\tau\right)$$

$$\Downarrow$$

$$\boxed{W = \frac{1}{2}\int (\vec{D}\cdot\vec{E}) d\tau}$$

$\boxed{E3}$ discuss § 4.4.4 if energy allows.